Hokkaido Mathematical Journal Vol. 13 (1984), p. 347-361

Transformation equations and the special values of Shimura's zeta functions

By Koji DOI, Haruzo HIDA and Yoshitaka MAEDA

(Received April 16, 1984)

Introduction

As is well known, the study of the transformation equations for modular forms has one of its origins in Klein's work [5], and many authors, e.g., Kiepert [4], Hurwitz [3], Fricke [1] and Herglotz [2], did certain contributions in this theory. For a modular form h of weight k on the congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbf{Z})$, the transformation equation for h is defined by

$$\Phi(X;\ h)=\prod_{\alpha\in\Gamma_0(N)\backslash SL_2(\mathbf{Z})}(X-h|_k\alpha)=0\,,$$

where $h|_k\alpha$ denotes the usual action of $\alpha\in SL_2(\mathbf{Z})$ of weight k (for the notation, see § 1). The above mentioned references are mainly concerned with $\Delta(Nz)$ and related functions as h for the discriminant function Δ.

For a long time, the importance of the investigation of these equations has been in the mind of the first author of this paper.

Now recently, Shimura [11] proved the algebraicity at certain integers of the zeta function defined by

$$D(s,f,g)=\sum_{n=1}^{\infty}a(n)\,b(n)\,n^{-s}\,,$$

where $f=\sum_{n=1}^{\infty}a(n)\,e(nz)$ $(e(z)=\exp(2\pi iz))$ is a primitive cusp form on $\Gamma_0(N)$ of weight k and $g=\sum_{n=0}^{\infty}b(n)\,e(nz)$ is an arithmetic modular form on $\Gamma_0(N)$ of weight l less than k. Then $D(m,f,g)$ for $(k+l)/2-1<m<k$ is an algebraic number times the Petersson self inner product of f and a power of π. We take as h the product of g and a certain Eisenstein series $E_{\lambda,N}^{*}$ which is utilized in his proof of the algebraicity of $D(m,f,g)$. Then the sum of μ-th power of all the roots of $\Phi(X;\ h)=0$ can be expressed as a finite linear combination of primitive forms of level 1 with the coefficients $D(m,f,g')$ for $g'=g^{\mu}(E_{\lambda,N}^{*})^{\mu-1}$. In fact, we have

THEOREM. *For an arbitrary element $g\in S_l(\Gamma_0(N))$ and for any positive integers μ and $\lambda>2$, we have*

$$\text{(i)}\quad \mathrm{Tr}\,(gE_{\lambda,N}^{*})^{\mu}=c\sum_{f\in P(k\mu)}\frac{D(k\mu-1,f,g^{\mu}(E_{\lambda,N}^{*})^{\mu-1})}{\pi^{k\mu}\langle f,f\rangle}f$$

with $c=3\cdot 4^{-(k\mu-1)}\Gamma(k\mu-1)$. *Here* $k=l+\lambda$; $P(k\mu)$ *is the set of all primitive forms in* $S_{k\mu}(SL_2(\mathbf{Z}))$; $S_\nu(\Gamma)$ *stands for the space of cusp forms of weight* ν *on* Γ *for* $\Gamma=\Gamma_0(N)$ *or* $SL_2(\mathbf{Z})$; $E^*_{\lambda,N}$ *is the Eisenstein series defined by*

$$E^*_{\lambda,N}(z)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)}(cz+d)^{-\lambda}\qquad \left(\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right),$$

where $\Gamma_\infty=\left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\middle| n\in\mathbf{Z}\right\}$ (*cf.* [11, (2.6)]); Tr *is the trace operator of* $S_{k\mu}(\Gamma_0(N))$ *to* $S_{k\mu}(SL_2(\mathbf{Z}))$ *defined by*

$$\mathrm{Tr}\,(h)=\sum_{\alpha\in\Gamma_0(N)\backslash SL_2(\mathbf{Z})}h|_{k\mu}\alpha\qquad \textit{for}\quad h\in S_{k\mu}(\Gamma_0(N)).$$

One can specialize the equation $\Phi(X;\,h)=0$ at any elliptic curve $\mathcal{E}$ defined over $\mathbf{Q}$ in the usual manner described in § 3. Suppose that $\mathcal{E}$ is uniformized by the Weierstrass $\wp$-function with modulus (ω_1, ω_2) and assume the following conditions:

(1) *the cusp form* g *has Fourier coefficients in* $\mathbf{Q}$;

(2) *the specialized equation of* $\Phi(X;\, gE^*_{\lambda,N})$ *at* $\mathcal{E}_{/\mathbf{Q}}$ *is irreducible in* $\mathbf{Q}[X]$;

(3) *the space* $S_{k\mu}(SL_2(\mathbf{Z}))$ *is spanned by an element* $f=\sum_{n=1}^\infty a(n)\,e(nz)$ *of* $P(k\mu)$ *and its conjugates*

$$f^\sigma(z)=\sum_{n=1}^\infty a(n)^\sigma e(nz)\qquad \textit{for}\quad \sigma\in\mathrm{Aut}\,(\mathbf{C}).$$

Under the assumption (2), let K_N be the field generated over $\mathbf{Q}$ by the root $(2\pi/\omega_2)^k g(\omega_1/\omega_2)\,E^*_{\lambda,N}(\omega_1/\omega_2)$ of the specialized equation and let K_f be the Hecke field generated over $\mathbf{Q}$ by all the Fourier coefficients of f. Then we have

COROLLARY. *We have the following equality*:

$$\text{(ii)}\quad \mathrm{Tr}_{K_N/\mathbf{Q}}\left\{(2\pi/\omega_2)^k g(\omega_1/\omega_2)\,E^*_{\lambda,N}(\omega_1/\omega_2)\right\}^\mu$$
$$=c\,\mathrm{Tr}_{K_f/\mathbf{Q}}\left\{\frac{D(k\mu-1,f,g^\mu(E^*_{\lambda,N})^{\mu-1})}{\pi^{k\mu}\langle f,f\rangle}(2\pi/\omega_2)^{k\mu}f(\omega_1/\omega_2)\right\},$$

where c *is the constant in the theorem.*

One can regard this equality (ii) as a *reciprocity law* for two distinct fields K_N and K_f. In fact, as may be well known and will be explained in § 3, the field K_N coincides with the field $\mathbf{Q}(j,j')$ generated by the j-invariant $j=j(\omega_1/\omega_2)$ of $\mathcal{E}$ and $j'=j(N\omega_1/\omega_2)$. Note that K_N is contained in the field of N-section points of $\mathcal{E}$. On the other hand, as proved by Shimura [11, Th. 3], the value $\dfrac{D(k\mu-1,f,g^\mu(E^*_{\lambda,N})^{\mu-1})}{\pi^{k\mu}\langle f,f\rangle}$ in the right-hand side of the equality (ii) is contained in the field K_f, and as will be seen in § 3, $(2\pi/\omega_2)^{k\mu}f(\omega_1/\omega_2)$ also belongs to K_f. Therefore the right-hand side of the equality (ii) is the trace of the number

$$\frac{D(k\mu-1,f,g^{\mu}(E^{*}_{\lambda,N})^{\mu-1})}{\pi^{k\mu}\langle f,f\rangle}(2\pi/\omega_2)^{k\mu}f(\omega_1/\omega_2)$$

from K_f to $\mathbf{Q}$, and meanwhile the left-hand side is the trace of the number

$$\left\{(2\pi/\omega_2)^k g(\omega_1/\omega_2)\, E^{*}_{\lambda,N}(\omega_1/\omega_2)\right\}^{\mu}$$

from K_N to $\mathbf{Q}$. This is why we call the equality (ii) the reciprocity law between K_N and K_f. The importance of these fields is needless to say.

Those who are familiar with the Eichler-Selberg trace formula may find a similarity between our result and the method of calculation of the eigenvalues of Hecke operators. Especially, the power sum of the roots of our equation is a sum of the special values of Shimura's zeta functions and that of eigenvalues of Hecke operators is a sum of class numbers of imaginary quadratic fields.

It should be noted that the method using power sums and Newton's formulas is classically adopted in the investigation of transformation equations (e.g., Hurwitz [3, pp. 580–590]).

Here is a summary of the paper: The theorem will be proved in § 4. For the convenience of the readers, we will give a short and very elementary account of the theory of transformation equations in §§ 1, 2 and 3, and thus one can read the result without any knowledge of this theory beforehand. If the reader is familiar with this subject, we recommend him to proceed into § 4 directly. A numerical example of the transformation equations will be given in § 5. More examples will be published in [7] by one of the authors of the present paper.

§ 1. Transformation equations of Fricke type

In this section, we are going to give an exposition of classical results on modular forms. The details may be found in Fricke [1].

Let Γ be a congruence subgroup of $SL_2(\mathbf{Z})$ containing

$$\Gamma(N)=\left\{\gamma\in SL_2(\mathbf{Z})\,|\,\gamma\equiv 1 \mod N\right\}$$

for a positive integer N, and $\mathcal{M}_k(\Gamma)$ be the space of modular forms of weight k on Γ. The element f of $\mathcal{M}_k(\Gamma)$ is a function on the upper half complex plane $\mathfrak{H}$ with the properties:

(1. 1_a) f is holomorphic on $\mathfrak{H}$;

(1. 1_b) Writing $(f|_k\gamma)(z)=\det(\gamma)^{k/2}f(\gamma(z))(cz+d)^{-k}$ for $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2^{+}(\mathbf{R})$ $=\{\alpha\in GL_2(\mathbf{R})|\det(\alpha)>0\}$, we have $f|_k\gamma=f$ for all $\gamma\in\Gamma$;

(1. 1_c) $f|_k\gamma$ has the Fourier expansion of the form: $\sum_{n=0}^{\infty}a(n)\,e(nz/N)$ at $i\infty$ for all $\gamma\in SL_2(\mathbf{Z})$ $(e(z)=\exp(2\pi iz))$.

The space $\mathcal{M}_k(\Gamma)$ is naturally isomorphic to the space of functions F on

$$\mathfrak{Z}=\left\{\omega={}^t(\omega_1,\omega_2)\in\mathbf{C}^2\,\middle|\,\omega_2\neq 0,\ \mathrm{Im}\,(\omega_1/\omega_2)>0\right\}$$

with the properties:

(1. 2_a) F is holomorphic on $\mathfrak{Z}$;

(1. 2_b) $F(\lambda\omega)=\lambda^{-k}F(\omega)$ for all $\lambda\in\mathbf{C}^\times$, $F(\gamma\omega)=F(\omega)$ for all $\gamma\in\Gamma$;

(1. 2_c) Writing $(F|\gamma)(\omega)=F(\gamma\omega)$ for $\gamma\in GL_2^+(\mathbf{R})$, we have $(F|\gamma)({}^t(z,1))=\sum_{n=0}^{\infty}a(n)\,e(nz/N)$ for all $\gamma\in SL_2(\mathbf{Z})$ $(z\in\mathfrak{H})$.

This isomorphism assigns $f(z)=F({}^t(z,1))\in\mathcal{M}_k(\Gamma)$ to F as above. In fact, we have $\gamma\begin{pmatrix}z\\1\end{pmatrix}=\begin{pmatrix}\gamma(z)\\1\end{pmatrix}(cz+d)$ for $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2^+(\mathbf{R})$ and $z\in\mathfrak{H}$; thus for F with (1. $2_{a,b,c}$) and $\gamma\in\Gamma$, we see

$$\begin{aligned}f\big(\gamma(z)\big)&=F\big({}^t(\gamma(z),1)\big)=F\left(\gamma\begin{pmatrix}z\\1\end{pmatrix}(cz+d)^{-1}\right)=(cz+d)^kF\left(\gamma\begin{pmatrix}z\\1\end{pmatrix}\right)\\&=(cz+d)^kF\big({}^t(z,1)\big)=(cz+d)^kf(z)\,.\end{aligned}$$

This proves (1. 1_b). The conditions (1. $1_{a,c}$) for f are obvious from (1. $2_{a,c}$) for F. The inverse correspondence can be defined as

$$F\big({}^t(\omega_1,\omega_2)\big)=\omega_2^{-k}f(z)\quad\text{for}\quad f\in\mathcal{M}_k(\Gamma)\qquad(z=\omega_1/\omega_2)\,.$$

We call the function F corresponding to f as above the homogeneous form of f. Classically the homogeneous form is used often (e.g. see [1]), and is rather more suited for our later use. This is because we recall here the definition of the homogeneous form. *We hereafter identify the space $\mathcal{M}_k(\Gamma)$ with that of the homogeneous forms on $\mathfrak{Z}$.*

Now we are going to give some explanation on the process of forming the transformation equation of f in $\mathcal{M}_k(\Gamma)$. Let R be a complete set of representatives for $\Gamma\backslash SL_2(\mathbf{Z})$, and define

(1. 3) $$\Phi(X;f)=\prod_{\alpha\in R}(X-f|_k\alpha)=\sum_{m=0}^{d}(-1)^m\sigma_m(f)\,X^{d-m}\,,$$

where $d=[SL_2(\mathbf{Z}):\Gamma]$ and $\sigma_m(f)$ is the m-th elementary symmetric function of variables $\{f|_k\alpha\}_{\alpha\in R}$. The equation $\Phi(X;f)=0$ will be called the transformation equation of f. We see easily that

(1. 4) $$\sigma_m(f)\in\mathcal{M}_{km}\big(SL_2(\mathbf{Z})\big)\,.$$

It is well known that $\sigma_m(f)$ can be expressed uniquely as an isobaric polynomial of g_2 and g_3, where $g_2\in\mathcal{M}_4(SL_2(\mathbf{Z}))$ and $g_3\in\mathcal{M}_6(SL_2(\mathbf{Z}))$ are defined by

$$g_2\left({}^t(\omega_1, \omega_2)\right) = 60 \sum'_{m,n\in\mathbf{Z}} (m\omega_1 + n\omega_2)^{-4}$$
$$= (2\pi/\omega_2)^4 \left\{ \frac{1}{12} + 20 \sum_{n=1}^{\infty} \left(\sum_{0<d\mid n} d^3\right) e(nz) \right\},$$

and

$$g_3\left({}^t(\omega_1, \omega_2)\right) = 140 \sum'_{m,n\in\mathbf{Z}} (m\omega_1 + n\omega_2)^{-6}$$
$$= (2\pi/\omega_2)^6 \left\{ \frac{1}{216} - \frac{7}{3} \sum_{n=1}^{\infty} \left(\sum_{0<d\mid n} d^5\right) e(nz) \right\}.$$

Let us explain this in more detail. As classically known, we can take a canonical basis of $\mathcal{M}_k(SL_2(\mathbf{Z}))$ for even $k \geqq 4$ as described below: We know that

$$r = \dim \mathcal{M}_k\left(SL_2(\mathbf{Z})\right) = [k/12] \text{ or } [k/12]+1$$

according as $k \equiv 2 \bmod 12$ or not.
Then $4a + 6b = k - 12(r-1)$ has the unique solution with non-negative integers a and b. Put

$$(1.5_a) \qquad h_i = (12g_2)^a (216g_3)^{b+2(r-1-i)} \Delta^i \qquad (0 \leqq i \leqq r-1),$$

where $\Delta = g_2^3 - 27g_3^2 = (2\pi/\omega_2)^{12} e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24}$. Write

$$h_i\left({}^t(\omega_1, \omega_2)\right) = (2\pi/\omega_2)^k \sum_{j=0}^{\infty} c(i,j)\, e(jz).$$

Then $c(i,j)$ are rational integers and the matrix of the Fourier coefficients has the convenient form:

$$(1.5_b) \qquad \left(c(i,j)\right)_{i,j=0,1,\cdots,r-1} = \begin{pmatrix} 1 & & \\ & 1 & * \\ 0 & \ddots & \\ & & 1 \end{pmatrix}.$$

This shows that $\{h_0, \cdots, h_{r-1}\}$ is a basis of $\mathcal{M}_k(SL_2(\mathbf{Z}))$; in particular, every modular form $h = \sum_{n=0}^{\infty} c(n)\, e(nz)$ of $\mathcal{M}_k(SL_2(\mathbf{Z}))$ is a linear combination of h_i with coefficients in $\Lambda = \mathbf{Z}[c(n) \mid 0 \leqq n < r]$. Thus we can express h as an isobaric polynomial of g_2 and g_3 with coefficients in Λ. The uniqueness of the polynomial follows from [12, Prop. 2.27 and its proof]. The integrality over Λ of the expression of h by the basis h_i will guarantee the integrality of the specialized equation discussed in § 3 (see the statement below (3.6) in § 3). In order to determine the coefficients of this isobaric polynomial, one may use the Fourier coefficients of $(2\pi/\omega_2)^{-4} g_2$, $(2\pi/\omega_2)^{-6} g_3$ and $(2\pi/\omega_2)^{-k} h$ instead of those of g_2, g_3 and h. *When we deal with those isobaric polynomials, we hereafter drop the factor of the power of* $2\pi/\omega_2$.

To relate our equations to those classically investigated, we are going to specialize the group Γ to

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbf{Z})\,\middle|\, c\equiv 0 \bmod N\right\}.$$

The above argument still holds even for meromorphic modular forms f of weight k, but in this case, the coefficients of $\Phi(X;f)$ become rational functions of g_2 and g_3. For example, let us take the usual J-invariant; namely, J is defined by

$$J(z)=12^3 g_2(z)^3/\Delta(z).$$

We may take $J(Nz)$ as f; then the equation $\Phi(X;\ J(Nz))=0$ is classically called the transformation (or modular) equation of level N (cf. Fricke [1, II. 3. 2, pp. 342–349] or Shimura [9, § 5]).

§ 2. Transformation equations of arithmetic modular functions

In this section, we first recall Shimura's theory of arithmetic modular functions and then explain its relation to transformation equations. The main reference is his book [12, Chap. 6].

For a positive integer N, we write $\mathscr{F}_N$ for the field of modular functions on $\mathfrak{H}/\Gamma(N)$ of the form g/f for some $f,\ g\in\mathscr{M}_k(\Gamma(N))$ $(k>0)$ with Fourier coefficients in $k_N=\mathbf{Q}(e(1/N))$. We further define for a subring A of $\mathbf{C}$ and $\Gamma=\Gamma_0(N)$, $\Gamma_1(N)$ or $\Gamma(N)$,

$$\mathscr{M}_k(\Gamma;\ A)=\left\{f\in\mathscr{M}_k(\Gamma)\,\middle|\, f(z)=\textstyle\sum_{n=0}^{\infty}a(n)\,e(nz/N),\ a(n)\in A \text{ for all } n\right\}.$$

Here $\Gamma_1(N)$ is a subgroup of $\Gamma_0(N)$ defined by

$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbf{Z})\,\middle|\, a\equiv d\equiv 1 \bmod N,\ c\equiv 0 \bmod N\right\}.$$

We can define an action of $\sigma\in\mathrm{Gal}(k_N/\mathbf{Q})$ on $\mathscr{M}_k(\Gamma;\ k_N)$ by

$$f^{\sigma}(z)=\textstyle\sum_{n=0}^{\infty}a(n)^{\sigma}e(nz/N) \quad\text{for}\quad f(z)=\sum_{n=0}^{\infty}a(n)\,e(nz/N).$$

Then this action is known to be well defined (cf. [10, § 4 Prop. 4]). Thus, for $h=g/f\in\mathscr{F}_N$ with $g,\ f\in\mathscr{M}_k(\Gamma(N);\ k_N)$, $h^{\sigma}=g^{\sigma}/f^{\sigma}$ is also contained in $\mathscr{F}_N$ and this gives an action of $\mathrm{Gal}(k_N/\mathbf{Q})$ on $\mathscr{F}_N$. In [12], $\mathscr{F}_N$ is defined as the field generated over $\mathbf{Q}$ by $J(z)$ and

$$f_a(\omega)=\frac{g_2(\omega)\,g_3(\omega)}{\Delta(\omega)}\wp(a\omega;\ \omega) \qquad (\omega\in\mathfrak{Z})$$

for all $a=(r/N,s/N)$ with $0\leqq r<N$, $0\leqq s<N$ and $(r,s)\neq(0,0)$, where

$$\wp(u;\ \omega)=u^{-2}+\textstyle\sum'_{l\in L(\omega)}\left[(u-l)^{-2}-l^{-2}\right]$$

for $L(\omega)=\mathbf{Z}\omega_1+\mathbf{Z}\omega_2$, $\omega={}^t(\omega_1,\ \omega_2)$.

By [12, Prop. 6. 9], this definition is equivalent to the one given above. The function f_a for $a\in\mathbf{Q}^2-\mathbf{Z}^2$ depends only on $a \bmod \mathbf{Z}$, and $f_{-a}=f_a$.

It is known by [12, Th. 6. 6] that $\mathscr{F}_N$ is a finite Galois extension of $\mathscr{F}_1=\mathbf{Q}(J)$, and $\mathrm{Gal}(\mathscr{F}_N/\mathbf{Q}(J))\cong GL_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$, where $\gamma\in GL_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$ acts on f_a and J through

$$f_a^{\gamma}=f_{a\gamma} \quad\text{and}\quad J^{\gamma}=J\,.$$

Then the action of $\sigma\in\mathrm{Gal}(k_N/\mathbf{Q})$ on $\mathscr{F}_N$ as defined above corresponds to the matrix

$$\begin{pmatrix}1 & 0\\ 0 & d\end{pmatrix}\in GL_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$$

when $e(1/N)^{\sigma}=e(d/N)$ (cf. [10, § 4 Prop. 4]). Furthermore, the field $\mathscr{F}_N$ contains k_N and, as seen in [12, Th. 6. 6], the element $\gamma\in GL_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$ acts on $e(1/N)$ as $e(1/N)^{\gamma}=e(\det(\gamma)/N)$. Since $f_{a\gamma}=f_a|\gamma$ by the definition of f_a, every $\gamma\in SL_2(\mathbf{Z})$ defines an automorphism of $\mathscr{F}_N$ and the corresponding element of $GL_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$ $(=\mathrm{Gal}(\mathscr{F}_N/\mathbf{Q}(J)))$ is the natural image $\gamma \bmod N$ in $GL_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$.

Let us put

$$U_0(N)=\left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\in GL_2(\mathbf{Z}/N\mathbf{Z})\,\middle|\, a, d\in(\mathbf{Z}/N\mathbf{Z})^{\times},\ b\in\mathbf{Z}/N\mathbf{Z}\right\},$$

and

$$U_1(N)=\left\{\pm\begin{pmatrix}1 & b\\ 0 & d\end{pmatrix}\in GL_2(\mathbf{Z}/N\mathbf{Z})\,\middle|\, d\in(\mathbf{Z}/N\mathbf{Z})^{\times},\ b\in\mathbf{Z}/N\mathbf{Z}\right\}.$$

Let $\mathscr{K}_N$ and $\mathscr{L}_N$ be the subfields of $\mathscr{F}_N$ corresponding to $U_0(N)/\{\pm 1\}$ and $U_1(N)/\{\pm 1\}$, respectively. Then by [12, Prop. 6. 9], we know

(2. 1) If $h=f/g$ with $g, f\in\mathscr{M}_k(\Gamma_0(N)\,;\,\mathbf{Q})$ (resp. $\mathscr{M}_k(\Gamma_1(N)\,;\,\mathbf{Q})$), then $h\in\mathscr{K}_N$ (resp. $\mathscr{L}_N$).

By [10, Th. 6], (2. 1) can be generalized as

(2. 2) *If $h=f/g$ with $g, f\in\mathscr{M}_k(\Gamma_0(N)\,;\,K)$* (resp. $\mathscr{M}_k(\Gamma_1(N)\,;\,K)$), *then $h\in K\mathscr{K}_N$* (resp. $K\mathscr{L}_N$) *for any algebraic number field K.*

For $\varphi\in\mathscr{K}_N$, if $\varphi^{\sigma}\neq\varphi$ for any non-trivial isomorphism σ of $\mathscr{K}_N$ into $\mathscr{F}_N$ over $\mathbf{Q}(J)$, then φ gives a generator of $\mathscr{K}_N$ over $\mathbf{Q}(J)$. Thus, by our construction of $\Phi(X;\,\varphi)$, we know that

(2. 3) *$\Phi(X;\,\varphi)$ is irreducible as an element of $\mathbf{Q}(J)[X]$ if $\varphi|\gamma\neq\varphi$ for $\gamma\in\Gamma_0(N)\backslash SL_2(\mathbf{Z})$ and $\gamma\neq 1$.*

In fact, one can take a representative set for $U_0(N)\backslash GL_2(\mathbf{Z}/N\mathbf{Z})$ in $\Gamma_0(N)\backslash SL_2(\mathbf{Z})$. In general, when f is an element of $\mathscr{M}_k(\Gamma_0(N)\,;\,\mathbf{Q})$, then

(2. 4) *$\Phi(X; f/g)$ is irreducible for a* $\mathbf{Q}$*-rational meromorphic modular form g of weight k on $SL_2(\mathbf{Z})$ if $f|_k\gamma\neq f$ for any $\gamma\in\Gamma_0(N)\backslash SL_2(\mathbf{Z})$ and $\gamma\neq 1$.*

Now, if we form the transformation equation $\Phi(X; f)=0$ for $f\in\mathscr{F}_N$, $\Phi(X; f)=0$ gives an equation of f over $\mathbf{Q}(J)$. Especially, when we take $J(Nz)$ as f, the classical transformation equation $\Phi(X; J(Nz))=0$ gives the defining equation of the field $\mathscr{K}_N$ over $\mathbf{Q}(J)$ (cf. [12, p. 157]).

§ 3. Specialization of transformation equations at elliptic curves

In this section, we specialize transformation equations at an elliptic curve defined over $\mathbf{Q}$.

It is known by Tate [13, Th. 3. 2 and Remark 3, p. 40] that every elliptic curve defined over $\mathbf{Q}$ has a minimal model over $\mathbf{Z}$, which is defined by the equation of the form:

(3. 1) $y^2+a_1xy+a_3y=x^3+a_2x^2+a_4x+a_6$

with $a_1, \cdots, a_6\in\mathbf{Z}$. Take the minimal model defined by the equation (3. 1). Then according to [13, (1. 2)], putting

$$(3.2)\quad \begin{cases} b_2=a_1^2+4a_2\,, \\ b_4=a_1a_3+2a_4\,, \\ b_6=a_3^2+4a_6\,, \\ b_8=a_1^2a_6-a_1a_3a_4+4a_2a_6+a_2a_3^2-a_4^2\,, \\ 12\bar{g}_2=b_2^2-24b_4\,, \\ 216\bar{g}_3=-b_2^3+36b_2b_4-216b_6\,, \\ \bar{\Delta}=-b_2^2b_8-8b_4^3-27b_6^2+9b_2b_4b_6=\bar{g}_2^3-27\bar{g}_3^2\,, \end{cases}$$

we know that the curve $\mathcal{E}$ defined by

$$y^2=4x^3-\bar{g}_2x-\bar{g}_3$$

is isomorphic over $\mathbf{Q}$ to the curve (3. 1). Thus

(3. 3) *$12\bar{g}_2$, $216\bar{g}_3$ and $\bar{\Delta}$ are all rational integers.*

This is why we have started from the minimal model (3. 1).

As is well known, by choosing a suitable basis $\{c_1, c_2\}$ of the singular homology group $H_1(\mathcal{E}; \mathbf{Z})$, we see that $\omega={}^t(\omega_1, \omega_2)$ for $\omega_1=\int_{c_1}dx/y$, $\omega_2=\int_{c_2}dx/y$ satisfies:

(3. 4_a) $\omega\in\mathfrak{Z}$;

(3. 4_b) $g_2(\omega)=\bar{g}_2$, $g_3(\omega)=\bar{g}_3$ and $\Delta(\omega)=\bar{\Delta}$.

Thus the Weierstrass functions $\wp(u;\omega)$ and $\wp'(u;\omega)$ uniformize the elliptic curve $\mathcal{E}$ (cf. [12, 4. 2]); namely, we know

$$(3.5)\quad \begin{aligned} &\mathbf{C}/L(\omega)\cong\mathcal{E}(\mathbf{C})\subset\mathbf{P}^2(\mathbf{C}),\\ &\qquad u\mapsto(\wp(u;\omega),\ \wp'(u;\omega),\ 1)\end{aligned}$$

where $L(\omega)=\mathbf{Z}\omega_1+\mathbf{Z}\omega_2$ and $\mathcal{E}(\mathbf{C})$ is the group of $\mathbf{C}$-rational points of $\mathcal{E}$.

Now we consider the transformation equation $\Phi(X;f)=0$ for a modular form f of $\mathcal{M}_k(\Gamma)$ as in § 1. For a subring Λ of $\mathbf{C}$, we assume

(3.6) *all the coefficients $\sigma_m(f)$ of the polynomial $\Phi(X;f)$ have Λ-rational Fourier expansions.*

Then every $\sigma_m(f)$ is expressed as a Λ-linear combination of $\{h_i\}$ in (1. 5_a). By substituting $\bar{g}_2$ and $\bar{g}_3$ for g_2 and g_3 in the isobaric polynomials of $\sigma_m(f)$, we obtain an equation $\Phi(X;f,\mathcal{E})=0$ over Λ. This equation will be called the specialized equation of $\Phi(X;f)=0$ at $\mathcal{E}$.

We suppose $\Gamma=\Gamma_0(N)$ for a positive integer N, and put

$$C=\left\{\left(\wp(a\omega_2;\omega),\ \wp'(a\omega_2;\omega),\ 1\right)\in\mathbf{P}^2(\mathbf{C})\,\middle|\,a=r/N,\ 0\leqq r<N,\ r\in\mathbf{Z}\right\}\subset\mathcal{E}(\mathbf{C})\,.$$

Then we define a field K_N by the fixed field in $\bar{\mathbf{Q}}$ by all automorphisms σ of $\bar{\mathbf{Q}}$ such that there is an isomorphism over $\bar{\mathbf{Q}}$ of $(\mathcal{E},C)$ onto $(\mathcal{E}^\sigma,C^\sigma)$. To consider the structure $(\mathcal{E},C)$ is equivalent to consider the pair $(\mathcal{E},\mathcal{E}/C)$ for the quotient $\mathcal{E}/C$ of $\mathcal{E}$ by the subgroup C. Thus K_N is generated over $\mathbf{Q}$ by the J-invariants of $\mathcal{E}$ and $\mathcal{E}/C$; namely, we have

$$K_N=\mathbf{Q}\left(J(\omega_1/\omega_2),\ J(N\omega_1/\omega_2)\right).$$

If every coefficient of $\Phi(X;h)$ for $h\in S_k(\Gamma_0(N))$ has $\mathbf{Q}$-rational Fourier expansion and if the specialized equation $\Phi(X;h,\mathcal{E})=0$ is irreducible over $\mathbf{Q}$, then K_N is generated over $\mathbf{Q}$ by the root $(2\pi/\omega_2)^k h(\omega_1/\omega_2)$ of $\Phi(X;h,\mathcal{E})=0$. Thus the definition of K_N in the corollary in Introduction coincides with that given here. This fact follows easily from the argument in the proof of the following proposition. It is well known (cf. [12, p. 157]) that

$$\mathcal{K}_N=\mathbf{Q}\left(J(z),\ J(Nz)\right).$$

This shows

(3.7) *K_N is the residue field of the valuation of $\mathcal{K}_N$ at the point $\omega_1/\omega_2\in\mathfrak{H}$.*

Put $\mathcal{E}[N]=\{x\in\mathcal{E}\,|\,Nx=0\}$. Then we see from (3. 5) that

(3.8) $\mathcal{E}[N]\cong(\mathbf{Z}/N\mathbf{Z})^2$ through $(\mathbf{Z}/N\mathbf{Z})^2\ni a\mapsto(\wp(a\omega;\omega),\ \wp'(a\omega;\omega),\ 1)\in\mathcal{E}[N]\,.$

Let F_N be the smallest field of rationality of $\mathcal{E}[N]$ over $\mathbf{Q}$. Then F_N is a finite Galois extension of $\mathbf{Q}$ and through the isomorphism (3. 8), we can identify $\mathrm{Gal}(F_N/\mathbf{Q})$ with a subgroup of $GL_2(\mathbf{Z}/N\mathbf{Z})$. Put

$$U_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GL_2(\mathbf{Z}/N\mathbf{Z})\middle| c=0\right\}.$$

Then we see from the construction of $\mathcal{K}_N$ in [12, Chap. 6] that

(3. 9) if $\mathrm{Gal}(F_N/\mathbf{Q})=GL_2(\mathbf{Z}/N\mathbf{Z})$, then K_N is the fixed field of $U_0(N)$.

PROPOSITION. *Assume* (3. 6) *for* $f\in\mathcal{M}_k(\Gamma_0(N))$. *Then* $\Phi(X; f, \mathcal{E})$ *belongs to* $\Lambda[X]$ *and* $\Phi(X; f, \mathcal{E})=0$ *has at least one root in* ΛK_N.

PROOF. The fact $\Phi(X; f, \mathcal{E})\in\Lambda[X]$ was already shown; so we are going to prove the second one. To reduce the problem to the case of modular functions, we want to find a modular form ψ of weight k with $\psi(\omega)\neq 0$ whose behavior under any automorphism over $\mathbf{Q}(J)$ is known. Put

$$(3.10)\quad \delta(\omega)=\Delta(\omega)^{1/6}=(2\pi/\omega_2)^2 e(z/6)\prod_{n=1}^{\infty}(1-e(nz))^4$$

$$\left(\omega={}^t(\omega_1, \omega_2)\in\mathfrak{Z},\ z=\omega_1/\omega_2\right).$$

Then it is known by Hurwitz [3, p. 566] that $\delta\in S_2(\Gamma(6))$. By the addition formula combined with the definition of the Siegel functions g_a for $a\in\mathbf{Q}^2$ as in Kubert-Lang [6, p. 29], we know

$$(3.11)\quad \wp\left(\frac{1}{2}\omega_1;\ \omega\right)-\wp\left(\frac{1}{2}\omega_2;\ \omega\right)=-\frac{g_{a+b}(\omega)\,g_{a-b}(\omega)}{g_a^2(\omega)\,g_b^2(\omega)}\delta(\omega)$$

$$\text{for}\quad a=\left(\frac{1}{2},\ 0\right),\qquad b=\left(0,\ \frac{1}{2}\right).$$

The function $\dfrac{g_{a+b}g_{a-b}}{g_a^2 g_b^2}$ is contained in $\mathcal{F}_6$ (cf. [6, K3, p. 28]). Thus $\delta(\omega)$ is contained in F_6. For f as in the proposition, we know $\varphi=f/\delta^{k/2}$ is contained in $\Lambda\mathcal{F}_{6N}$. Any automorphism σ of $\Lambda\mathcal{F}_{6N}$ over $\mathcal{F}=\Lambda\mathcal{K}_N$ can be represented by an element $\gamma\in\Gamma_0(N)$ and $d\in(\mathbf{Z}/6N\mathbf{Z})^\times$ so that $\sigma=\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}(\gamma \bmod 6N)\in GL_2(\mathbf{Z}/6N\mathbf{Z})$. Then we see $\varphi^\sigma=(f|\gamma)/(\delta^{k/2}|\gamma)=f/(\delta^{k/2}|\gamma)$ from the definition. On the other hand, we see $\delta|\gamma=\zeta(\gamma)\,\delta$ for a 6-th root $\zeta(\gamma)$ of unity by [3, p. 566]. Any automorphism σ' of ΛF_{6N} over $F=\Lambda K_N$ can be naturally lifted to an automorphism σ of $\Lambda\mathcal{F}_{6N}$ over $\mathcal{F}$ by Hilbert's theory of decomposition groups. The automorphism σ is represented by γ and d as above, and we see from (3. 11) that

$$\delta(\omega)^{\sigma'}=(\delta|\gamma)(\omega)=\zeta(\gamma)\,\delta(\omega).$$

Thus $f(\omega)=\delta(\omega)^{k/2}\varphi(\omega)$ is invariant under all the automorphisms of ΛF_{6N} over F. Note that $\delta(\omega)\neq 0$ by (3.10). Then this shows the proposition.

Let us now give some remarks. When f is a modular form of level 1 with algebraic Fourier coefficients, then it is clear from the definition of the basis $\{h_i\}$ as in (1.5_a) that

(3.12) *the isobaric polynomial $P(g_2, g_3)$ of f in g_2 and g_3 has coefficients in the field K_f,*

where K_f is the field generated over $\mathbf{Q}$ by all the Fourier coefficients of f. Furthermore, if $f(z)=\sum_{n=0}^{\infty}a(n)\,e(nz)\in\mathcal{M}_k(SL_2(\mathbf{Z}))$, it is known that $f^\sigma(z)=\sum_{n=0}^{\infty}a(n)^\sigma e(nz)$ again belongs to $\mathcal{M}_k(SL_2(\mathbf{Z}))$ for any automorphism σ of $\mathbf{C}$. Especially,

(3.13) *the isobaric polynomial of f^σ is given by $P^\sigma(g_2, g_3)$,*

where σ acts on $P(g_2, g_3)$ through its coefficients.

Now we consider the transformation equation $\Phi(X;\, gE^*_{\lambda,N})=0$ as in the introduction. Note that if $h\in\mathcal{M}_\nu(\Gamma_0(N);\,\mathbf{Q})$, then by virtue of [11, Th. 3], for any primitive form $f\in S_\kappa(SL_2(\mathbf{Z}))$ and for any $m\in\mathbf{Z}$ with $(\kappa+\nu)/2-1<m<\kappa$,

$$(3.14_a)\quad \frac{D(m,f,h)}{\pi^\kappa\langle f,f\rangle}\in K_f\,;$$

$$(3.14_b)\quad \left\{\frac{D(m,f,h)}{\pi^\kappa\langle f,f\rangle}\right\}^\sigma=\frac{D(m,f^\sigma,h)}{\pi^\kappa\langle f^\sigma,f^\sigma\rangle}\quad\text{for any automorphism } \sigma \text{ of } \mathbf{C}\,.$$

Now take an elliptic curve $\mathcal{E}$ defined over $\mathbf{Q}$. Then the sum S_μ of μ-th power of all the roots of $\Phi(X;\, gE^*_{\lambda,N}, \mathcal{E})=0$ is expressed by the theorem as

$$S_\mu=c\sum_{f\in P(k\mu)}\frac{D(k\mu-1,f,g^\mu(E^*_{\lambda,N})^{\mu-1})}{\pi^{k\mu}\langle f,f\rangle}\left(\frac{2\pi}{\omega_2}\right)^{k\mu}f(\omega_1/\omega_2)\,.$$

Note that $E^*_{\lambda,N}$ has $\mathbf{Q}$-rational Fourier expansion. Then for any automorphism σ of $\mathbf{C}$, we know from (3.13) and (3.14_b) that if $g\in S_l(\Gamma_0(N);\,\mathbf{Q})$, then

$$(3.15)\quad \left\{\frac{D(k\mu-1,f,g^\mu(E^*_{\lambda,N})^{\mu-1})}{\pi^{k\mu}\langle f,f\rangle}\left(\frac{2\pi}{\omega_2}\right)^{k\mu}f(\omega_1/\omega_2)\right\}^\sigma$$
$$=\frac{D(k\mu-1,f^\sigma,g^\mu(E^*_{\lambda,N})^{\mu-1})}{\pi^{k\mu}\langle f^\sigma,f^\sigma\rangle}\left(\frac{2\pi}{\omega_2}\right)^{k\mu}f^\sigma(\omega_1/\omega_2)\,.$$

This shows $S_\mu\in\mathbf{Q}$ for every μ; namely, $\Phi(X;\, gE^*_{\lambda,N}, \mathcal{E})\in\mathbf{Q}[X]$.

When $h\neq h|\gamma$ for $h=gE^*_{\lambda,N}$ with any $\gamma\in SL_2(\mathbf{Z})-\Gamma_0(N)$,

(3.16) *$\Phi(X;\, gE^*_{\lambda,N}, \mathcal{E})$ is irreducible over $\mathbf{Q}$ except for finitely many elliptic curves $\mathcal{E}$.*

§ 4. Proof of the theorem

Let $\mathfrak{F}$ be a fundamental domain for $\mathfrak{H}/SL_2(\mathbf{Z})$ and $m(\mathfrak{F})$ the volume of $\mathfrak{F}$ relative to the measure $y^{-2}dxdy$. We know $m(\mathfrak{F})=\pi/3$. Let R be a complete set of representatives for $\Gamma_0(N)\backslash SL_2(\mathbf{Z})$. Then $\mathfrak{F}_0=\bigcup_{\alpha\in R}\alpha^{-1}(\mathfrak{F})$ is a fundamental domain for $\mathfrak{H}/\Gamma_0(N)$. For $f\in S_\kappa(\Gamma_0(N))$ and $h\in\mathcal{M}_\kappa(\Gamma_0(N))$, the normalized Petersson inner product is defined by

$$\langle f, h\rangle = m(\mathfrak{F}_0)^{-1}\int_{\mathfrak{F}_0}\overline{f(z)}\,h(z)\,y^{\kappa-2}dxdy \qquad (z=x+iy)\,,$$

where $m(\mathfrak{F}_0)$ is the volume of $\mathfrak{F}_0$ relative to the measure $y^{-2}dxdy$.

Take an arbitrary element h of $S_\nu(\Gamma_0(N))$ and let $E^*_{\lambda,N}$ be the Eisenstein series of weight $\lambda>2$ as in the introduction, then $E^*_{\lambda,N}\in\mathcal{M}_\lambda(\Gamma_0(N))$. Put $\kappa=\lambda+\nu$. Then for a primitive form f on $SL_2(\mathbf{Z})$ of weight κ, we have

$$\begin{aligned} & m(\mathfrak{F})\left\langle f, \mathrm{Tr}\,(hE^*_{\lambda,N})\right\rangle \\ & = \int_{\mathfrak{F}}\bar{f}\left(\textstyle\sum_{\alpha\in R} hE^*_{\lambda,N}|_\kappa\alpha\right) y^{\kappa-2}dxdy \\ & = \int_{\mathfrak{F}_0}\bar{f}hE^*_{\lambda,N}y^{\kappa-2}dxdy \\ & = (4\pi)^{-(\kappa-1)}\Gamma(\kappa-1)\,D(\kappa-1,f,h)\,. \end{aligned}$$

The last equality follows from the equality at $s=\kappa-1$ of [11, (2. 3)]. On the other hand, we can write

$$\mathrm{Tr}\,(hE^*_{\lambda,N})=\textstyle\sum_{f\in P(\kappa)}c(f)f \quad \text{with } c(f)\in\mathbf{C}\,,$$

where $P(\kappa)$ is the set of all primitive forms in $S_\kappa(SL_2(\mathbf{Z}))$. By the orthogonality for primitive forms under the Petersson inner product, $c(f)$ is expressed as

$$c(f)=\left\langle f, \mathrm{Tr}\,(hE^*_{\lambda,N})\right\rangle\Big/\langle f,f\rangle\,.$$

Thus we have

$$\begin{aligned} (4.1)\quad \mathrm{Tr}\,(hE^*_{\lambda,N}) & = (4\pi)^{-(\kappa-1)}\Gamma(\kappa-1)\,m(\mathfrak{F})^{-1}\textstyle\sum_{f\in P(\kappa)}\dfrac{D(\kappa-1,f,h)}{\langle f,f\rangle}f \\ & = 3\cdot 4^{-(\kappa-1)}\Gamma(\kappa-1)\textstyle\sum_{f\in P(\kappa)}\dfrac{D(\kappa-1,f,h)}{\pi^\kappa\langle f,f\rangle}f\,. \end{aligned}$$

Now, let g be an element of $S_l(\Gamma_0(N))$. For a positive integer μ, we take $g^\mu(E^*_{\lambda,N})^{\mu-1}$ as h in (4. 1). Then we have

$$\mathrm{Tr}\,(gE^*_{\lambda,N})^\mu = 3\cdot 4^{-(k\mu-1)}\Gamma(k\mu-1)\textstyle\sum_{f\in P(k\mu)}\dfrac{D(k\mu-1,f,g^\mu(E^*_{\lambda,N})^{\mu-1})}{\pi^{k\mu}\langle f,f\rangle}f\,,$$

where $k=l+\lambda$. Thus proves the theorem.

§ 5. Numerical examples

We shall give an example of the transformation equation $\Phi(X;\, gE^*_{\lambda,N})=0$ and the specialized equations $\Phi(X;\, gE^*_{\lambda,N}, \mathcal{E})=0$ at several elliptic curves $\mathcal{E}$. A few more examples of this type of equations will be published in [7].

Let us take a primitive form $h\in S_4(\Gamma_0(5))$. Since we know $\dim S_4(\Gamma_0(5))=1$, h is uniquely determined. Put $g=-5\cdot 13h$ to guarantee **Z**-rationality of the equation. Then the transformation equation $\Phi(X;\, gE^*_{4,5})=0$ is given by

$$\begin{aligned}\Phi(X;\, gE^*_{4,5}) = {} & X^6-25(12g_2)\Delta X^4-1440\Delta^2X^3+155(12g_2)^2\Delta^2X^2\\ & +\left\{(12g_2)(216g_3)^2\Delta^2+18096(12g_2)\Delta^3\right\}X\\ & +\left\{65(216g_3)^2\Delta^3+538240\Delta^4\right\}\\ = {} & 0\,.\end{aligned}$$

The coefficients of this equation are calculated through the Newton formula by the power sums $\mathrm{Tr}\,(gE^*_{4,5})^\mu$ given below:

μ	$\mathrm{Tr}\,(gE^*_{4,5})^\mu$
1	0
2	$50(12g_2)\Delta$
3	$4320\Delta^2$
4	$630(12g_2)^2\Delta^2$
5	$-5(12g_2)(216g_3)^2\Delta^2+89520(12g_2)\Delta^3$
6	$7610(216g_3)^2\Delta^3+16815360\Delta^4$

In what follows, we are going to give the specialized equations $\Phi(X;\, gE^*_{4,5}, \mathcal{E})=0$ at several elliptic curves $\mathcal{E}$ defined over **Q**. We first list the curves where we specialize the above transformation equation $\Phi(X;\, gE^*_{4,5})=0$:

Case A: $y^2=4x^3-\dfrac{2^2}{3}x+\dfrac{19}{3^3}$ (11A);

Case B: $y^2=4x^3-2^2x+1$ (37A);

Case C: $y^2=4x^3-\dfrac{2^3\cdot 5}{3}x+\dfrac{251}{3^3}$ (37B);

Case D: $y^2=4x^3+2^3\cdot 3x-2^3$;

Case E: $y^2=4x^3+1$ (27A).

The curve in Case A is isogeneous to the modular curve $X_0(11)_{/\mathbf{Q}}$ ($\cong\mathfrak{H}/\Gamma_0(11)$). This curve is referred in [14] as 11A. The curves in Case B and Case C correspond the distinct non-isogeneous factors of the jacobian variety of $X_0(37)_{/\mathbf{Q}}$. The curve in Case D is found in Serre [8, 5.9.2], which has potential everywhere good reduction. The curve in Case E has complex multiplication under $\mathbf{Q}(\sqrt{-3})$. The specialized equations $\Phi(X;\ gE^*_{4,5},\mathcal{E})=0$ at these elliptic curves are listed as:

$\mathcal{E}$	$\Phi(X;\ gE^*_{4,5},\mathcal{E})$
Case A	$X^6+4400X^4-174240X^3+4801280X^2-340643072X+5881529280$ $=(X-22)(X^5+22X^4+4884X^3-66792X^2+3331856X-267342240)$ Discriminant of the irreducible factor of degree 5 $=-2^{24}\cdot 5^3\cdot 11^8\cdot 19^2\cdot 389^2\cdot 142939^2$ Constant term of the irreducible factor of degree 5 $=-2^5\cdot 3\cdot 5\cdot 11^2\cdot 4603$
Case B	$X^6-44400X^4-1971360X^3+488897280X^2+47063460096X+1162360730560$ Discriminant $=2^{36}\cdot 3^{12}\cdot 5^5\cdot 11^6\cdot 37^{12}\cdot 42044237^2$ Constant term $=2^6\cdot 5\cdot 37^3\cdot 71711$
Case C	$X^6-148000X^4-1971360X^3+5432192000X^2+1029841968640X+14284097373120$ Discriminant $=2^{36}\cdot 5^5\cdot 37^{12}\cdot 97^2\cdot 251^4\cdot 158512865466953^2$ Constant term $=2^6\cdot 3\cdot 5\cdot 37^3\cdot 293749$
Case D	$X^6-111974400X^4-348285173760X^3+3109490031329280X^2+193955142847074140 16X+307561897831601641 88160$ Discriminant $=2^{156}\cdot 3^{102}\cdot 5^5\cdot 523^2\cdot 1993^2$ Constant term $=2^{30}\cdot 3^{20}\cdot 5\cdot 31\cdot 53$
Case E	$X^6-1049760X^3+226351350720$ Discriminant $=2^{36}\cdot 3^{66}\cdot 5^5\cdot 11^6\cdot 17^6$ Constant term $=2^6\cdot 3^{12}\cdot 5\cdot 11^3$

In the above table, all the factors of the specialized equations are irreducible over **Q**. The factors of the discriminants and the constant terms are primes if they are less than 10^{10}; otherwise, we do not know whether they are prime or not.

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Koji Doi
Haruzo Hida
Yoshitaka Maeda
Department of Mathematics
Hokkaido University
Sapporo 060, JAPAN