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Lifting modules, extending modules and their applications to QF-rings

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A right R-module M is said to be an extending module if, for any submodule A of M, there exists a direct summand A^* of M such that A^* is an essential extention of A. Dually, M is said to be a lifting module provided that, for any submodule A of M, there exists a direct summand A^* of M which is a co-essential submodule of A in M, i.e., $A^* \subseteq A$ and A/A^* is small in M/A^*.

In this paper we study the following two conditions:

($\sharp$) *Every injective R-module is a lifting module.*

$(\sharp)^{\sharp}$ *Every projective R-module is an extending module.*

A major reason why we are interested in these ($\sharp$) and $(\sharp)^{\sharp}$ comes from the fact that these conditions are closely related to the following conditions due to Harada [13]~[15]:

(*) *Every non-small R-module contains a non-zero injective submodule.*

$(*)^*$ *Every non-cosmall R-module contains a non-zero projective direct summand.*

Indeed, we show the following theorems which are main results of this paper.

THEOREM I. *The following conditions are equivalent for a given ring* R:

1) *R satisfies* ($\sharp$).

2) *R is a right artinian ring with* (*).

3) *R is a right perfect ring and the family of all injective R-modules is closed under taking small covers.*

4) *Every R-module is expressed as a direct sum of an injective module and a small module.*

THEOREM II. *The following conditions are equivalent for a given ring* R:

1) *R satisfies* $(\sharp)^{\sharp}$.

2) *R is a ring with the ACC on right annihilator ideals and satisfies* $(*)^*$.

3) *The family of all projective R-modules is closed under taking essential extensions.*

4) *Every R-module is expressed as a direct sum of a projective module and a singular module.*

It should be noted that rings satisfying the conditions in Theorem I or Theorem II are completely characterized in terms of ideals by Harada [13]~[15]. To be specific, let R be a semi-perfect ring and let $\{e_i\} \cup \{f_i\}$ be a complete set of orthogonal primitive idempotents of R, where e_iR is a non-small module and f_iR is a small module. It is shown in [13, Theorem 5] that R satisfies the condition 2) in Theorem I iff, for each e_i, there exists n_i such that e_iR/e_iS_t is injective for $0 \leq t \leq n_i$ and $e_iR/e_iS_{n_i+1}$ is a small module, where $\{S_i\}$ is the ascending Loewy chain of R as a right R-module. Dually, it is shown in [15, Theorem 3. 6] that R satisfies the condition (*)* iff 1) each e_iR is injective, 2) each f_jR can be enbedded in some e_iR and 3) for each e_i, there exists n_i such that e_iJ_t is projective for $t \leq n_i$ and $e_iJ_{n_i+1}$ is a singular module, where $\{J_i\}$ is the descending Loewy chain of R.

We call, in this paper, that R is a *right* H-ring if it satisfies one of equivalent conditions in Theorem I; while R is a *right co-H*-ring if it satisfies one of the equivalent conditions in Theorem II. *Left* H-rings and *left co-H*-rings are symmetrically defined and both right and left H-rings (resp. both right and left co-H-rings) are simply called H-rings (resp. *co*-H-rings). It is shown in Theorem 4. 3 that a ring R is a QF-ring iff it is a right H-rings with $J(R)=Z(R)$, and iff it is a right co-H-ring with $J(R)=Z(R)$, where $J(R)$ and $Z(R)$ are the Jacobson radical and the singular right ideal of R, respectively. Generalized uniserial rings are also H-rings and co-H-rings (cf. [23]). For an algebra R over a field of finite dimension, R is a right H-ring iff it is a left co-H-ring (Theorem 5. 1). Combining Theorems I and II with the Colby-Rutter's theorem [4, Theorem 1. 3], we see that right H-rings and right co-H-rings are semiprimary QF-3 rings. For a right non-singular ring R, it is shown in Theorem 4. 6 that R is a right H-ring iff it is a right co-H-ring, and iff it is Morita equivalent to a finite direct sum of upper triangular matrix rings over division rings. We can give two typical examples of right H-rings and right co-H-rings which are constructed by local QF-rings. Using one of these examples, we show that right H-rings (resp. right co-H-rings) are not always left H-rings (resp. left co-H-rings).

I. Preliminaries

Throughout this paper we assume that R is an associative ring with identity and all R-modules considered are unitary right R-modules. Let M

be an R-module. We use $E(M)$, $J(M)$, $\mathrm{Soc}(M)$ and $Z(M)$ to denote the injective hull, the Jacobson radical, the socle and the singular submodule of M, respectively. Furthermore, by $\{J_i(M)\}_I$ and $\{S_i(M)\}_I$, we denote the descending Loewy chain and the ascending Loewy chain of M, respectively;

$$\begin{array}{ll} J_0(M)=M & S_0(M)=0 \\ J_1(M)=J(M) & S_1(M)=\mathrm{Soc}(M) \\ \cdots\cdots\cdots & \cdots\cdots\cdots \end{array}$$

For submodules A, B of M with $A\subseteq B$, we write $A\subseteq_e B$ to denote that A is an essential submodule of B; while we use $A\subseteq_c B$ in M to denote that A is co-essential in B, i. e., B/A is a small submodule of M/A. For a subset X of M, $\mathrm{Ann}_r(X)$ (resp. $\mathrm{Ann}_l(X)$) means its right (resp. left) annihilator ideal $\{r\in R|Xr=0\}$ (resp. $\{r\in R|rX=0\}$).

For two R-modules M and N, we use the symbol $M\lesssim N$ to stand for 'M is isomorphic to a submodule of N'. For given set S, $|S|$ denotes its cardinal number. The term 'ACC' means the ascending chain condition. For a cardinal τ, τM means the direct sum of τ-copies of an R-module M.

Definition. We say that an R-module M is an *extending* module if, for any submodule A of M, there exists a direct summand A^* of M such that $A\subseteq_e A^*$. Dually, we say that M is a *lifting* module if, for any submodule A of M, there exists a direct summand A^* of M such that $A^*\subseteq_c A$ in M.

Definition ([17]). An R-module M is said to have the *extending property of uniform modules* if, for any uniform submodule A of M, there exists a direct summand A^* of M with $A\subseteq_e A^*$. In the case when M has a decomposition $M=\sum_I\oplus M_\alpha$, M is said to have the *extending property of finite contained uniform modules* with respect to $M=\sum_I\oplus M_\alpha$ if, for any uniform submodule A of M with $A\subseteq\sum_F\oplus M_\beta$ for some finite subset F of I, there exists a direct summand A^* of M such that $A\subseteq_e A^*$.

Definition ([18], [21]). An R-module M is said to be *continuous* if M is an extending module and satisfies the condition: For any direct summand A of M, every monomorphic image of A to M is a direct summand of M. M is said to be *quasi-continuous* if M is an extending module and satisfies the condition: For any direct summands A_1, A_2 of M, the condition $M_e\supseteq A_1\oplus A_2$ implies $M=A_1\oplus A_2$.

Definition ([22], cf [20]). An R-module M is said to be *semiperfect* if

M is a lifting module and satisfies the condition: For any direct summand A of M and any epimorphism φ from M to M/A, $\ker\varphi$ is a direct summand of M. M is said to be *quasi-semiperfect* if M is a lifting module and satisfies the condition: For any direct summands A_1, A_2 of M, if $M=A_1+A_2$ and $A_1\cap A_2$ is small in M then $M=A_1\oplus A_2$.

We note that quasi-injective$\Rightarrow$continuous$\Rightarrow$quasi-continuous; while semi-perfect$\Rightarrow$quasi-semiperfect, and that, when R is a right perfect ring, quasi-projective$\Rightarrow$semiperfect.

DEFINITION ([13]~[15], cf [24]). An R-module M is said to be a small module if it is small in its injective hull, and is said to be a non-small module if it is not a small module. Dually, M is said to be a cosmall module if, for any projective module P and any epimorphism $f: P\to M$, $\ker f$ is an essential submodule of P, and M is said to be a non-cosmall module if it is not a cosmall module.

The following results are used in this paper.

THEOREM A ([6, 20. 3 A, 20. 6 A], cf [23]). *For a given quasi-injective R-module M, the following conditions are equivalent*:

1) *M is Σ-quasi-injective.*
2) *$\chi_0 M$ is quasi-injective.*
3) *$E(M)$ is Σ-injective.*
4) *$\chi_0 E(M)$ is injective*
5) *The ACC holds on* $\{\mathrm{Ann}_r(X)\mid X\subseteq E(M)\}$.

Further, when this is so, M is expressed as a direct sum of completely indecomposable modules.

THEOREM B ([7], [26]). *Quasi-injective R-modules satisfy the exchange property.*

THEOREM C. *If M is a Σ-quasi-injective R-module, then M satisfies the following condition*:

(K) *For any independent family $\{A_\alpha\}_I$ of submodules of M, if $\sum_I\oplus A_\alpha$ is a locally direct summand of M, i.e., $\sum_F\oplus A_\beta\langle\oplus M$ for any finite subset F of I, then $\sum_I\oplus A_\alpha$ is just a direct summand of M.*

PROOF. This is clear from Theorems A, B and [12, Theorem 3. 2. 5].

We quote the Colby-Rutter's characterizations of semiprimary QF-3 rings ([4, Theorem 1. 3]) as follows:

THEOREM D. *The following conditions are equivalent for a given ring R*:

1) *R is ring perfect and contains a faithful Σ-injective right ideal.*

2) *R is right perfect and the injective hull of every projective R-module is projective.*

3) *R is right perfect and the projective cover of every injective R-module is injective.*

4) *R is right and left perfect and contains faithful injective right and left ideals, respectively.*

When this is so, then R satisfies ACC on right, and also left, annihilator ideals. The conditions 1)$\sim$4) are right-left symmetric, and a ring R satisfying one of these conditions is just a semiprimary QF-3 ring.

2. *H*-ring

In this section, we shall prove Theorem I mentioned in introduction. Therefore, we are concerned with the following:

(♯) *Every injective R-module is a lifting module.*

(*) *Every non-small R-module contains a non-zero injective submodule.*

(*ICC*) *The family of all injective R-modules is closed under taking small covers, i.e., for any exact sequence $P \xrightarrow{\phi} E \to 0$ where E is injective and $\ker\phi$ is small in P, P is injective.*

(*ISD*) *Every R-module is expressed as a direct sum of an injective module and a small module.*

The condition (*) is due to Harada ([13], [15]). It is easy to see that (*) is equivalent to the following condition: Let E be an injective R-module and A a submodule of E such that A is not small in E. Then, A contains a non-zero direct summand of E. On the other hand, we know that an R-module M is a lifting module iff, for any submodule A of M, there exists a decomposition $M=A^*\oplus A^{**}$ such that $A^*\subseteq A$ and $A\cap A^{**}$ is small in A^{**}. As a result, we have

PROPOSITION 2.1. *(♯) implies (*).*

PROPOSITION 2.2. *(♯) implies (ICC).*

PROOF. Let $P \xrightarrow{\phi} E \to 0$ be an exact sequence such that E is injective and $\ker\phi$ is small in P. Then we can take an epimorphism $\phi' : E(P)\to E$ with $\phi'|P=\phi$. By (♯), we have a decomposition $P=X\oplus Y$ such that X is injective and Y is small in $E(P)$. Then $\phi'(Y)=\phi(Y)$ is small in E. So we get from $E=\phi(X)+\phi(Y)$ that $E=\phi(Y)$. Therefore, $P=X+\ker\phi$ and hence $P=X$ since $\ker\phi$ is small in P. Thus, P is injective

PROPOSITION 2.3. *($\sharp$) is equivalent to (ISD).*

PROOF. Obviuos.

LEMMA 2.4. *If an R-module M satisfies the condition (K) in Theorem C, then M is expressed as a direct sum of indecomposable modules.*

PROOF. Assume that M does not contain a non-zero indecomposable direct summand. Then, using (K), we can see that every non-zero direct summand of M is expressed as a direct sum of countably infinite non-zero submodules. In particular, we express $M=N_1\oplus N_2$ with each $N_i\neq 0$. We pick a non-zero element x in N_1. By Zorn's lemma, we can take a maximal independent family $\{M_\alpha\}_I$ of submodules of M such that $x\notin M'=\sum_I\oplus M_\alpha$ and $M'=\sum_I\oplus M_\alpha$ is a locally direct summand of M. Then, by (K), $M=M\oplus M''$ for some submodule M''. Since $x\notin M'$, we see $M''\neq 0$. Therefore, M'' is written as a direct sum of countably infinite non-zero submodules; say $M''=\sum_J\oplus T_\beta$. Since $x\in M=M'\oplus\sum_J\oplus T_\beta$, there exists a finite subset F of J with $x\in M\oplus\sum_{F}\oplus T_\gamma$. Then, if $x\in M'\oplus\sum_{J-F}\oplus T_\beta$, M' must contain x, a contradiction. So, $x\notin M'\oplus\sum_{J-F}\oplus T_\beta$. However, this contradicts the maximality of M'. Thus, we conclude that M and every non-zero direct summand of M contain non-zero indecomposable direct summands. From this fact and (K), M is clearly expressed as a direct sum of indecomposable modules.

LEMMA 2.5. *If M is a continuous lifting R-module, then it satisfies the condition (K) in Theorem C.*

PROOF. Let $\{A_\alpha\}_I$ be an independent family of submodules of M such that $A=\sum_I\oplus A_\alpha$ is a locally direct summand of M. Since M is a lifting R-module, we have a decomposition $M=A^*\oplus A^{**}$ such that $A=A^*\oplus(A\cap A^{**})$ and $A\cap A^{**}$ is small in A^{**}. We want to show $A=A^*$. Assume that $A\cap A^{**}\neq 0$ and take a non-zero element x in $A\cap A^{**}$. Then $xR\subseteq A_{\alpha_1}\oplus\cdots\oplus A_{\alpha_n}$ for some $\{\alpha_1,\cdots,\alpha_n\}\subseteq I$. Since $B=A_{\alpha_1}\oplus\cdots\oplus A_{\alpha_n}$ is an extending module (cf. [21, Proposition 1.4]), there exists a direct summand $X\langle\oplus B$ with $xR\subseteq_e X$. Then, we see that $X\cap A^*=0$; whence there exists a submodule $Y\subseteq A\cap A^{**}$ with $X\simeq Y$. However, the continuity of M shows that $Y\langle\oplus M$, which contradicts that Y is small in M. Accordingly, $A\cap A^{**}=0$.

By Proposition 2.1, Lemmas 2.4 and 2.5 and [15, Lemma 2.1], we have

PROPOSITION 2.6. *Assume that ($\sharp$) holds. Then, every injective R-*

module is expressed as a direct sum of cyclic hollow modules. Therefore, R is a right artinian ring, by Faith-Walker's theorem ([5], [6, 20. 17]).

PROPOSITION 2.7. *If R is a right noetherian ring with (*) then it satisfies (#).*

PROOF. Let E be an injective R-module and A a non-small submodule of E. We can take a maximal independent family $\{A_\alpha\}_I$ of non-zero injective submodules of A. Put $A'=\sum_I \oplus A_\alpha$. Since R is right noetherian, A' is also injective (cf. [5, 20. 1]) ; whence we have $E=A'\oplus A''$ for some submodule A''. So, we may show that $A\cap A''$ is small in A''. If not, then we can take a non-zero injective submodule of $A\cap A''$ since A'' also satisfies (*). But this contradicts the maximality of $\{A_\alpha\}_I$.

LEMMA 2.8. *Consider an exact sequence: $H=\sum_I \oplus H_\alpha \xrightarrow{\phi} E \to 0$, where each H_α is a cyclic hollow injective R-module with non-zero socle and E is an indecomposable injective R-module but not cyclic hollow. Then, $\ker\phi$ contains* Soc(H).

PROOF. Put $K=\ker\phi$ and $S_\alpha=\mathrm{Soc}(H_\alpha)$ for each $\alpha\in I$. If K does not contain Soc(H) then there exists $\alpha\in I$ such that $S_\alpha\not\subseteq K$. In this case, $(S_\alpha+K)/K\simeq S_\alpha$ and hence $S_\alpha\subsetneqq E$. This implies that $E(S_\alpha)=H_\alpha\simeq E$; whence E is cyclic hollow, a contradiction.

NOTATION. For convenience's sake of the statement of the lemma below, we say that an R-module M satisfies (F) if M satisfies the ACC on $\{\mathrm{Ann}_r(X)\mid X\subseteq M\}$.

LEMMA 2.9. *Let e be an idempotent of R. If eR satisfies (F) then $eR/e\mathrm{Ann}_l(J^k)$ satisfies (F) for all integer $k\geq 1$, where $J=J(R)$.*

PROOF. We put $\overline{eR}=eR/e\mathrm{Ann}_l(J^k)$ and $\overline{\overline{eR}}=eR/e\mathrm{Ann}_l(J^{K+1})$. Assuming that $\overline{eR}$ satisfies (F), we want to show that $\overline{\overline{eR}}$ satisfies (F). For any subset X of eR, we note that

$$\begin{aligned}\mathrm{Ann}_r(\bar{X}) &= \left\{r\in R\mid Xr\subseteq e\mathrm{Ann}_l(J^k)\right\}\\ &= \left\{r\in R\mid XrJ^k=0\right\},\\ \mathrm{Ann}_r(\bar{\bar{X}}) &= \left\{r\in R\mid Xr\subseteq e\mathrm{Ann}_l(J^{k+1})\right\}\\ &= \left\{r\in R\mid XrJ^{k+1}=0\right\}.\end{aligned}$$

Now, assume that $\overline{\overline{eR}}$ does not satisfy (F). Then we can take subsets $\{X_i\}$ of eR such that $X_1\supseteq X_2\supseteq\cdots$ and

$$\mathrm{Ann}_r(\overline{\overline{X}}_1) \subsetneqq \mathrm{Ann}_r(\overline{\overline{X}}_2) \subsetneqq \cdots .$$

Then, there exist $r_2, r_3, \cdots$ in R such that $X_i r_i J^{k+1}=0$ but $X_{i-1} r_i J^{k+1}=0$ for $i=2, 3, \cdots$. Hence we can take $t_i \in J$ such that $X_i r_i t_i J^k=0$ but $X_{i-1} r_i t_i J^k \neq 0$ for $i=2, 3, \cdots$. This shows that $\mathrm{Ann}_r(\bar{X}_1) \subsetneqq \mathrm{Ann}_r(\bar{X}_2) \subsetneqq \cdots$, a contradiction. Thus $\overline{\overline{eR}}$ must satisfy (F).

PROPOSITION 2.10. *If R is a right perfect ring with (ICC), then the following hold:*

1) *R is a semiprimary QF-3 ring.*

2) *Every indecomposable injective R-module is a cyclic hollow module. More precisely, if E is an indecomposable injective R-module, then there exists a primitive idempotent e in R and an integer $k \geq 0$ such that E is isomorphic to $eR/e\mathrm{Ann}_l(J^k)$, where $J=J(R)$.*

3) *Every cyclic hollow non-small R-module is injective.*

PROOF. 1) This follows from Theorem D.

2) Let E be an indecomposable injective R-module. Consider a projective cover: $P \xrightarrow{\phi} E \to 0$, and express P as $P=\sum_I \oplus e_\alpha R$, where each e_α is a primitive idempotent of R. Now, (ICC) says that P is injective; so is each $e_\alpha R$. Assume that E is not cyclic hollow. Then, $\ker\phi$ contains $S_1(P)=\sum_I \oplus e_\alpha S_1(R)$ by Lemma 2.8. So, ϕ induces an epimorphism $\phi_1: P/S_1(P)=\sum_I \oplus e_\alpha R/e_\alpha S_1(R) \to E$. Then, $\ker\phi_1$ is small in $P/S_1(P)$. Therefore, again (ICC) shows that $P/S_1(P)$ and each $e_\alpha R/e_\alpha S_1(R)$ are injective. Then, again, by Lemma 2.8, $\ker\phi_1$ contains $S_2(P)/S_1(P)$ or, equivalently, $\ker\phi$ contains $S_2(P)$. Hence ϕ induces an epimorphism ϕ_2: $P/S_2(P)=\sum_I \oplus e_\alpha R/e_\alpha S_2(R) \to E$. Thus, the same argument inductively works and we see that $\ker\phi$ contains $S_\tau(P)$ for all ordinal τ. However, as R is a left perfect ring, this implies that $\ker\phi=P$, a contradiction. Therefore E must be a cyclic hollow module. Hence we can take $e \in \{e_\alpha\}_I$ such that $\phi(eR)=E$. Put $\psi=\phi|eR$. Since R is a semiprimary ring by 1), we can take an integer t such that $S_t(R)=R$; so there must exist k such that $\ker\psi \supseteq eS_k(R)$ but $\ker\psi \not\supseteq eS_{k+1}(R)$. Then, ψ induces an isomorphism: $eR/S_k(R) \simeq E$. Since $S_k(R)=\mathrm{Ann}_l(J^k)$, the proof of 2) is completed.

3) Let M be a cyclic hollow non-small R-module, and put $E=E(M)$. Since R is a left perfect ring, $\mathrm{Soc}(E) \subseteq_e E$. Let us express $\mathrm{Soc}(E)$ as $\mathrm{Soc}(E)=\sum_I \oplus S_\alpha$, where each S_α is a simple module. Then, by Theorem A, Lemma 2.9 and 2), we see that $\mathrm{E}(S_\alpha)$ is a cyclic hollow Σ-injective module. Hence, $E=\sum_I \oplus \mathrm{E}(S_\alpha)$ and I is a finite set. Put $I=\{1, \cdots, n\}$, and let π_i: $E=\mathrm{E}(S_1)$

$\oplus\cdots\oplus E(S_n)\to E(S_i)$ be the projection. Since M is a non-small module, there must exist i such that $\pi_i(M)=E(S_i)$. Thus M is injective by (ICC).

We have now our first main theorem.

THEOREM 2.11. *The following conditions are equivalent for a given ring R:*

1) *R satisfies (♯).*

2) *R is a right artinian ring with (*).*

3) *R is a right perfect ring with (ICC).*

4) *R satisfies (ISD).*

5) *R is a right and left perfect ring with the condition: For any primitive idempotent e in R with eR non-small, there exists an integer t satisfying*

a) *$eR/eS_k(R)$ is injective for all $0\le k\le t$, and*

b) *$eR/eS_{t+1}(R)$ is a small module.*

When this is so, R is then a semiprimary QF-3 ring.

PROOF. For a ring with (*), we know from Harada [13] and [15] that the following conditions are equivalent:

i) R is right artinian.

ii) R is right noetherian.

iii) R is right and left perfect.

Now, 2)⇔5) is due to Harada [15, Theorem 2.3]. 1)⇔2) follows from Propositions 2.1 and 2.6. 2)⇒1) follows from Proposition 2.7. 1)⇔4) is just Proposition 2.3. 1)⇒3) follows from Propositions 2.1 and 2.2. The proof is completed if we prove 3)⇒5). Assume that 3) holds. By 1) of Proposition 2.10, R is right and left perfect. Using 3) of Proposition 2.10, we can easily show that a) and b) in 5) hold.

DEFINITION. In honor of Harada [13] and [15], we call that a ring R is a right H-ring if it satisfies one of the equivalent conditions in Theorem 2.11. Left H-rings are symmetrically defined and right and left H-rings are simply called H-rings.

REMARK. Let R be a right artinian ring whose indecomposable injective R-modules are finitely generated modules. Then, in view of the proofs of all results in this section, we see that the following conditions are equivalent:

1) R is a right H-ring.

2) Every finitely generated injective R-module is a lifting module.

3) Every finitely generated non-small R-module contains a non-zero injective submodule.

4) The family of all finitely generated injective R-modules is closed under taking small covers.

5) Every finitely generated R-module is expressed as a direct sum of an injective module and a small module.

3. Co-H-ring

This section is concerned with Theorem II mentioned in introduction. Therefore, the following conditions are studied:

(#)# *Every projective R-module is an extending module.*

(*)* *Every non-cosmall R-module contains non-zero projective direct summand.*

(*PEC*) *The family of all projective R-modules is closed under taking essential extensions.*

(*PSD*) *Every R-module is expressed as a direct sum of a projective module and a singular module.*

The condition(*)* is due to Harada and semiperfect rings with this condition has been setteled in terms of ideal theoretic properties ([14], [15]).

Proposition 3. 1. (#)# *implies* (*PSD*).

Proof. Let M be an R-module. Consider an exact sequence: $P\xrightarrow{\phi} M\to 0$, where P is a free R-module. By (#)#, there exists a decomposition $P=P_1\oplus P_2$ with $\ker\phi\subseteq_e P_1$. Then $M=\phi(P_1)\oplus\phi(P_2)$, $P_2\simeq\phi(P_2)$ and $P_1/\ker\phi\simeq\phi(P_1)$. As a result, $\phi(P_2)$ is projective and $\phi(P_1)$ is a singular module.

Lemma 3. 2. ([15], [24]). *The following statements hold about non-cosmall modules*:

1) *An R-module M is non-cosmall iff it does not coincide with its singular module,*

2) *If an R-module M contains a non-zero projective submodule, then it is non-cosmall.*

Proposition 3. 3. (*PSD*) *implies* (*PEC*).

Proof. Let P be a non-zero projective R-module, and consider an R-module M with $P\subseteq_e M$. By (PSD), M is written as $M=Q\oplus Z$, where Q is projective and Z singular. Let π: $M=Q\oplus Z\to Q$ be the projection. By (PSD), $\pi(P)$ is expressed as $\pi(P)=T\oplus V$, where T is projective and V singular. Let ω: $\pi(P)=T\oplus V\to T$ be the projection. Since T is projective, we have a decomposition $P=P_1\oplus P_2$ such that $P_1\simeq T$ by $\omega\pi$ and $\omega\pi(P_2)=0$. Then $P_2\subsetneqq V\oplus Z$. Since $V\oplus Z$ is a singular module, we see $P_2=0$ by Lemma

3. 2. Thus $P \simeq T$ by $\omega\pi$. Since $P \subseteq_e M$, this easily shows that $Z=0$ and hence $M=Q$; so M is projective.

LEMMA 3. 4. *Assume that R satisfies (PEC). Then, every projective injective R-module contains an indecomposable cyclic Σ-injective module. So, in particular, every indecomposable projective injective R-module is cyclic Σ-injective.*

PROOF. Let P be a projective and injective R-module. We take a cardinal number τ such that $\tau > \max\{\aleph_0, |R|\}$, and consider the injective hull $E=\mathrm{E}(\tau P)$. Then, we see from Theorem B that E is written as a direct sum cyclic submodules, say $E=\sum_I \oplus x_\alpha R$. Then, clearly, $|I| \geq \tau$. As a result, there must exist a subset J of I such that $|J| \geq \aleph_0$ and $x_\alpha R \simeq x_\beta R$ for any α, $\beta \in J$. So, according to Theorem A, $x_\alpha R$ is Σ-injective for every $\alpha \in J$ and is expressed as a direct sum of indecomposable cyclic modules. Thus, at any rate, we can take a uniform submodule X of E such that $\mathrm{E}(X)$ is indecomposable cyclic Σ-injective. Clearly, $\mathrm{E}(X) \lesssim \tau P$ and hence we see that $\mathrm{E}(X) \lesssim P$ by Theorem B.

LEMMA 3. 5. *Assume that R satisfies (PEC), and let $\{P_\alpha\}_K$ be a family of indecomposable projective injective R-modules. Then, $P=\sum_K \oplus P_\alpha$ is Σ-injective.*

PROOF. We can assume that $P_\alpha \not\simeq P_\beta$ for any $\alpha \neq \beta$. We take a cardinal τ with $\tau > \max\{\aleph_0, |R|\}$, and consider $E=\mathrm{E}(\tau P)$. Then, as in the proof of Lemma 3. 4, E is written as

$$E=\sum_I \oplus x_\alpha R$$

where $|I| \geq \tau$ and we see that there exists an infinite subset J of I such that $x_\alpha R \simeq x_\beta R$ for any α, $\beta \in J$. We note that each $x_\alpha R$ contains an indecomposable injective module isomorphic to some P_β in $\{P_\beta\}_K$ by Lemma 3. 4 and Theorem B.

Here, we consider the partitions $I=\bigcup_\Gamma I_\gamma$ and $\Gamma=\Gamma_1 \cup \Gamma_2$ such that, for any $\alpha \in I_\gamma$ and $\beta \in I_\sigma$,

$$\begin{aligned} x_\alpha R \simeq x_\beta R &\quad \text{if} \quad \gamma=\sigma, \\ x_\alpha R \not\simeq x_\beta R &\quad \text{if} \quad \gamma \neq \sigma, \\ |I_\gamma| \geq \aleph_0 &\quad \text{if} \quad \gamma \in \Gamma_1 \\ |I_\gamma| < \infty &\quad \text{if} \quad \gamma \in \Gamma_2 . \end{aligned}$$

Then, we see that $|\Gamma_2|\leq|R|$. Put $I_i=\bigcup_{\alpha\in\Gamma_i} I_\alpha$, $i=1,2$. By Theorem A, for any $\alpha\in I_1$, $x_\alpha R$ is expressed as a direct sum of indecomposable Σ-injective modules. Therefore, we can assume that, for any $\alpha\in I_1$, $x_\alpha R$ is an indecomposable Σ-injective module isomorphic to some member in $\{P_\beta\}_K$.

We put $E_i=\sum_{I_i}\oplus x_\alpha R$, $i=1,2$. Then,

$$\chi_0 E_1=\sum_{\Gamma_1}\chi_0(\sum_{I_\gamma}\oplus x_\alpha R)\simeq\sum_{\Gamma_1}(\sum_{I_\gamma}\oplus x_\alpha R)=E_1$$

Hence, E_1 is Σ-injective by Theorem A. From this fact, we can assume that, for any $\beta\in I_2$, $x_\beta R$ does not contain any direct summand isomorphic to some member in $\{x_\alpha R\}_{I_1}$. Here, consider

$$K_1=\left\{\beta\in K|P_\beta \text{ is isomorphic to some member in } \{x_\alpha R\}_{I_1}\right\}.$$

We claim that $K=K_1$. Assume $K_2=K-K_1\neq\phi$ and put $Q=\sum_{K_2}\oplus P_\delta$. By Theorem B, we see that any indecomposable direct summand of E_i is isomorphic to some member in $\{P_\beta\}_{K_i}$ for $i=1,2$. From this fact we conclude that $\tau Q\cap E_1=0$; whence $\tau Q\subsetneqq E_2=\sum_{\Gamma_2}\sum_{I_\gamma}\oplus x_\beta R$. However, this shows that $|\Gamma_2|\geq\tau$, a contradiction. Consequently $K=K_1$ and hence P is isomorphic to a direct summand of E_1. Thus, P is Σ-injective.

By Lemmas 3.4 and 3.5, we have

PROPOSITION 3.6. *Assume that R satisfies (PEC). Then, for any projective R-module P, $E(P)$ is projective Σ-injective and, moreover, P is expressed as a direct sum of indecomposable cyclic modules.*

REMARK. From the above proposition, we see that if R satisfies (PEC) then the identity of R is written as a sum of primitive orthogonal idempotents.

LEMMA 3.7. *Assume that R satisfies (PEC) and let $\{f_j\}$ a complete set of primitive idempotents. Then, each f_jR is Σ-quasi-injective and $Z(f_jR)=Z(E(f_jR))$. More precisely, there exist a subset $\{e_i\}\subseteq\{f_j\}$ and integers $\{n_i\}$ such that*

1) *each e_iR is injective,*
2) *e_iJ^t is cyclic projective for $t\leq n_i$ and $e_iJ^{n_i+1}=Z(e_iR)$,*
3) *for any $f\in\{f_j\}$, there exists $e\in\{e_i\}$ such that $E(fR)\simeq eR$,*

where $J=J(R)$.

PROOF. Let $f\in\{f_j\}$. By Proposition 3.6, $\mathrm{E}(fR)$ is written as a direct sum of indecomposable Σ-injective cyclic projective modules, say $\mathrm{E}(fR)=p_1R\oplus\cdots\oplus p_mR$. We show $m=1$. Consider the projection π_i: $\mathrm{E}(fR)=\sum_{i=1}^{n}\oplus$

$p_i R \to p_i R$, $i=1, \cdots, n$ Then $fR \subseteq_e \sum_{i=1}^{n} \oplus \pi_i(fR)$. Hence, by (PEC), each $\pi_i(fR)$ is projective. Therefore, $fR \simeq \pi_1(fR)$, whence we see that $fR \cap \sum_{i=2}^{n} \oplus p_i R = 0$. Thus, $\mathrm{E}(fR)$ is indecomposable Σ-injective and cyclic projective. So, in particular, fR is uniform. Moreover, using Theorem B, we see that there exist $e \in \{f_j\}$ and an isomorphism: $\mathrm{E}(fR) \overset{\phi}{\simeq} eR$.

Now, assume that $\phi(fR) \subsetneqq eR$. Then $\phi(fR) \subseteq_e eJ$*; whence eJ is indecomposable Σ-quasi-injective projective by (PEC) and Lemma 3.4. Again, by Theoren B, eJ is isomorphic to some $f_i R$. Then eJ^2 is a unique maximal submodule of eJ and hence it is Σ-quasi-injective (cf. Theorem A). If $eJ \neq \phi(fR)$ then $\phi(fR) \subseteq_e eJ^2$ and we see, as above, that eJ^2 is Σ-quasi-injective cyclic projective and is isomorphic to some $f_j R$. This procedure terminates. For, if otherwise then eJ^t is Σ-quasi-injective projective and isomorphic to some $f_j R$ for $t=0, 1, \cdots$. Then there must exist distinct t_1, t_2 such that $eJ^{t_1} \simeq eJ^{t_2}$. However, then, by the fully invarientness of eJ^{t_1} and eJ^{t_2} we see $eJ^{t_1} = eJ^{t_2}$, a contradiction. Thus there exists an integer n such that eJ^t is indecomposable Σ-quasi-injective and isomorphic to some $f_j R$ for $t \leq n$ and $eJ^n = \phi(fR)$; so fR is Σ-quasi-injective.

We further observe eJ^{n+i}. Since eJ^n is Σ-quasi-injective and is isomorphic to some $f_j R$, we also see that eJ^{n+1} is also Σ-quasi-injective. If eJ^{n+1} is projective then eJ^{n+1} is Σ-quasi-injective and isomorphic to some $f_j R$ by the same arqument above. If eJ^{n+2} is projective, similarly eJ^{n+2} is Σ-quasi-injective and isomorphic to some $f_j R$. This procedure also terminates by the same reason above. Hence there must exists s such that eJ^{n+i} is Σ-quasi-injective cyclic projective for $1 \leq i \leq s$ and eJ^{n+s+1} is not projective.

Put $k=n+s$. In view above, we see $Z(eR) = Z(eJ^n) = Z(eJ^k) \subseteq eJ^{k+1}$. It remains only to prove $Z(eR) = eJ^{k+1}$. Assuming $Z(eR) \neq eJ^{k+1}$ we take $x \in eJ^{k+1}$ such that xR is not singular, i.e., $\mathrm{Ann}_r(x)$ is not essential in R. Then, noting that each $f_j R$ is uniform, there exists $f_j R$ such that $\mathrm{Ann}_r(x) \cap f_j R = 0$. But this implies $f_j R \lesssim xR \subseteq eJ^{k+1}$ and hence eJ^{k+1} is projective by (PEC), a contradiction. Thus $Z(eR) = eJ^{k+1}$.

THEOREM 3.8. *If R satisfies (PEC) then R is a semiprimary ring satisfying* (*)* *and the ACC on right annihilator ideals.*

PROOF. Note that uniform quasi-injective R-modules are completely indecomposable. From this fact, Lemma 3.7, [15, Theorem 3.6], Proposition

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*) For a primitive idempotent e, eR is completely indecomposable iff $eJ(R)$ is a unique maximal submodule of eR ([27]).

3.6 and Theorem A we conclude that R is a semiperfect ring satisfying (*)* and the ACC on right annihilator ideals. Now, to prove the remainder, it suffices to show that R is right perfect (cf. [1, Proposition 29.1]). By Lemma 3.7, every projective R-module is written as a finite direct sum of quasi-injective modules. Therefore, every projective R-module satisfies the exchange property (Theorem B). So, R is a right perfect ring by [16] or [28].

PROPOSITION 3.9. *The condition* (*)* *is equivalent to the condition: For any projective R-module P and any submodule A of P, if A is not essential in P then there exists a proper direct summand $B\langle\oplus P$ with $A\subseteq B$.*

PROOF. Assume that (*)* holds and let P be a projective R-module and A a submodule of P which is not essential in P. Then, P/A is non-cosmall and hence P/A is written as $P/A=X\oplus Y$, where X is non-zero projective. Since the canonical epimorphism ϕ: $P\to X\to 0$ splits, we get $P=\ker\phi\oplus Q$ for some submodule Q. Then $\ker\phi\supseteq A$ and $Q\neq 0$.

Conversely, let M be a non-cosmall R-module and consider a sequence $F\xrightarrow{\phi}M\to 0$ where F is a free R-module. Then $\ker\phi$ is not essential in F and hence we have a decomposition $F=F_1\oplus F_2$ such that $\ker\phi\subseteq F_1$ and $F_2\neq 0$. Then we see that $M=\phi(F_1)\oplus\phi(F_2)$ and $F_2\simeq\phi(F_2)$ by ϕ.

COROLLARY 3.10. *If* (*)* *holds then every indecomposable projective R-module is uniform.*

PROPOSITION 3.11. *Assume that R satisfies* (*)* *and the identity of R is a sum of primitive orthogonal idempotents $\{f_j\}$. Then each f_jR is quasi-injective and moreover there exist a subset $\{e_i\}\subseteq\{f_j\}$ and integers $\{n_i\}$ satisfying*

1) *each e_iR is injective,*

2) *e_iJ^t is cyclic projective for $t\leq n_i$ and $e_iJ^{n_i+1}=Z(e_iR)$,*

3) *for any $f\in\{f_j\}$ there exists $e\in\{e_i\}$ such that $E(fR)\simeq eR$; so $Z(fR)=Z(E(fR))$, where $J=J(R)$.*

PROOF. By Corollary 3.10 each f_jR is uniform. Let $f\in\{f_j\}$. Then $E(fR)$ is indecomposable projective by Lemma 3.2. Hence using the exchange property of $E(fR)$ (Theorem B) we see that there exists $e\in\{f_j\}$ and an isomorphism $E(fR)\simeq eR$.

Assume $\phi(fR)\subsetneqq eR$. Then $Z(eR)=Z(eJ)\subseteq eJ$ and hence, again, by Lemma 3.2 and Theorem B, we see that eJ is quasi-injective and projective, and is isomorphic to some f_jR. Similarly if $\phi(fR)\subsetneqq eJ$ then eJ^2 is quasi-injective and projective, and is isomorphic to some f_jR. This procedure terminates as in the proof of Lemma 3.7. Therefore there exists n such that eJ^t is projective and quasi-injective for any $t \leq n$, $eJ^n = \phi(fR) \simeq fR$ and $eJ^{n+1} = Z(eR) = Z(\phi(fR))$.

THEOREM 3.12. *If R satisfies* (*)* *and the ACC on right annihilator ideals then R is a semiprimary ring with Σ-quasi-injective singular submodule $Z(R)$; so $Z(P)$ is quasi-injective for every projective R-module P.*

PROOF. Since R satisfies the ACC on right annihilator ideals, the identity of R is written as a sum of orthogonal primitive idempotents $\{f_j\}$. According to Proposition 3.11 each f_jR is uniform and quasi-injective; so is completely indecomposable. Thus R is a semi-perfect ring. Furthermore, each f_jR is Σ-quasi-injective by Theorem A. Hence, by the same proof as in the proof of Theorem 3.8 we see that R is a semiprimary ring. The remainder is clear from Proposition 3.11 and Theorem A.

LEMMA 3.13. *Assume that R satisfies* (*)* *and the ACC on right annihilator ideals. Let P be a projective R-module and let $P = \sum_I \oplus P_\alpha$ be an indecomposable decomposition. (Such a decomposition exists by Theorem 3.12.) Then any uniform submodule of P is finitely contained with respect to $P = \sum_I \oplus P_\alpha$, i.e., for any uniform submodule A of P there exists a finite subset F of I with $A \subseteq \sum_F \oplus P_\alpha$.*

PROOF. By Proposition 3.11 and Theorem A we see that each P_α is uniform, each $\mathrm{E}(P_\alpha)$ is cyclic and $\mathrm{E}(P) = \sum_I \oplus \mathrm{E}(P_\alpha)$. Let A be a uniform submodule of P. Then we can take $\alpha \in I$ such that $A \oplus \sum_{I-\{\alpha\}} \oplus P_\beta \subseteq_e P$. Put $Q = \sum_{I-\{\alpha\}} \oplus P_\beta$, and denote, by π_α and π_Q, the projections: $P = P_\alpha \oplus Q \to P_\alpha$ and $P = P_\alpha \oplus Q \to Q$, respectively. Then the mapping ψ: $\pi_\alpha(A) \to Q$ given by $\psi(\pi_\alpha(a)) = \pi_Q(a)$ is a homomorphism and $A = \{x + \psi(x) \mid x \in \pi_\alpha(A)\}$. ψ is then extended to a homomorphism ψ: $\mathrm{E}(P_\alpha) \to \mathrm{E}(\sum_{I-\{\alpha\}} \oplus P_\beta) = \sum_{I-\{\alpha\}} \oplus \mathrm{E}(P_\beta)$. Since $\mathrm{E}(P_\alpha)$ is cyclic there exists a finite subset $F \subseteq I$ satisfying $\psi(\mathrm{E}(P_\alpha)) \subseteq \sum_{F'} \oplus \mathrm{E}(P_\beta)$. This shows $\psi(A) \subseteq \sum_F \oplus P_\alpha$ and hence $A \subseteq P_\alpha \oplus \sum_{F'} \oplus P_\beta$.

LEMMA 3.14. *Assume that R satisfies* (*)* *and the identity of R is a sum of orthogonal primitive idempotents. Then every projective R-module has the extending property of finitely contained uniform modules with respect to any indecomposable decomposition of P.*

PROOF. By Proposition 3.11 we see that R is a semiperfect ring. Let P be a projective module. Since R is semiperfect, P has an indecomposable decomposition $P=\sum_I \oplus P_\alpha$ (cf. [1, 27. 1]). Our statement is the following: For any uniform submodule A of P such that $A\subseteq \sum_F \oplus P_\alpha$ for some finite subset $F\subseteq I$, there exists a direct summand A^* of P with $A\subseteq_e A^*$. However, in view of [17, Theorem 10], it suffices to show the following condition: Let f and g be primitive idempotents of R, and let A a submodule of fR and ϕ: $A\to gR$ a homomorphism. Then, if ϕ is a monomorphism then there exists either ψ: $fR\to gR$ or $gR\to fR$ with $\psi|A=\phi$ or $\psi|\phi(A)=\phi^{-1}$. If ϕ is a non-monomorphism, then ϕ is extended to a homomorphism : $fR\to gR$.

This condition is verified by Proposition 3. 11. Actually, ϕ is extended to a homomorphism ψ: $\mathrm{E}(fR)\to \mathrm{E}(gR)$. If ϕ is a monomorphism then ψ is an isomorphism and, by Proposition 3. 11, $\psi(fR)\subseteq gR$ or $\psi^{-1}(gR)\subseteq fR$. On the other hand, if ϕ is a non-monomorphism then, so is ψ and, again by Proposition 3. 11, $\psi(\mathrm{E}(fR)\subseteq \mathrm{Z}(\mathrm{E}(gR))=\mathrm{Z}(gR)\subseteq gR$; whence $\psi(fR)\subseteq gR$. Thus the proof is completed.

By Lemmas 3. 13 and 3. 14, we have the following result.

THEOREM 3. 15. *Assume that R satisfies* (*)* *and the ACC on right annihilator ideals. Then every projective R-module has the extending property of uniform modules.*

LEMMA 3. 16. *We assume that R is a right perfect ring with* (*)*. *Let P be a projective R-module and A a submodule of P. Then there exist decompositions $P=P^*\oplus Q$ and $A=A^*\oplus Z$ such that A^* is projective with $A^*\subseteq_e P^*$ and Z a singular module with $Z\subseteq Q$.*

PROOF. Let $\{f_1, \cdots, f_m\}$ be a complete set of orthogonal primitive idempotents of R. By Proposition 3. 11 or [15, Theorem 3. 1], there exists a subset $\{e_1, \cdots, e_s\}\subseteq\{f_i\}$ and integers $\{n_1, \cdots, n_s\}$ such that

1) each e_iR is injective,

2) e_iJ^t is cyclic uniform projective for all $t\leq n_i$ and $e_iJ^{n_i+1}=\mathrm{Z}(e_iR)$,

3) every indecomposable projective R-module is isomorphic to some e_iJ^t, where $J=\mathrm{J}(R)$.

We can assume that $\{e_1, \cdots, e_s\}$ is the representative set of indecomposable projective injective R-modules, i. e., $e_iR\not\approx e_jR$ for any $i\neq j$. Then $\{e_iJ^t|t\leq n_i, i\leq s\}$ is the representative set of indecomposable projective R-modules. For convinience's sake of the proof, we say that an indecomposable projective R-module is type (e_i, t) if it is isomorphic to e_iJ^t.

Now, let A be a submodule of P. If $A=\mathrm{Z}(A)$, then there is nothing to prove; so assume $A\neq \mathrm{Z}(A)$. Then, by [15, Proposition 3. 2], A contains a non-zero projective direct summand. In particular, A contains a non-zero indecomposable projective summand.

We can take $e_{i_1}\in\{e_i\}$ and $t_1^{i_1}\leq n_{i_1}$ such that

1) A contains a direct summand of type $(e_{i_1}, t_1^{i_1})$, but

2) A does not contain any direct summand of type (e_k, t) for $k<i_1$ and $t\leq n_k$ and any direct summand of type (e_{i_1}, t) for $t<t_1^{i_1}$.

Using Zorn's lemma and Theorem 3. 15, we can take a maximal independent family $\{P_\alpha\}_{I(e_{i_1},t_1^{i_1})}$ of indecomposable direct summands of P such that

3) $\sum_{I(e_{i_1},t_1^{i_1})}\oplus P_\alpha$ is a locally direct summand of P, and

4) $A_\alpha=P_\alpha\cap A$ is a projective direct summand of A of type $(e_{i_1}, t_1^{i_1})$ such that $A_\alpha\subseteq_e P_\alpha$ for all $\alpha\in I(e_{i_1}, t_1^{i_1})$.

We put $I=I(e_{i_1}, t_1^{i_1})$, $P^{(e_{i_1},t_1^{i_1})}=\sum_I\oplus P_\alpha$ and $A^{(e_{i_1},t_1^{i_1})}=\sum_I\oplus A_\alpha$. Then, $P^{(e_{i_1},t_1^{i_1})}\langle\oplus P$ by Theorem 3. 12 and [22, Proposition 3. 2]. We also show $A^{(e_{i_1},t_1^{i_1})}\langle\oplus A$. For this purpose, let Q be a submodule of P such that

$$P=P^{(e_{i_1},t_1^{i_1})}\oplus Q\,.$$

By π_α, we denote the projection : $P=P^{(e_{i_1},t_1^{i_1})}\oplus Q\rightarrow P_\alpha$ for all $\alpha\in I$. Since $P_\alpha\cap A=A_\alpha$ and A does not contain any direct summand of type (e_{i_1}, t) for $t<t_1^{i_1}$, we can verify $\pi_\alpha(A)=A_\alpha$ for all $\alpha\in I$ (cf. Proposition 3. 11). As a result, we get

$$A=(\sum_I\oplus A_\alpha)\oplus(Q\cap A)$$

as desired.

We put $B=Q\cap A$. Then B has no direct summand of type $(e_{i_1}, t_1^{i_1})$ by the maximality of $\{P_\alpha\}_I$ and Theorem 3. 15. If $Z(B)=B$ the proof is completed. If $Z(B)\neq B$, then the same argument above works on $B\subseteq Q$ instead of $A\subseteq P$. In this case, the following two cases are considered.

The first case is that there exist $t_2^{i_1}>t_1^{i_1}$ and B has a direct summand of type $(e_{i_1}, t_2^{i_1})$ but does not contain any direct summand of type (e_{i_1}, t) for $t<t_2^{i_1}$. Then we can obtain decompositions

$$Q=\sum_{I(e_{i_1},t_2^{i_1})}\oplus P_\alpha\oplus S\,,$$

$$B=\sum_{I(e_{i_1},t_2^{i_1})}\oplus(P_\alpha\cap B)\oplus C$$

such that $C\subseteq S$, each P_α is indecomposable with $A_\alpha=P_\alpha\cap B\subseteq_e P_\alpha$, each A_α is projective and C does not contain a direct summand of type (e_{i_1}, t) for any $t<t_2^{i_1}$. Then we put $P^{(e_{i_1},t_2^{i_1})}=\sum_{I(e_{i_1},t_2^{i_1})}\oplus P_\alpha$ and $A^{(e_{i_1},t_2^{i_1})}=\sum_{I(e_{i_1},t_2^{i_1})}\oplus A_\alpha$.

The secound case is that B does not contain any direct summand of type (e_{i_1}, t). In this case we have $i_2 > i_1$ and $t_1^{i_2} < n_{i_2}$ such that B contains a direct summand of type $(e_{i_2}, t_1^{i_2})$ but not contain a direct summand of type (e_k, t) for $k < i_2$ and of type (e_{i_2}, t) for $t < t_1^{i_2}$. Then we also obtain decompositions

$$Q = \sum_{I(e_{i_2}, t_1^{i_2})} \oplus P_\alpha \oplus S,$$

$$B = \sum_{I(e_{i_2}, t_1^{i_2})} \oplus (P_\alpha \cap B) \oplus C$$

such that $S \supseteq C$, each P_α is indecomposable with $P_{\alpha e} \supseteq A_\alpha = P_\alpha \cap B$, each A_α is projective and C has no direct summand of type $(e_{i_2}, t_1^{i_2})$. Then we put $P^{(e_{i_2}, t_1^{i_2})} = \sum_{I(e_{i_2}, t_1^{i_2})} \oplus P_\alpha$ and $A^{(e_{i_2}, t_1^{i_2})} = \sum_{I(e_{i_2}, t_1^{i_2})} \oplus A_\alpha$.

Continuing this procedure we get

$$1 \leq i_1 < i_2 < \cdots < i_k \leq s$$

$$0 \leq t_1^{i_j} < t_2^{i_j} < \cdots < t_{l_{i_j}}^{i_j}, \qquad j = 1, \cdots, k,$$

and

$$P = \sum_{j=1}^{l_{i_1}} \oplus P^{(i_1, t_j^{i_1})} \oplus \sum_{j=1}^{l_{i_2}} \oplus P^{(i_2, t_j^{i_2})} \oplus \cdots \oplus \sum_{j=1}^{l_{i_k}} \oplus P^{(i_k, t_j^{i_k})} \oplus W,$$

$$A = \sum_{j=1}^{l_{i_1}} \oplus A^{(i_1, t_j^{i_1})} \oplus \sum_{j=1}^{l_{i_2}} \oplus P^{(i_2, t_j^{i_2})} \oplus \cdots \oplus \sum_{j=1}^{l_{i_k}} \oplus P^{(i_k, t_j^{i_k})} \oplus V$$

such that each $A^{(i_u, t_j^{i_u})}$ is projective and of type $(e_{i_u}, t_j^{i_u})$, $P^{(i_u, t_j^{i_u})}{}_e \supseteq A^{(i_u, t_j^{i_u})}$ for each $(i_u, t_j^{i_u})$ and $V = Z(V) \subseteq W$. This completes the proof.

THEOREM 3.17. *If R satisfies* $(*)^*$ *and the ACC on right annihilator ideals then R satisfies* $(\#)^\#$.

PROOF. Let P be a projective R-module and A a submodule of P. For our assertion we can assume $A \subseteq Z(P)$ by Lemma 3.16. Theorem 3.12 says that $Z(P)$ is quasi-injective. This fact enable us to further assume that $A \langle \oplus Z(P)$. By Proposition 3.11 and Theorem 3.12 we infer that $Z(P)$ is written as a direct sum of completely indecomposable uniform modules. As a result, any non-zero direct summand of A has a non-zero completely indecomposable uniform direct summand (cf. Theorem B).

By Zorn's lemma we can take maximal independent families $\{P_\alpha\}_I$ of indecomposable direct summands of P and $\{A_\alpha\}_I$ of indecomposable direct summands of A such that $P_{\alpha e} \supseteq A_\alpha$ for all $\alpha \in I$ and both $\sum_I \oplus P_\alpha$ and $\sum_I \oplus A_\alpha$ are locally direct summands of P and A, respectively. Then we conclude

$\sum_I \oplus P_\alpha \langle \oplus P$ by Theorem 3.12 and [22, Proposition 3.2] and $\sum_I \oplus A_\alpha \langle \oplus A$ by Theorem C; put

$$P = \sum_I \oplus P_\alpha \oplus Q\,,$$

$$A = \sum_I \oplus A_\alpha \oplus B\,.$$

Our proof is established by showing $B=0$. Assume $B\neq 0$ and take a non-zero uniform direct summand $C\langle\oplus B$. Let us consider an indecomposable decomposition $Q=\sum_J \oplus P_\beta$ and let π_γ: $P=\sum_K \oplus P_\gamma \to P_\gamma$ be the projection for all $\gamma\in K=I\cup J$. According to Lemma 3.13 we can take a finite subset $F=\{\gamma_1, \cdots, \gamma_n\}\subseteq K$ such that $C\subseteq P_{\gamma_1}\oplus\cdots\oplus P_{\gamma_n}$. Put $J^*=F\cap J$.Since C is uniform and $C\cap\sum_I \oplus P_\alpha=0$ we see that there exists $\beta_0\in J^*$ such that $\pi_{\beta_0}|C$ is monomorphic. Here consider the mapping ψ: $\pi_{\beta_0}(C)\to \sum_{J-\{\beta_0\}} \oplus\pi_\beta(C)$ given by $\pi_{\beta_0}(c)\to \sum_{J-\{\beta_0\}} \pi_\beta(c)$ and put $X=\{x+\psi(x)\,|\, x\in\pi_{\beta_0}(C)\}$. Then X is uniform and $\sum_I \oplus A_\beta \oplus C \subseteq_e \sum_I \oplus P_\beta \oplus C = \sum_I \oplus P_\beta \oplus X$. Now using Theorem 3.15 we get a uniform direct summand $Y\langle\oplus \sum_J \oplus P_\beta$ with $X\subseteq_e Y$. Thus we have a situation that

$$\sum_I \oplus A_\alpha \oplus C \langle \oplus A\,,$$

$$\sum_I \oplus A_\alpha \oplus C \subseteq_e \sum_I \oplus P_\alpha \oplus Y \langle \oplus P\,.$$

This contradicts the maximality of $\{P_\alpha\}_I$ and $\{A_\alpha\}_I$. Thus we have $B=0$ as desired.

We are now in a position to state our secound main theorem of this paper, which is mentioned in introduction.

THEOREM 3.18. *The following conditions are equivalent for a given ring R:*

1) *R satisfies* (♯)$^{\sharp}$.
2) *R satisfies* (*PSD*).
3) *R satisfies* (*PEC*).
4) *R satisfies* (*)* *and the ACC on right annihilator ideals.*

When this is so, then R is a semiprimary QF-3 ring.

PROOF. 1)$\Rightarrow$2)$\Rightarrow$3) follows from Propositions 3.1 and 3.3. 3)$\Rightarrow$4)$\Rightarrow$1) follows from Theorems 3.8 and 3.17. In view of Theorem D, the condition 3) implies that R is a semiprimary QF-3 ring.

REMARK. Let R be a semiperfect ring with a complete set $\{e_i\}\cup\{q_i\}$

of primitive orthogonal idempotents such that each e_iR is non-small and each g_iR is small. As used in Lemma 3.16, Harada has shown in [15, Theorem 3.6] that R satisfies (*)* iff it satisfies the following conditions:

1) each e_iR is injective,

2) for any g_j, there exists e_i such that $g_jR \subsetneq e_iR$.

3) for each e_i, there exists n_i such that e_iJ^t is projective for $0 \leq t \leq n_i$ and $e_iJ^{n_i+1}$ is a singular module, where $J = \mathrm{J}(R)$.

Further, in this case, it is shown that every submodule e_iB in e_iR either is contained in $e_iJ^{n_i+1}$ or equal to some e_iJ^t, $0 \leq t \leq n_i+1$.

It should be noted that we used the idea of this result in our Lemma 3.7 and Proposition 3.11.

As a dual of a right H-ring, we give

DEFINITION. We say that a ring R is a right co-H-ring if it satisfies one of the equivalent conditions in Theorem 3.18. Left co-H-rings are symmetrically defined and right and left co-H-rings are simply called co-H-rings.

REMARK. Let R be a right noetherian ring whose indecomposable injective R-modules are finitely generated modules. Then, in view of the proof of the results in this section, we see that the following conditions are equivalent:

1) R is a right co-H-ring.

2) Every finitely generated projective R-module is an extending module.

3) Every finitely generated non-cosmall R-module contains a non-zero projective direct summand.

4) The family of all finitely generated projective R-modules is closed under taking essential extensions (cf. [19]).

5) Every finitely generated R-module is expressed as a direct sum of a projective module and a singular module.

4. Application

As a first application of H-rings and co-H-rings, we study quasi-Frobenius rings (abbreviated QF-rings).

A ring R is said to be QF if it satisfies one of the following equivalent conditions:

1) R is a right self-injective ring and satisfies the ACC on right annihilator ideals.

2) Every injective R-module is projective.

3) Every projective R-module is injective.

As is well known, these conditions 1)$\sim$3) are right-left symmetric.

LEMMA 4.1. ([22, Theorem 4.11]). *Let M be a quasi-semiperfect R-module, and let $\{A_\alpha\}_I$ be a faimly of indecomposable direct summands of M with $M=\sum_I A_\alpha$. If $M=\sum_I A_\alpha$ is an irredundant sum, then $M=\sum_I \oplus A_\alpha$.*

THEOREM 4.2. *The following conditions are equivalent for a given ring R:*

1) *R is QF.*
2) *Every injective R-module is semiperfect.*
3) *Every injective R-module is quasi-semiperfect.*
4) *Every projective R-module is continuous.*
5) *Every projective R-module is quasi-continuous.*

PROOF. The implications 1)$\Rightarrow$2)$\Rightarrow$3) and 1)$\Rightarrow$4)$\Rightarrow$5) are clear.

3)$\Rightarrow$1) By Proposition 2.6, R is right artinian. Hence, combining Lemma 4.1 to [14, Proposition 3], we see that R is QF.

5)$\Rightarrow$1) By [21, Proposition 1.9], it follows from 5) that every projective R-module is quasi-injective and hence injective. As a result, R is QF.

The theorem above suggests the following characterizations of QF-rings.

THEOREM 4.3. *The following conditions are equivalent for a given ring R:*

1) *R is QF.*
2) *R is a right H-ring with $Z(R)=J(R)$.*
3) *R is a right co-H-ring with $Z(R)=J(R)$.*

So, the conditions 2) and 3) are right-left symmetric.

PROOF. If R is QF, then the injectivity of R implies that $\mathrm{Z}(R)=\mathrm{J}(R)$ ([35]). Clearly QF-rings satisfies (#) and (#)* (cf. [22, Theorem 2.1]). Hence 1)$\Rightarrow$2) and 1)$\Rightarrow$3) follow.

2)$\Rightarrow$1). By Proposition 2.6, R is right artinian. Let e be a primitive idempotent of R. It is enough to show that eR is small in $\mathrm{E}(eR)$ by (ISD). By Proposition 2.6, $E=\mathrm{E}(eR)$ is expressed as $E=E_1\oplus\cdots\oplus E_n$ with each E_i cyclic hollow. By π_i, we denote the projection: $E=E_1\oplus\cdots\oplus E_n\to E_i$, $i=1,\cdots,n$. Since eR is small in E, clearly, $\pi_i(eR)\neq E_i$ for all i. Noting that each E_i is cyclic hollow, we see that there exists a primitive idempotent f_i such that E_i is a homomorphic image of f_iR, $i=1,\cdots,n$. Therefore, it follows from $\mathrm{Z}(f_iR)=\mathrm{J}(f_iR)$ that $\mathrm{Z}(E_i)\supseteq\mathrm{J}(E_i)$, $i=1,\cdots,n$. Hence $eR\subseteq\pi_1(eR)\oplus\cdots\oplus\pi_n(eR)\subseteq\mathrm{Z}(E_1)\oplus\cdots\oplus\mathrm{Z}(E_n)$; so eR is a singular module, a contradiction. Thus, eR must be an injective module.

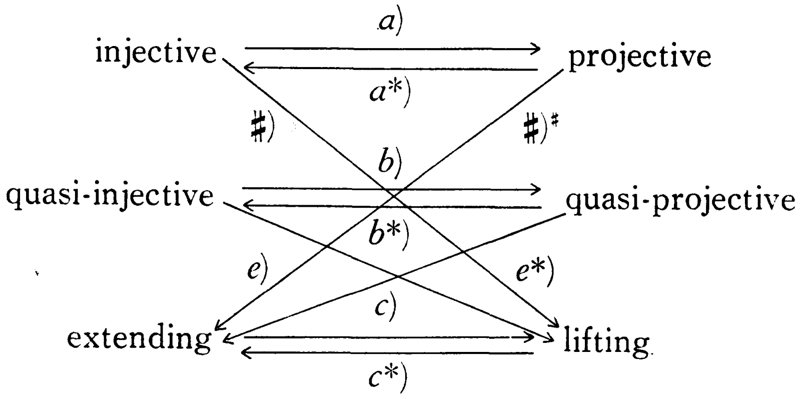
3)$\Rightarrow$1). By Theorem 3.18, R satisfies the ACC on right annihilator ideals. Hence we may show that eR is an injective module for every primitive idempotent e. Indeed, this fact is easily seen by $Z(R)=J(R)$ and [15, Theorem 3.6] or 3) of Lemma 3.7.

THEOREM 4.4. *If R is a commutative ring, then the following conditions are equivalent*:

1) *R is QF.*
2) *R is a H-ring.*
3) *R is a co-H-ring.*

PROOF. 1)$\Rightarrow$2) and 1)$\Rightarrow$3) follow from Theorem 4.3. 2)$\Rightarrow$1) follows from Proposition 2.6 and [6, 25.4.18A]. 3)$\Rightarrow$1) is clear from the proof of Theorem 4.3 and the fact that, for any primitive idempotent e and f, $fR\subsetneqq eR$ implies $fR=eR$.

REMARK. As we saw above, QF-rings are H-rings and co-H-rings. In order to state more information about connection among right H-rings, right co-H-rings and classical artinian rings, we consider the following implications:



As is well known, R is $QF\Leftrightarrow a)\Leftrightarrow a^*)$; R is uniserial$\Leftrightarrow b)\Leftrightarrow b^*)$ ([2], [10]). It is shown in [23] that R is generalized uniserial$\Leftrightarrow c)\Leftrightarrow R$ is a right perfect ring with $c^*)\Leftrightarrow e)\Leftrightarrow e^*)$. Rings with $\sharp)$ are just right H-rings; while rings with $\sharp)^\sharp$ are just co-H-rings.

From this remark, we have immediately

THEOREM 4.5. *If R is a generalized uniserial rings, then it is a H-ring and also a co-H-ring.*

In the rest of this section, we study right non-singular right H-rings and right non-singular right co-H-rings.

Now, if R is a right co-H-ring, we see from (PSD) that every non-singular R-module is projective. We first note that the converse also holds when R is a right non-singular right co-H-ring. For, in this case, a submodule A of a projective R-module P is a closed submodule of P iff P/A is non-singular. Therefore, by the Goodearl's work [11, Chapter 5] or [14, Corollary 1] and Theorem 3.18, a right non-singular right co-H-ring is completely determined as it is Morita equivalent to a finite direct sum of upper triangular matrix rings over division rings.

Right non-singular right H-rings also have the same structure as the following shows.

THEOREM 4.6. *If R is a right non-singular ring, then the following conditions are equivalent:*

1) *R is a right H-ring.*

2) *R is a right co-H-ring.*

3) *R is Morita equivalent to a finite direct sum of upper triangular matrix rings over division rings.*

PROOF. 2)$\Leftrightarrow$3) holds as noted above. 3)$\Rightarrow$1), 2) hollows from Theorem 4.5. We shall show 1)$\Rightarrow$3). Assume 1). By [4, Theorem 3.2] together with Theorem 2.11, to show 3), we may show that R is a right hereditary ring. Further, by [15, Propositions 2.5 and 2.8] and Proposition 2.10, it suffices to show that, for a primitive idempotent e such that eR is injective, every non-zero homomorphic image of eR is non-small. To prove this, let A be a right ideal of R with $eR/eA\neq 0$ and assume that eR/eA is small in $E=\mathrm{E}(eR/eA)$. As in the proof of Proposition 2.10, E is written as

$$E=e_1R/e_1A_1\oplus\cdots\oplus e_nR/e_nA_n$$

where all e_i are primitive idempotents and all A_i are right ideals. Put $E_i=e_iR/e_iA_i$, $i=1,\cdots,n$. Here, consider the diagram:

$$\begin{tikzcd} & eR \arrow[d, "\psi"] & \\ & eR/eA \arrow[d, hook] & \\ P=\sum_{i=1}^{n}\oplus e_iR \arrow[r, "\eta"] & E \arrow[r] & 0 \end{tikzcd}$$

where η and ψ are the canonical homomorphisms. Since eR is projective, we get a homomorphism $\phi: eR\rightarrow P$ with $\eta\phi=\psi$. We can assume that $\pi_1\phi\neq 0$, where π_1 is the projection: $P\rightarrow e_1R$. Since both eR and e_1R are non-singular and eR is injective we see that π_1 is an isomorphism. As a result, we obtain

$$E=eR/eA+(E_2\oplus\cdots\oplus E_n)$$

from which we conclude $E=E_2\oplus\cdots\oplus E_n$, a contradiction. Thus eR/eA must be a non-small module. The proof is completed.

5. Examples

The purpose of this section is to give two typical examples of H- and co-H-rings.

As we saw in Theorems 4.4 and 4.5, QF-rings and generaliged uniserial rings are H- and co-H-rings. From these facts and Figure mentioned in section 4, the following problems arise:

1) Are right H-rings left H-rings?
2) Are right co-H-rings left co-H-rings?
3) Are right H-rings right co-H-rings?
4) Are right co-H-rings left H-rings?

In the case when R is an algebra over a field of finite dimension, these problems are equivalent, as the following shows.

THEOREM 5.1. *Let R be an algebra over a field K of finite dimension. Then R is a right co-H-ring iff it is a left H-ring.*

PROOF. For an R-module M, we denote its dual by M^*, that is,

$$M^* = \mathrm{Hom}_K(M, K)\,.$$

For a homomorphism f from an R-module M to an R-module N, f^* denotes the correponding homomorphism: $N^* \to M^*$. The following facts are well known:

1) Every indecomposable injective R-module is finitely generated.
2) A finitely generated R-module P is projective iff P^* is injective.
3) Let $0 \to N \to M$ be an exact sequence of finitely generated R-modules. Then, Im F is essential in M iff $\mathrm{Ker} f^*$ is small in M^*.

Using these facts, we can easily see that the family of all finitely generated right R-modules is closed under taking essential extensions iff the family of all finitely generated injective left R-modules is taking small covers. Therefore, our proof is established by the remarks after Theorems 2.11 and 3.18.

We use the following two lemmas.

LEMMA 5.2 ([9]). *Let R be a one sided artinian ring, and let e and f be primitive idempotents of R. If (eR, Rf) is an injective pair, that is.*

$$\mathrm{Soc}\,(eR_R) \simeq fR/fJ \quad \textit{and} \quad \mathrm{Soc}\,({}_RRf) \simeq Re/Je$$

where $J=J(R)$, then both eR_R and ${}_RRf$ are injective.

LEMMA 5.3 ([24]). *Let R be a right artinian ring, and let M be a right R-module. Then, M is a small module iff* $M\operatorname{Soc}({}_RR)=0$.

From now on, in order to construct two examples of right H- and right co-H-rings, we consider a local QF-ring Q ($\neq 0$). For the sake of convenience, we put $J=\mathrm{J}(Q)$, $S=\operatorname{Soc}(Q_Q)(=\operatorname{Soc}({}_QQ))$, $\bar{Q}=Q/S$ and $\bar{a}=a+S$ for any a in Q. Note that J canonically becomes a two-sided Q-module, since $SJ=SJ=0$. Here, we define $V(Q)$, $W(Q)$ and $T(Q)$ as follows:

$$V(Q)=\begin{pmatrix} Q & Q \\ J & Q \end{pmatrix}=\left\{\begin{pmatrix} a & b \\ d & c \end{pmatrix}\middle| a, b, c\in Q,\ d\in J\right\}$$

$$W(Q)=\begin{pmatrix} Q & \bar{Q} \\ J & \bar{Q} \end{pmatrix}=\left\{\begin{pmatrix} a & \bar{b} \\ d & \bar{c} \end{pmatrix}\middle| a, b, c\in Q,\ d\in J\right\}$$

$$T(Q)=\begin{pmatrix} Q & \bar{Q} \\ J & Q \end{pmatrix}=\left\{\begin{pmatrix} a & \bar{b} \\ d & c \end{pmatrix}\middle| a, b, c\in Q,\ d\in J\right\}$$

Then, these become rings by usual addition and multiplication of matrices. We put

$$1_V=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad e_V=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad f_V=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ in } V(Q)$$

$$1_W=\begin{pmatrix} 1 & \bar{0} \\ 0 & \bar{1} \end{pmatrix},\quad e_W=\begin{pmatrix} 1 & \bar{0} \\ 0 & \bar{0} \end{pmatrix},\quad f_W=\begin{pmatrix} 0 & \bar{0} \\ 0 & \bar{1} \end{pmatrix} \text{ in } W(Q)$$

$$1_T=\begin{pmatrix} 1 & \bar{0} \\ 0 & 1 \end{pmatrix},\quad e_T=\begin{pmatrix} 1 & \bar{0} \\ 0 & 0 \end{pmatrix},\quad f_T=\begin{pmatrix} 0 & \bar{0} \\ 0 & 1 \end{pmatrix} \text{ in } T(Q)$$

Then, 1_V, 1_W and 1_T are identity elements of $V(Q)$, $W(Q)$ and $T(Q)$, respectively, and $\{e_V, f_V\}$, $\{e_W, f_W\}$ and $\{e_T, f_T\}$ are sets of orthogonal primitive idempotents and $1_V=e_V+f_V$, $1_W=e_W+f_W$ and $1_T=e_T+f_T$.

REMARK. Let R be a ring, and let $\{e_1, \cdots, e_n\}$ be a set of orthogonal idempotents of R with $1=e_1+\cdots+e_n$. Then, as is easily seen, R is left artinian iff ${}_{e_iRe_i}e_iRe_j$ is artinian for all e_i and e_j. By this result, we see that $V(Q)$, $W(Q)$ and $T(Q)$ are right and left artinian.

THEOREM 5.4. *$T(Q)$ is a QF-ring.*

PROOF. Put $T=T(Q)$, $e=e_T$ and $f=f_T$. Note that

$$\operatorname{Soc}(eT_T)=\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}=\operatorname{Soc}({}_TTe) \text{ and } \operatorname{Soc}(fT_T)=\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}=\operatorname{Soc}({}_TTf).$$

So, it is easy to see that both (eT, Te) and (fT, Tf) are injective pairs.

Hence eT_T and fT_T are injective by Lemma 5. 2. Hence T is QF.

THEOREM 5. 5. *$V(Q)$ is a H- and co-H-ring.*

PROOF. Put $V=V(Q)$, $e=e_V$ and $f=f_V$. Since V is left-right symmetric, we may show that V is left H and right co-H. Note that

$$\mathrm{Soc}\,(V_V)=\begin{pmatrix}0 & S\\ 0 & S\end{pmatrix} \quad\text{and}\quad \mathrm{Soc}\,({}_VV)=\begin{pmatrix}S & S\\ 0 & 0\end{pmatrix}.$$

We put $X=\mathrm{Soc}\,(V_V)$ and $Y=\mathrm{Soc}\,({}_VV)$. Since $fVY=0$ and $XVe=0$, fV is a small right V-module and Ve is a small left V-module by Lemma 5. 3. Moreover, we see that (eV, Vf) is an injective pair; whence eV_V and ${}_VVf$ are injective,. We can also easily see that

$$\mathrm{J}_1(eV_V)\simeq fV \quad\text{and}\quad \mathrm{J}_2(eV_V)=Z(eV_V)$$

Therefore, V is a right co-H-ring by Theorem 3. 18 and its remark. Next, in order to show that V is a left H-ring, note that

$$\mathrm{S}_1({}_VVf)=\begin{pmatrix}0 & S\\ 0 & 0\end{pmatrix} \quad\text{and}\quad \mathrm{S}_2({}_VVf)=\begin{pmatrix}0 & S\\ 0 & S\end{pmatrix}$$

and

$$X\big(Vf/\mathrm{S}_1({}_VVf)\big)\neq 0 \quad\text{and}\quad X\big(Vf/\mathrm{S}_2({}_VVf)\big)=0\,.$$

Hence, by Lemma 5. 3, $Vf/\mathrm{S}_1({}_VVf))$ is a non-small left V-module and Vf/S_2 $({}_VVf))$ is a small left V-module. Therefore we may show that $M=VF/\mathrm{S}_1$ $({}_VVf)$ is an injective left V-module (cf. Theorem 2. 11). Since $\mathrm{S}_1({}_VVf)=$ $\mathrm{S}_1(fV_V)$, $\mathrm{S}_1({}_VVf)$ is a two sided ideal of V. Let us consider the factor ring

$$\bar{V}=V/\mathrm{S}_1({}_VVf)\,.$$

As $\bar{V}$ is canonically isomorphic to the ring

$$T=T(Q)=\begin{pmatrix}Q & Q/S\\ J & Q\end{pmatrix}$$

we can identify $\bar{V}$ with T. Put $f^*=f_T$. By Theorem 5. 4, Tf^* is injective as a left T-module. We want to see that Tf^* is injective as a left V-module. (note $Tf^*=M$). Let I be a left ideal of V with $I\subseteq_e V$, and let ϕ be a homomorphism from ${}_VI$ to ${}_VTf^*$. Consider the diagram:

$$\begin{array}{ccccc}0 & \longrightarrow & I & \xrightarrow{\;i\;} & V\\ & & \phi\big\downarrow & & \\ & & Tf^* & & \end{array}$$

where i is the inclusion map. Note that

$$S_1({}_V Vf)=\begin{pmatrix}0 & S\\ 0 & 0\end{pmatrix} \quad\text{and}\quad \phi\big(S_1({}_V Vf)\big)\subseteq\begin{pmatrix}0 & Q/S\\ 0 & 0\end{pmatrix}$$

On the other hand, the socle of Tf^* is

$$\begin{pmatrix}0 & 0\\ 0 & S\end{pmatrix}$$

As a result, we see $\phi(S_1({}_V Vf))=0$ and hence ϕ induces the canonical homomorphism $\bar{\phi}$: $\bar{I}=I/S_1({}_V Vf))\to\bar{V}=T$. Since Tf^* is injective as a left T-module, we obtain a homomorphism η: $\bar{V}=T\to Tf^*$ satisfying $\eta\bar{\imath}=\bar{\phi}$, where $\bar{\imath}$ is the inclusion map: $\bar{I}\to\bar{V}=T$. If we denote the canonical V-homomorphism: $V\to\bar{V}=T$ by δ, then the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "\phi"'] & V \arrow[dl, "\eta\delta"] \\ & Tf^* & \end{tikzcd}$$

is commutative. Thus $M=Tf^*$ is injective as a left V-module.

THEOREM 5.6. *$W(Q)$ is a left H- and right co-H-ring.*

PROOF. Put $W=W(Q)$, $e=e_W$ and $f=f_W$. Noting $\mathrm{Soc}\,(eW_W)=\mathrm{Soc}\,({}_W We)$ $=\begin{pmatrix}S & 0\\ 0 & 0\end{pmatrix}$, we see that (eW, We) is an injective pair; whence eW_W and ${}_W We$ are injective. Further, we see $eJ_1(W)\simeq fW$ and $eJ_2(W)=Z(eW_W)$. Hence W is a right co-H-ring by Theorem 3.18 and its remark.

Next, we put

$$X=\mathrm{Soc}\,(W_W)=\begin{pmatrix}S & 0\\ S & 0\end{pmatrix}$$

Since $XWf=0$ and $X(We/\mathrm{Soc}\,({}_W We))=0$, by Lemma 5.3, Wf and We/Soc $({}_W We)$ are small left W-module. Thus W is a left H-ring by Theorem 2.11.

REMARK. In general, $W(Q)$ is neither a right H-ring nor a left co-H-ring. Therefore, the concepts of right H-rings and right co-H-rings are not right-left symmetric; in particular, the conditions (*) and (*)* are not right-left symmetric (cf. [15]). Actually, assume that $W(Q)$ is a left co-H-ring, and put $W=W(Q)$, $e=e_W$ and $f=f_W$. Then, W is an artinian QF-3 ring but not QF. As we saw in the proof of Theorem 5.6, ${}_W We$ is injective. Hence, by the assumption there must exist an isomorphism ϕ from Wf to $J({}_W We)$. Then

$$\phi\left(\begin{pmatrix}0 & 0\\ 0 & Q\end{pmatrix}\right)=\begin{pmatrix}0 & 0\\ J & 0\end{pmatrix}$$

This shows that J is a cyclic left ideal of Q. However, in general, J need not be a cyclic left ideal. For example, consider the ring

$$Q=K[x, y]/(x^2, y^2)$$

where K is a field. As is well known, Q is a local QF-algebra over K of dimension 4. Put $I=(x^2, y^2)$, $\alpha=x+I$, $\beta=y+I$ and $\gamma=xy+I$. Clearly $\mathrm{J}(Q)=Q\alpha+Q\beta$ and $\mathrm{Soc}(Q)=Q\gamma$. Now, we can easily see that $\mathrm{J}(Q)$ is not a cyclic ideal. Accordingly, $W(Q)$ for this Q is not a left co-H-ring, and at the same time, it is not a right H-ring by Theorem 5. 1.

REMARK. Let Q be as in the above remark. Again, we put $W=W(Q)$, $e=e_W$ and $f=f_W$. Assume that $\mathrm{Soc}({}_W Wf)$ is a simple socle. Then, since W is QF-3, ${}_W Wf$ must be embedded in ${}_W We$. However, this implies that $\mathrm{J}(Q)$ is a cyclic left ideal of Q, which is impossible as we noted above. Thus $\mathrm{Soc}({}_W Wf)$ is not a simple socle. As a result, this W also gives a counter example of a ring which solves a problem raised by Fuller [8].

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