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Non-existence of higher order non-singular holomorphic immersions

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0. Introduction

In [6] Pohl formulated and studied the higher order complex analytic geometry and recently in [10] Watanabe studied higher order non-singular holomorphic embeddings of algebraic manifolds into Grassmann manifolds. In this note we study non-existences of higher order non-singular holomorphic immersions of complex projective spaces and their non-singular complex hypersurfaces into complex projective spaces by means of Chern classes. Our main results are Theorem 2.2 and Corollary 3.3. It is well known that non-singular complex algebraic curves of degree >2 in a complex projective plane have inflection points. The statement (iii) of Corollary 3.3 is a generalization of this fact to a case of higher dimension and higher order. Let P_m be the m-dimensional complex projective space and for $q\geq 2$, we denote a non-singular complex hypersurface of degree q in P_{n+1} by $V_n(q)$. In [2] Feder proved the following theorem.

THEOREM 0.1. *If* $f: P_n\rightarrow P_N$ *is a holomorphic immersion and* $N<2n$, *then* $\deg(f)=1$, *where* $\deg(f)$ *is a degree of* f (see Section 2 of this note).

Furthermore in [7] Samsky proved the following theorem.

THEOREM 0.2. *If* $f: V_n(q)\rightarrow P_N$ *is a holomorphic immersion and* $N<2n$, *then* $\deg(f)=1$, *where* $\deg(f)$ *is a degree of* f (see Section 3 of this note).

In our terminology, holomorphic immersions may be regarded as first order non-singular holomorphic mappings or holomorphic mappings without 0-th order inflection points (see Section 1 of this note). Hence the statement (i) of Theorem 2.2 (Corollary 3.3 resp.) is a result for the higher order case of the above Theorem 0.1 (0.2 resp.). The proofs much depend upon symmetric power operations in K-theory which Suzuki [8, 9] firstly used in KO-theory to show non-existences of higher order non-singular differentiable immersions of real (and complex resp.) projective spaces into euclidean or real (and complex resp.) projective spaces. The author is grateful to Mr. Watanabe for enlightening conversations and advices.

1. Preliminaries.

Let $\eta \to M_n$ be a holomorphic vector bundle of rank m over a complex manifold M_n of complex dimension n and let $\phi: \eta \to C^{N+1}$ be a holomorphic mapping which is complex linear on each fibre of η. Then we call ϕ a *realization of* η. We say that the realization ϕ is *non-singular at* $x \in M_n$ if $\phi|_x$ is of maximal rank, where $\phi|_x$ is a restriction of ϕ to the fibre η_x at x and that ϕ is *non-singular* if ϕ is non-singular at each $x \in M_n$. ϕ is non-singular if and only if ϕ is injective or surjective on each fibre of η as $m \leqq N+1$ or $m \geqq N+1$, respectively. For $k \geqq 0$, we put

$$\mu(n, k) = \binom{n+k-1}{k}, \qquad \nu(n, k) = \binom{n+k}{k} - 1 .$$

We denote the k-fold symmetric tensor product of η by

$$O^k \eta \, (O^0 \eta = 1, \ O^1 \eta = \eta) ,$$

where 1 is a trivial complex line bundle over M_n. It is a holomorphic vector bundle over M_n of rank $\mu(m, k)$. Let $\xi \to M_n$ be a holomorphic line bundle over M_n. Now we introduce a holomorphic vector bundle $\Delta_p \xi$ over M_n of rank $\nu(n, k)+1$ which is called the p-th derivative of ξ and defined by Pohl in [6]. Its precise definition and detailed discussion are described in [6], [10] but we explain roughly it. For "n-multi indices", i.e., n-tuples of non-negative integers $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$, we put $\alpha + \beta = (\alpha_1 + \beta_1, \cdots, \alpha_n + \beta_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha ! = \alpha_1 ! \cdots \alpha_n !$. Suppose that $(U; z^1, \cdots, z^n)$ is a holomorphic local chart of M_n and that (e) is a holomorphic local frame field on U of ξ, where e is a holomorphic section of $\xi|_U$ such that $e_x \neq 0$ for each $x \in U$. For each n-multi index $\alpha = (\alpha_1, \cdots, \alpha_n)$ with $|\alpha| > 0$, we set

$$D_z^\alpha = \frac{\partial^{|\alpha|}}{(\partial z^1)^{\alpha_1} \cdots (\partial z^n)^{\alpha_n}} .$$

Then the holomorphic local frame field on U of $\Delta_p \xi$ is of the form:

$$(D_z^\alpha \cdot e \, ; \ |\alpha| \leqq p) ,$$

where the following properties:

$$D_z^\alpha \cdot e = e \ (\text{for } |\alpha| = 0) ,$$

$$D_z^\gamma \cdot (\sigma + \tau) = D_z^\gamma \cdot \sigma + D_z^\gamma \cdot \tau ,$$

$$D_z^\gamma \cdot (h\sigma) = \sum_{\alpha+\beta=\gamma} \frac{(\alpha+\beta)!}{\alpha! \ \beta!} (D_z^\alpha h) \, D_z^\beta \cdot \sigma$$

hold for each n-multi index γ, any holomorphic sections σ, τ of ξ_U and any holomorphic function h on U. Moreover suppose that $(V; w^1, \cdots, w^n)$ is another holomorphic local chart of M_n such that $U\cap V\neq\phi$ and that (f) is another holomorphic local frame field on V of ξ. Let g_{VU}, J_{VU} be transition functions of ξ, $\tau(M_n)$ on $U\cap V$, respectively, where $\tau(M_n)$ is the holomorphic tangent bundle of M_n. Let $\Delta_p g_{VU}: U\cap V\to GL(\nu(n, p)+1; \boldsymbol{C})$ be the transition function of $\Delta_p\xi$. Then we have that

$$(e)=(f)\,g_{VU}, \qquad (D_z^i;\ 1\leqq i\leqq n)=(D_w^i;\ 1\leqq i\leqq n)\,J_{VU},$$

$$(D_z^\alpha\cdot e;\ |\alpha|\leqq p)=(D_w^\alpha\cdot f;\ |\alpha|\leqq p)\,\Delta_p g_{VU},$$

$$\Delta_p g_{VU}=\begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0p} \\ 0 & A_{11} & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{p-1\,p} \\ 0 & \cdots & 0 & A_{pp} \end{pmatrix},$$

where $A_{jj}=O^j J_{VU}\otimes g_{VU}$ is the transition function of $O^j\tau(M_n)\otimes\xi\,(0\leqq j\leqq p)$; A_{jk} is a matrix of type $(\mu(n, j), \mu(n, k))$ whose components are holomorphic functions involving partial derivatives of g_{VU} of order $\leqq k-j$ $(0\leqq j<k\leqq p)$. In [3] Feldman pointed out that $\Delta_p\xi$ is regarded as $J_p(\xi^*)^*$, where ξ^* is the dual bundle of ξ and $J_p(\xi^*)$ denotes the holomorphic vector bundle of p-jets of sections of ξ^*. We have the following holomorphic short exact sequence (see [6, § III. 1).

$$0\longrightarrow\Delta_{p-1}\xi\longrightarrow\Delta_p\xi\longrightarrow O^p\tau(M_n)\otimes\xi\longrightarrow 0,$$

where $\Delta_0\xi=\xi$. Hence $\Delta_p\xi$ is topological isomorphic to

$$\left(\sum_{j=0}^{p} O^j\tau(M_n)\right)\otimes\xi,$$

where $\sum$ denotes Whitney sum. By $\sum_{j=0}^{p} O^j\tau(M_n)=O^p(\tau(M_n)+1)$, we have the following proposition.

PROPOSITION 1.1. *$\Delta_p\xi$ is topologically isomorphic to*

$$O^p\big(\tau(M_n)+1\big)\otimes\xi.$$

Next we assume that a holomorphic line bundle ξ over M_n and its realization ϕ: $\xi\to\boldsymbol{C}^{N+1}$ are given. We introduce the canonical realization $D_p(\phi): \Delta_p\xi\to\boldsymbol{C}^{N+1}$ induced by ϕ. Its detailed discussion is described in [6; Theorem 3.12]. Let $(D_z^\alpha\cdot e;\ |\alpha|\leqq p)$ be the above holomorphic local frame field on U of $\Delta_p\xi$. Then $D_p(\phi)$ is given by

$$D_p(\phi)\big|_x\Big((D_z^\alpha\cdot e)_x\Big)=\Big(D_z^\alpha\big(\phi(e)\big)\Big)_x\in \boldsymbol{C}^{N+1}(|\alpha|\leqq p)$$

at each $x\in U$. Let P_N be the N-dimensional complex projective space and let $\pi:\gamma_N\to P_N$ be the universal complex line bundle over P_N. We may think of γ_N as consisting of all pairs (y, v), where $y\in P_N$ is a complex line through the origin of $\boldsymbol{C}^{N+1}$ and v is a vector of y. The projection π is defined by $\pi(y, v)=y$. Let $\sigma:\gamma_N\to \boldsymbol{C}^{N+1}$ be the mapping $(y, v)\mapsto v$. Then σ is a realization of γ_N. Now let $f: M_n\to P_N$ be a holomorphic immersion of a complex manifold M_n of complex dimension n into P_N $(n\leqq N)$. $f^{-1}\gamma_N$ denotes the pull-back of γ_N by f. Then we have a bundle mapping $\hat{f}: f^{-1}\gamma_N\to\gamma_N$ over f. Clearly the mapping $\sigma\circ\hat{f}: f^{-1}\gamma_N\to \boldsymbol{C}^{N+1}$ is a realization of $f^{-1}\gamma_N$. We say that the immersion $f: M_n\to P_N$ is *non-singular of order p at $x\in M_n$* if the canonical realization $D_p(\sigma\circ\hat{f}): \Delta_p f^{-1}\gamma_N\to \boldsymbol{C}^{N+1}$ induced by $\sigma\circ\hat{f}$ is non-singular at x and that f is *non-singular of order p* if f is non-singular of order p at each $x\in M_n$. f is non-singular of order p if and only if $D_p(\sigma\circ\hat{f})$ is injective or surjective on each fibre of $\Delta_p f^{-1}\gamma_N$ as $\nu(n, p)\leq N$ or $\nu(n, p)\geq N$, respectively. If $\nu(n. p)\leq N$ and f is non-singular of orader p, then it is non-singular of order k for $1\leq k\leq p$. A holomorphic immersion into the complex projective space is first order non-singular. Suppose that $k\geq 1$, $\nu(n, k)\leq N$ and that the holomorphic immersion f is non-singular of order k. Unless f is non-singular of order $k+1$ at $x\in M_n$, we say that x is a *k-th order inflection point of f*. If $p\geq 2$, $\nu(n, p)\leq N$ and unless the holomorphic immersion f is non-singular of order p, then f has at least one inflection point of order $\leq p-1$. For $k\geq 1$ and a holomorphic immersion f: $M_n\to P_N$ $(n\leq N)$, we denote by

$$\delta_k(f):\ \Delta_k f^{-1}\gamma_N\longrightarrow \hat{\boldsymbol{C}}^{N+1}$$

the holomorphic homomorphism that the canonical realization $D_k(\sigma\circ\hat{f})$ induces, where $\hat{\boldsymbol{C}}^{N+1}$ denotes a product bundle $M_n\times \boldsymbol{C}^{N+1}$. Moreover if f is non-singular of order p, we denote the cokernel or the kernel of $\delta_p(f)$ by Coker $\delta_p(f)$ or Ker $\delta_p(f)$ as $\nu(n, p)\leq N$ or $\nu(n, p)\geq N$, respectively. Then Coker $\delta_p(f)$, Ker $\delta_p(f)$ are holomorphic vector bundles of rank $N-\nu(n, p)$, $\nu(n, p)-N$, respectively. We will give an example of a p-th order non-singular holomorphic embedding. Let $(\zeta_0: \zeta_1: \cdots: \zeta_n)$ be homogeneous coordinates for the n-dimensional complex projective space P_n. One gets a holomorphic embedding

$$v_p:\ P_n\longrightarrow P_{\nu(n,p)}$$

by mapping $(\zeta_0: \zeta_1: \cdots: \zeta_n)$ into $(M_0(p): M_1(p): \cdots: M_{\nu(n,p)}(p))$, where $M_0(p)$, $M_1(p), \cdots, M_{\nu(n,p)}(p)$ are all possible distinct monomials of degree p in ζ_0,

$\zeta_1, \cdots, \zeta_n$. It is easily shown that v_p is non-singular of order p. It is called a Veronese embedding.

Last we explain the symmetric power operations in K-theorey that we use in Section 2 and 3. Let X be a finite connected CW-complex and $e(X)$ a set of all isomorphism classes of complex vector bundles over X. $e(X)$ is a commutative semiring with 1 in which the addition and multiplication are induced by the Whitney sum and tensor product of complex vector bundles over X. For $k \geq 1$ and a complex line bundle ξ over X, ξ^k, ξ^0, ξ^{-1}, ξ^{-k} denote a k-fold tensor product of ξ, trivial complex line bundle 1, dual bundle of ξ, k-fold tensor product of ξ^{-1}, respectively. For $[\eta] \in e(X)$, we put $O^j[\eta] = [O^j \eta]$ $(j \geq 0)$, where [ ] denotes an isomorphism class of a complex vector bundle over X. Then the operations O^j $(j \geq 0)$ have the following properties:

i) $O^0(x) = 1$, $O^1(x) = x$ for $x \in e(X)$,

ii) $O^k(x+y) = \sum_{i+j=k} O^i(x)\, O^j(y)$ for $x, y \in e(X)$,

iii) $O^j([\xi]) = [\xi^j] = [\xi]^j$ for $[\xi] \in e(X)$, where ξ is a complex line bundle.

Let $K(X)$ be a ring completion of $e(X)$ and let $\theta : e(X) \to K(X)$ be a natural semiring homomorphism. We set

$$O_t(x) = \sum_{j=0}^{\infty} t^j O^j(x)$$

for an indeterminate t and each $x \in e(X)$. Let $A(X)$ denote the multiplicative group of formal power series in t with coefficients in $K(X)$ and constant term 1. Then the properties i), ii) assert that O_t defines a homomorphism of $e(X)$ into $A(X)$. Hence we get a homomorphism $O_t : K(X) \to A(X)$. Taking the coefficients of O_t, this defines operators $O^j : K(X) \to K(X)$ $(j \geq 0)$ which are called the symmetric power operators. Properties i), ii) continue to hold for these O^j but property iii) holds only in $\theta(e(X))$. Hereafter by a complex vector bundle itself we denote its isomorphism class too. Note that for a complex line bundle ξ,

$$O_t(\xi) = (1 - t\xi)^{-1}.$$

2. The case of complex projective spaces

Let $f: P_n \to P_N$ be a holomorphic mapping. If $f^{-1}\gamma_N$ is topologically isomorphic to γ_n^d, we say that f is *of degree d* and denote the degree of f by $\deg(f)$. For holomorphic mappings $f, g : P_n \to P_N$, if $\deg(f) \neq \deg(g)$, then f is not homotopic to g. It is shown by Feder in [2] that for any $d > 0$, there exists a holomorphic immersion or embedding $f: P_n \to P_N$ of degree d as $2n \leq N$ or $2n+1 \leq N$, respectively. It is well known that $\tau(P_n)+1=(n+1)\gamma_n^{-1}$ in $K(P_n)$. Hence we have

$$\begin{aligned} O_t\left(\tau(P_n)+1\right) &= O_t\left((n+1)\gamma_n^{-1}\right) = (1-t\gamma_n^{-1})^{-(n+1)} \\ &= \sum_{p=0}^{\infty} t^p \binom{n+p}{p} \gamma_n^{-p} = \sum_{p=0}^{\infty} t^p O^p\left(\tau(P_n)+1\right). \end{aligned}$$

Therefore we have the following lemma.

LEMMA 2.1. *In* $K(P_n)$

$$O^p\left(\tau(P_n)+1\right) = \binom{n+p}{p} \gamma_n^{-p}.$$

Let $\alpha \in H^2(P_n; Z)$ be the first Chren class of γ_n^{-1}. Then the additive order of α^m is infinite for $1 \leq m \leq n$ and $\alpha^{n+1}=0$. Now we prove the following theorem which is one of the main results.

THEOREM 2.2. *Suppose that* $p \geq 2$ *and that* $f: P_n \rightarrow P_N$ *is a holomorphic immersion of degree* $d>0$.

(i) *As* $\nu(n,p) \leq N < \nu(n,p)+n$, *if* $d \neq p$, *then* f *has at least one inflection point of order* $\leq p-1$.

(ii) *As* $\nu(n,p)-n < N \leq \nu(n,p)$, *if* f *is non-singular of order* p, *then* $d=p$.

PROOF. Since (ii) is proved in the same manner as (i), we prove only (i). It follows from Proposition 1.1 and Lemma 2.1 that

$$\Delta_p f^{-1}\gamma_N = \binom{n+p}{p} \gamma_n^{d-p}$$

in $K(P_n)$. Suppose that f is non-singular of order p. Then Coker $\delta_p(f)$ is a complex vector bundle of rank $N-\nu(n,p)<n$. Moreover we have

$$\text{Coker } \delta_p(f) = \hat{C}^{N+1} - \binom{n+p}{p} \gamma_n^{d-p}$$

in $K(P_n)$. Hence the total Chern class of Coker $\delta_p(f)$ is given by

$$c\left(\text{Coker } \delta_p(f)\right) = \left(1-(d-p)\,\alpha\right)^{-(\nu(n,p)+1)}.$$

Thus the n-th Chern class of it is given by

$$c_n\left(\text{Coker } \delta_p(f)\right) = \binom{\nu(n,p)+n}{n} (d-p)^n \alpha^n.$$

Hence $c_n(\mathrm{Coker}\,\delta_p(f))\neq 0$. This contradicts that the rank of $\mathrm{Coker}\,\delta_p(f)$ is less than n. q. e. d.

REMARK. If $\nu(n,p)-n<N<\nu(n,p)+n$, then the above result is best possible. In fact there exists the p-th order non-singular embedding f: $P_n\to P_N$ of degree p. It is made of the Veronese embedding v_p after the same manner as the proof of Theorem 1. 2 in [2]. S. Watanabe informed me that if $N\leq\nu(n,p)-n$ or $\nu(n,p)+n\geq N$, then for $d>p\geq 2$, there exists a p-th order non-singular embedding f: $P_n\to P_N$ of degree d (see [10]).

3. The case of complex hypersurfaces

For $q\geq 2$, we denote a non-singular complex hypersurface of degree q in P_{n+1} by $V_n(q)$. Let j: $V_n(q)\to P_{n+1}$ be a canonical inclusion. We write $\xi_n=j^{-1}\gamma_{n+1}$, where $j^{-1}\gamma_{n+1}$ is a pull-back of γ_{n+1} by j. Then $\xi_n\to V_n(q)$ is a holomorphic line bundle. F. Hirzebruch has shown that the holomorphic normal bundle of $V_n(q)$ in P_{n+1} is given by $\nu(V_n(q))=\xi_n^{-q}$ (see [4; p. 69]). Hence we get $\tau(V_n(q))+1=(n+2)\,\xi_n^{-1}-\xi_n^{-q}$ in $K(V_n(q))$. Therefore we have

$$\begin{aligned} O_t\Big(\tau\big(V_n(q)\big)+1\Big) &= O_t\big((n+2)\,\xi_n^{-1}\big)\big(O_t(\xi_n^{-q})\big)^{-1} \\ &=(1-t\xi_n^{-1})^{-(n+2)}(1-t\xi_n^{-q}) \\ &=\left\{\sum_{k=0}^{\infty} t^k\binom{n+1+k}{k}\xi_n^{-k}\right\}(1-t\xi_n^{-q}) \\ &=1+\sum_{p=1}^{\infty} t^p\left\{\binom{n+1+p}{p}\xi_n^{-p}-\binom{n+p}{p-1}\xi_n^{-p-q+1}\right\}. \end{aligned}$$

Hence the following lemma has been shown.

LEMMA 3. 1. *In $K(V_n(q))$, for $p>0$*

$$O^p\Big(\tau\big(V_n(q)\big)+1\Big)=\binom{n+1+p}{p}\xi_n^{-p}-\binom{n+p}{p-1}\xi_n^{-p-q+1}.$$

Let $\beta\in H^2(V_n(q);Z)$ be the first Chern class of ξ_n^{-1}. Then the additive order of β^m is infinite for $0\leq m\leq n$ and $\beta^{n+1}=0$. Let f: $V_n(q)\to P_N(n+1\leq N)$ be a holomorphic immersion. If $f^{-1}\gamma_N$ is topologically isomorphic to ξ_n^d, we say that f is *of degree d* and write $\deg(f)=d$. It follows from Theorem (4) in [7] that for any $d>0$, there exists a holomorphic immersion or embedding f: $V_n(q)\to P_N$ of degree d as $2n\leq N$ or $2n+1\leq N$, respectively. For holomorphic immersions f, g: $V_n(q)\to P_N$, if $\deg(f)\neq\deg(g)$, then f is not homotopic to g. S. Watanabe informed me that if $N\leq\nu(n,p)-n$ or $\nu(n,p)+n\geq N$, then for $d>p\geq 2$, there exists a p-th order non-singular holomorphic embedding $f\colon V_n(q)\to P_N$ of degree d (see [10]). We prove the following theorem.

THEOREM 3.2. *Suppose that* $\nu(n,p)-n<N<\nu(n,p)+n$ *and that there exists a p-th order non-singular holomorphic immersion* $f\colon V_n(q)\to P_N$ *of degree* $d>0$. *Put*

$$A=\binom{n+p}{p-1}, \qquad B=\binom{n+1+p}{p},$$

$$a=q-1-(d-p), \qquad b=d-p.$$

(i) *If* $N\geq\nu(n,p)$, *then for any* m *with* $N-\nu(n,p)<m\leq n$,

$$(1)_m \qquad \sum_{k=0}^{m}\binom{A}{m-k}\binom{B-1+k}{k}a^{m-k}b^k=0.$$

(ii) *If* $N\leq\nu(n,p)$, *then for any* m *with* $\nu(n,p)-N<m\leq n$,

$$(2)_m \qquad \sum_{k=0}^{m}\binom{A-1+k}{k}\binom{B}{m-k}a^k b^{m-k}=0.$$

PROOF. Since (ii) is proved in the same manner as (i), we prove only (i). It follows from Proposition 1.1 and Lemma 3.1 that in $K(V_n(q))$,

$$\Delta_p f^{-1}\gamma_N=B\xi_n^b-A\xi_n^{-a}.$$

Since $\nu(n,p)\leq N$, Coker $\delta_p(f)$ is a complex vector bundle of rank $N-\nu(n,p)$. Moreover we have

$$\text{Coker } \delta_p(f)=\hat{C}^{n+1}-B\xi_n^b+A\xi_n^{-a}$$

in $K(V_n(q))$. Hence its total Chern class is given by

$$c\big(\text{Coker } \delta_p(f)\big)=\frac{(1+a\beta)^A}{(1-b\beta)^B}.$$

Thus for any m with $N-\nu(n,p)<m\leq n$, its m-th Chern class

$$c_m\big(\text{Coker } \delta_p(f)\big)=\sum_{k=0}^{m}\binom{A}{m-k}\binom{B-1+k}{k}a^{m-k}b^k\beta^m$$

must vanish. q.e.d.

COROLLARY 3.3. *Suppose that* $p\geq 2$ *and that* $f\colon V_n(q)\to P_N$ *is a holomorphic immersion of degree* $d>0$.

(i) *As* $\nu(n,p)\leq N<\nu(n,p)+n$, *if* $p\leq d<p+q$, *then* f *has at least one inflection point of order* $\leq p-1$.

(ii) *As* $\nu(n,p)-n<N\leq\nu(n,p)$, *if* f *is non-singular of order* p, *then*

$0<d<p$ *or* $p+q\leq d$.

(iii) *As* $N=\nu(n,p)$, *unless* $n=1$ *and* $q=2$, *then for any* $d>0$, f *has at least one inflection point of order* $\leq p-1$.

PROOF. (i), (ii) As $p\leq d<p+q$, since $a=q-1-(d-p)\geq 0$, $b=d-p\geq 0$ and $a+b=q-1>0$, the left sides of $(1)_n$, $(2)_n$ are not vanishing.

(iii) For any $d>0$, unless $n=1$ and $q=2$, then the left side of $(1)_1$ or $(1)_2$ is not vanishing. q. e. d.

REMARK. As $n=1$, $q>2$, $p=2$ and $d=1$, (iii) of Corollary 3. 3 is a classical fact on non-singular complex algebraic curves in P_2.

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