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Convex programming on spaces of measurable functions

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1. Introduction. In [3] and [4], J. Zowe has considered convex programming with values in ordered vector spaces. His hypotheses are so restrictive that his theory does not apply to the case of the function spaces L_p $(\infty>p>0)$. For L_∞ and $C(X)$ with X a Stonean space, his method is very useful. We shall consider in this note convex operators whose values are in much more general spaces than the usual function spaces such as L_p. Functions assuming the value $+\infty$ introduce certain complications, to which we must address ourselves.

For simplicity, we consider only convex operators defined on the real line $\mathbb{R}$ with values in the space of measurable functions with values in $\mathbb{R}\cup\{+\infty\}$. The general case will be treated in a later publication.

In this note, we present a generalization of the Fenchel-Moreau theorem and also of the Fenchel theorem. It is appropriate to consider the $P(\Omega)$ of measurable functions, whose definition will be found in Section 2.

2. Preliminary lemmas

Let F be an extended real-valued function on the real numbers $\mathbb{R}$, possibly assuming the value $+\infty$, but not the value $-\infty$. Let D be a (dense) countable subfield of $\mathbb{R}$. Such an extended real-valued function F defined on D is said to be *D-convex* if

$F(\alpha x+\beta y)\leqq\alpha F(x)+\beta F(y)$

for $\alpha, \beta\in D$ with $\alpha+\beta=1$, $\alpha, \beta\geq 0$ and $x, y\in D$.

An $\mathbb{R}$-convex function will be called convex, as usual.

We first present a number of lemmas.

LEMMA 1. *Every finite-valued D-convex function defined on D is continuous in D: that is $x_n\rightarrow x(x_n, x\in D)$ implies that $F(x_n)\rightarrow F(x)$.*

PROOF. If the sequence $F(x_n)$ does not converge for $x_n\rightarrow x$, the convexity of F implies that either $F(y)=+\infty$ for all $y>x$ or $F(y)=+\infty$ for all $y<x$. Since F is finite-valued, F is continuous.

Curiously enough, a D-convex function F defined on all of $\mathbb{R}$, i. e. a function satisfying $F(\alpha x+\beta y)\leqq\alpha F(x)+\beta F(y)$ for $\alpha, \beta\in D$ with $\alpha+\beta=1$, $\alpha, \beta\geq 0$ and $x, y\in\mathbb{R}$, is not necessarily continuous on $\mathbb{R}$. (For example, a discontinuous real-valued solution of the functional equation $\phi(x+y)=\phi(x)+\phi(y)$ will do.) Nevertheless we have the following lemma.

LEMMA 2. *Every finite-valued D-convex function can be extended in one and only one way to a (continuous) convex function on* $\mathbb{R}$.

We omit the proof.

We define convex functions on n-dimensional space $\mathbb{R}^n$ in the same way; they assume values in $\mathbb{R}\cup\{+\infty\}$.

Let F be a convex function on $\mathbb{R}^n$ with values in $\mathbb{R}\cup\{+\infty\}$. The effective domain of F, defined as $\mathscr{D}_F=\{x\,;F(x)<\infty\}$ is plainly a convex set.

LEMMA 3. *If $\mathscr{D}_F$ contains at least 2 points, then $\mathscr{D}_F$ has an interior point with respect to the affine hull A_F of $\mathscr{D}_F$ and the restriction of F to A_F is continuous at every interior point of $\mathscr{D}_F$ with respect to A_F. In particular, if F is always finite valued, then F is continuous at everywhere in* $\mathbb{R}^n$.

PROOF. See [1] Theorem 3 p. 188.

LEMMA 4. *A convex function F on $\mathbb{R}^n$, is lower semi-continuous at x_0 in $\mathbb{R}^n$ if and only if the restriction map F_l of F on each line l through x_0 is lower semi-continuous at x_0.*

PROOF. If F is not lower semi-continuous at x_0 and $F(x_0)<+\infty$, then there exists $\varepsilon>0$ and a sequence $\{x_n\}_{n=1}^{\infty}\subset\mathscr{D}_F(n=1,2,\cdots)$ with $x_n\to x_0$ such that $F(x_n)<F(x_0)-\varepsilon$. We can assume that there is at least one n such that x_n is an interior point of $\mathscr{D}_F$ with respect to A_F (see Lemma 3). Let $[x_0,x_n]$ be the interval from x_0 to x_n. Then, for $w\in(x_0,x_n]$, $F(w)\leqq F(x_0)-\varepsilon$ since $(x_0,x_n]$ is contained in the closed convex hull of $\{x_m\}_{m=0}^{\infty}$ and F is continuous on $(x_0,x_n]$. But this means that F_l is not lower semi-continuous at x_0 where l is the line containing the interval $[x_0,x_n]$. The converse is proved similarly.

Let Ω be a finite measure space with measure μ.
Two measurable sets A and B are identified if

$$\mu(A\oplus B)=\mu(A\backslash B)+\mu(B\backslash A)=0.$$

The collection of measurable sets of Ω then constitutes a complete Boolean lattice.

Let $S(\Omega)$ be the set of all measurable functions on Ω which are finite-valued almost everywhere. We identify f and $g\in S(\Omega)$ if they differ only on a set of μ-measure zero.

Let $P(\Omega)$ be the totality of all measurable functions on Ω assuming values in $\mathbb{R}\cup\{+\infty\}$. Plainly $P(\Omega)$ is a convex set.

We identify f and $g\in P(\Omega)$ if they differ only on a set of μ-measure zero. We define $Q(\Omega)=\{f\,;-f\in P(\Omega)\}$. Thus $f\in Q(\Omega)$ is a measurable function on Ω assuming values in $\mathbb{R}\cup\{-\infty\}$. Finally, let $U(\Omega)$ as the totality of all measurable functions on Ω with values in $\mathbb{R}\cup\{+\infty\}\cup\{-\infty\}$. it is obvious that

$$U(\Omega)\supset P(\Omega)\cup Q(\Omega)$$

and

$$L_{\infty}(\Omega)\subset L_{p}(\Omega)\subset L_{1}(\Omega)\subset S(\Omega)\subset \begin{matrix}P(\Omega)\\ Q(\Omega)\end{matrix}\subset U(\Omega)$$

for $p\geqq 1$.

With the usual ordering $S(\Omega)$ is a Dedekind complete vector lattice and if $S(\Omega)\ni f=\bigvee_{a} f_{a}$ for $f_{a}\in S(\Omega)$, there exists a countable subfamily $\{a_{n}\}$ of $\{a\}$ for which

$$\bigvee_{a} f_{a}=\bigvee_{n} f_{a_{n}},$$

$\bigvee_{a} f_{a}$ defines the supremum of $\{f_{a}\}$ in the complete vector lattice $S(\Omega)$.

The set $P(\Omega)$ is complete under the supremum operation i. e. if $f_{a}\in P(\Omega)$, the supremum $f=\bigvee_{a} f_{a}$ exists in $P(\Omega)$. Likewise, if $f_{a}\in Q(\Omega)$ the infimum $f=\bigwedge_{a} f_{a}$ (exists) in $Q(\Omega)$. Again, we can select a countable family $f_{a_{n}}$ with $f=\bigvee_{n} f_{a_{n}}$ or $f=\bigwedge_{n} f_{a_{n}}$.

LEMMA 5. *Let F be a convex operator from $\mathbb{R}$ into $Y=S(\Omega)$. Then there exist a subset A of Ω of measure zero and a function $F(\alpha, t)$ defined on $\mathbb{R}\times\Omega$ such that for each fixed $t\in\Omega\backslash A$, $\mathbb{R}\ni\alpha\rightarrow F(\alpha, t)$ is a convex function on $\mathbb{R}$ and for each fixed $\alpha\in\mathbb{R}$, $\Omega\ni t\rightarrow F(\alpha, t)$ is a measurable function on Ω which is identified with $F(\alpha)$ as an element of $S(\Omega)$ and F is continuous which is to say*

$$\alpha_{n}\rightarrow\alpha \text{ implies } F(\alpha_{n})\rightarrow F(\alpha) \text{ a. e.}$$

PROOF. Let $D=\left\{\frac{m}{2^{n}}(m \text{ is an integer and } n \text{ is a natural number})\right\}$.

For each fixed $\alpha\in D$, we write $t\rightarrow F(\alpha, t)$ for the function $F(\alpha)$ in $S(\Omega)$. This function is by definition measurable and finite a. e.. We will write $F(\alpha)=\{F(\alpha, t)\}$. Since

$$F(\alpha\alpha_{1}+\beta\beta_{1})\leqq\alpha F(\alpha_{1})+\beta F(\beta_{1})$$

for each 4-tuple $\alpha, \beta, \alpha_{1}, \beta_{1}\in D$ for which $\alpha+\beta=1$, $\alpha, \beta\geqq 0$, we have

$$F(\alpha\alpha_{1}+\beta\beta_{1}, t)\leqq\alpha F(\alpha_{1}, t)+\beta F(\beta_{1}, t)$$

except on a set $A(\alpha, \beta, \alpha_{1}, \beta_{1})$ of measure zero. As the number of 4-tuples $(\alpha, \beta, \alpha_{1}, \beta_{1})$ is countable, we see that $F(\alpha, t)$ is D-convex and finite except on the set $A_{0}=\cup A(\alpha, \beta, \alpha_{1}, \beta_{1})$ which has measure zero. Lemma 2 and Lemma 3 show that we can extend F to a finite valued convex function $F_{1}(\alpha, t)$ on all $\alpha\in\mathbb{R}$ for each $t\in\Omega\backslash A_{0}$. For each $t\in A_{0}$, we can define an arbitrary finite valued convex function $F_{1}(\alpha, t)$.

We can prove that $\{F_{1}(\alpha, t)\}=F(\alpha)$ for all $\alpha\in\mathbb{R}$. Suppose $F(\alpha)=$

$\{F(\alpha, t)\}$ and $\{F(\alpha, t)\} \neq \{F_1(\alpha, t)\}$ for some $\alpha \in \mathbb{R}$. Then the measurable set of Ω, $A=\{t ; F(\alpha, t) \neq F_1(\alpha, t)\}$ has positive measure. Since $F(\alpha, t)$ are finite a. e. on A there exists $\alpha_1 \in D$ near α such that $F(\alpha_1, t)=+\infty$ for t in a set of positive measure. Since this is impossible, we have $\{F_1(\alpha, t)\}=\{F(\alpha, t)\}$. That is, for each $\alpha \in \mathbb{R}$ we have $F_1(\alpha, t)=F(\alpha, t)$ a. e. in $t \in \Omega$.

A convex function F on the real numbers is said to be *right side finite* (or *left side infinite*) if there exists α_0 (or β_0) such that $F(\alpha)<+\infty$ for $\alpha \geqq \alpha_0$ (or $F(\alpha)=+\infty$ for $\alpha \leqq \beta_0$). In the same way, we can define a left side finite and right side infinite convex function, and both sides infinite convex function.

LEMMA 6. *Let F be a convex operator from $\mathbb{R}$ into $P(\Omega)$ such that $F(\alpha_0) \in S(\Omega)$ for some $\alpha_0 \in \mathbb{R}$. There exist pairwise disjoint measurable sets $A_i (i=1, 2, 3, 4)$ of Ω with $A_1 \cup A_2 \cup A_3 \cup A_4=\Omega$, such that for $t \in A_1$, $F(\alpha)$ is finite for all $\alpha \in \mathbb{R}$; for $t \in A_2$, $F(\alpha)$ is right side finite and left side infinite; for $t \in A_3$, $F(\alpha)$ is left side finite and right side infinite; for $t \in A_4$, $F(\alpha)$ is both sides infinite.*

PROOF. We consider the complete Boolean lattice of measurable subsets of Ω identifying sets whose symmetric difference has measure zero. We use the symbols $\vee$ and $\wedge$ to denote supremum and infimum respectively, in this complete Boolean lattice. We may suppose as above that $F(\alpha)$ is represented by a measurable function on $\Omega: t \rightarrow F(\alpha, t)$. For $\alpha \in \mathbb{R}$, write $A_\alpha=\{t ; F(\alpha, t)=+\infty\}$ and

$$A=\bigvee_{\alpha} A_\alpha .$$

We define $B=\bigvee_{\alpha>\alpha_0} A_\alpha$ and $C=\bigvee_{\alpha<\alpha_0} A_\alpha$. Plainly we have $B \cup C=A$. Finally let $A_1=\Omega \backslash A$, $A_2=A \backslash B$, $A_3=A \backslash C$ and $A_4=B \cap C$. Then A_1, A_2, A_3 and A_4 are as required.

LEMMA 7. *Let D be a countable subfield of $\mathbb{R}$. Then every D-convex function F on D with values in $\mathbb{R} \cup\{+\infty\}$ can be extended in exactly one way to a convex function which is continuous on $\mathbb{R}$ except at most two points of $\mathbb{R}$. Second, there exists a lower semi-continuous convex function on $\mathbb{R}$ that coincides with F on D except at most two points of D.*

We omit the proof.

LEMMA 8. *Let F be a convex operator from $\mathbb{R}$ to $P(\Omega)$ such that $F(\alpha_0) \in S(\Omega)$ for some $\alpha_0 \in \mathbb{R}$. First, there exists a measurable function $f(t) \in P(\Omega)$ such that if $f(t)<\alpha$ for all t in a set A, of positive measurable in Ω, then $F(\alpha)$ is $+\infty$ on A. Second, there exists a measurable function $g(t) \in Q(\Omega)$ such that $\alpha<g(t)$ for every t in A implies $F(\alpha)$ is $+\infty$ on A.*

PROOF. We define $f_\alpha(t)$ with

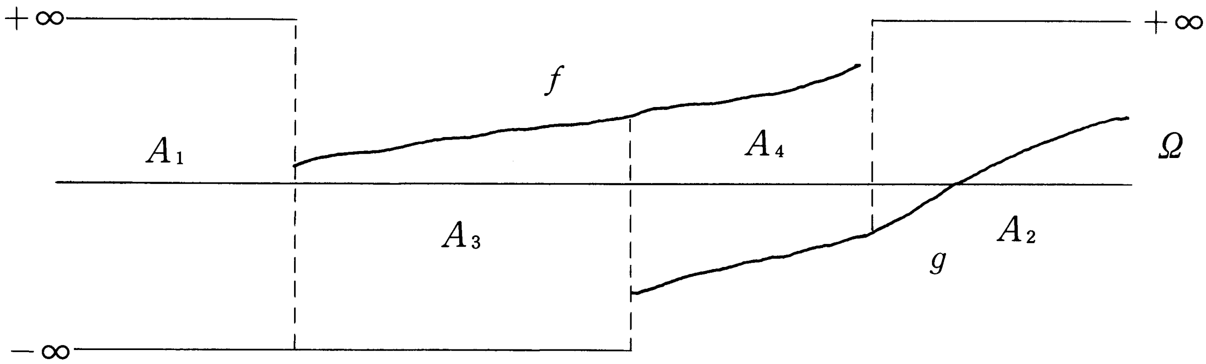
$$f_\alpha(t)=\begin{cases}\alpha & \text{if } t\in A_\alpha\\ +\infty & \text{if } t\notin A_\alpha\end{cases}$$

where $A_\alpha=\{t\,;F(\alpha)=F(\alpha,t)=+\infty\}$.
Then $f_\alpha(t)\geqq\alpha_0$ a. e. $t\in\Omega$ if $\alpha>\alpha_0$ and $f_{\alpha_0}(t)=+\infty$ for all $t\in\Omega$.

Now define

$$f=\bigwedge_{\alpha>\alpha_0} f_\alpha.$$

As is well known, there is a decreasing sequence of step functions $f_n\downarrow f\,(n=1,2,\cdots)$ where f_n is the infimum of some finite number of $f'_\alpha s$ and it is easy to see that f satisfies the conditions of the lemma. The construction of $g(t)$ is similar. The following figure shows how f and g behave.



Remark concerning Lemma 8. It is easy to see that the set of numbers $\alpha\in\mathbb{R}$ in which $\mu\{t\,;f(t)=\alpha\}>0$ is countable, as is the set of numbers $\alpha\in\mathbb{R}$ for which $\mu\{t\,;g(t)=\alpha\}>0$.

Now, we have the following lemma which is a generalization of Lemma 5.

LEMMA 9. *Let F be a convex operator from $\mathbb{R}$ into $P(\Omega)$ such that there exists α_0 with $F(\alpha_0)\in S(\Omega)$. Then, there exist a subset A of Ω of measure zero and a function $F(\alpha,t)$ defined on $\mathbb{R}\times\Omega$ such that for each fixed $t\in\Omega\backslash A$, $\mathbb{R}\ni\alpha\to F(\alpha,t)$ is a convex function on $\mathbb{R}$ and for each fixed $\alpha\in\mathbb{R}$, $\Omega\ni t\to F(\alpha,t)$ is a measurable function on Ω which is identified with $F(\alpha)$ as an element of $P(\Omega)$.*

PROOF. For simplicity, we shall consider the case in which $\Omega=A_3$ (notation is in Lemma 8). The other cases are proved in the same way. By the Remark concerning Lemma 8, we find the countable set $\{\alpha_n\}\subset\mathbb{R}$ for which $\mu\{t\,;f(t)=\alpha_n\}>0$. Let D be a countable subfield of $\mathbb{R}$ that contains all $\alpha_n(n=1,2,\cdots)$. We first determine a countable number of measurable functions $F(\alpha,t)$ $(\alpha\in D)$ such that $\{F(\alpha,t)\}=F(\alpha)$ for $\alpha\in$D. Then, by the method used in Lemma 5, there exists a subset A_0 of measure zero in Ω such that $F(\alpha,t)$ is a D-convex function of $\alpha\in D$ for all $t\in\Omega\backslash A_0$ and $\alpha>$

$f(t)$ implies that $F(\alpha, t)=+\infty$ for $t\in\Omega\backslash A_0$.

If $\alpha\neq\beta_n(n=1, 2, \cdots)$, then it is easy to see that $F(\alpha)=\lim_{n\to\infty} F(\beta_n, t)$ a. a. $t\in\Omega$ for each sequence $\beta_n\to\alpha$. We extend $F(\alpha, t)$ $(\alpha\in D)$ to the whole field $\mathbb{R}$ by Lemma 7 for $t\in\Omega\backslash A_0$. We denote this extension by $F(\alpha, t)$ for all $\alpha\in\mathbb{R}$. This function is equal to $F(\alpha)$ as an element of $P(\Omega)$, and the function $F(\alpha, t)$ is as required.

3. Duality theorems

Let $L(\mathbb{R}, S(\Omega))=L(\mathbb{R};\Omega)$ be the totality of all linear operator from $\mathbb{R}$ to $S(\Omega)$. The following is a special case of Lemma 5.

LEMMA 10. *Let* $T\in L(\mathbb{R};\Omega)$. *Then there exists a measurable function* $T(t)$ $(t\in\Omega)$ *on* Ω *such that* $T(\alpha)=\{\alpha T(t)\}$.

Let F *be a convex operator from* $\mathbb{R}$ *to* $P(\Omega)$, *and suppose that there is at least a number* $\alpha_0\in\mathbb{R}$ *such that* $F(\alpha_0)\in S(\Omega)$. *For* $\alpha\in R$ *with* $F(\alpha)\in S(\Omega)$, *we can define the subdifferential* $\partial F(\alpha)$ *of* F *at* α *as follows*:

$\partial F(\alpha)=\{T\in L(R;\Omega); F(\alpha)-T(\alpha)\leq F(\beta)-T(\beta)$ *for all* $\beta\in\mathbb{R}\}$, *where* $L(\mathbb{R};\Omega)=L(\mathbb{R}; S(\Omega))\cong S(\Omega)$ (*Lemma* 10).

THEOREM 1. *Let* $F(\alpha)=\{F(\alpha, t)\}$ *be a representation as in Lemma* 9 *such that for each* $t\in\Omega$ *except a subset of measure zero,* $\alpha\to F(\alpha, t)$ *is a convex function on* $\mathbb{R}$ *and for each* $\alpha\in\mathbb{R}$, $t\to F(\alpha, t)$ *is a measurable function on* Ω *which is identified with* $F(\alpha)$ *as an element of* $P(\Omega)$. *Then* $\partial F(\alpha)\neq\phi$ *iff* $\partial F(\cdot, t)(\alpha)\neq\phi$ *for a. a.* $t\in\Omega$, *where*

$\partial F(\cdot, t)(\alpha)=\{\xi\in\mathbb{R}; F(\alpha, t)-\xi\alpha\leqq F(\beta, t)-\xi\beta$ *for all* $\beta\in\mathbb{R}\}$

PROOF. We define a sequence of measurable functions $\{\phi_n\}\subset P(\Omega)$ as follows:

$$\phi_n(t)=\begin{cases} n\left\{F(\alpha+\frac{1}{n}, t)-F(\alpha, t)\right\} & \text{if } f(t)>\alpha \\ n\left\{F(\alpha, t)-F(\alpha-\frac{1}{n}, t)\right\} & \text{if } f(t)=\alpha, g(t)<\alpha \\ 0 & \text{if } f(t)=g(t)=\alpha \end{cases}$$

where f and g are as in Lemma 8.
If $\partial F(\cdot, t)(a)\neq\phi$, then limit

$$\phi(t)=\lim_{n\to\infty}\phi_n(t)$$

exists as an element of $S(\Omega)$, and $\phi(t)\in\partial F(\cdot, t)(\alpha)$ for a. a. $t\in\Omega$. Hence the operator $T\in L(\mathbb{R};\Omega)$ defined by the formula

$$Tx(t)=T(x)(t)=x\cdot\phi(t)$$

is in $\partial F(\alpha)$.

If $\partial F(\alpha)\neq\phi$, plainly we have $\partial F(\cdot, t)(\alpha)\neq 0$ for a. a. $t\in\Omega$.

LEMMA 11. *Let* F *be a convex operator from* $\mathbb{R}$ *to* $P(\Omega)$ *such that* $F(\alpha_0)$

$\in S(\Omega)$ for some $\alpha_0 \in \mathbb{R}$. By Lemma 9, we can find a representation of F as functions $F(\alpha, t)$ with $F(\alpha)=\{F(\alpha, t)\}$ such that $F(\alpha, t)$ is a convex function of $\alpha \in \mathbb{R}$ for a. a. $t \in \Omega$. The set $\{\alpha_n\}$ such that

$\mu\{t \in \Omega\,;\, F(\cdot, t)$ *is discontinuous at* $\alpha_n\}>0$

is countable.

This lemma follows easily from Lemma 8.

LEMMA 12. *Let F be as in Lemma 11. For $T(t) \in S(\Omega)$, the composite function $F(T(t), t)$ of $t \in \Omega$ is an element of $P(\Omega)$.*

PROOF. Suppose first that $F(\xi, t)$ be a countinuous function of $\xi \in \mathbb{R}$ for a. e. $t \in \Omega$, i. e. $F(\alpha) \in S(\Omega)$ for all $\alpha \in \mathbb{R}$. For $T(t) \in S(\Omega)$, there exists a sequence of simple functions $T_n(t)$ with

$$\lim_{n\to\infty} T_n(t)=T(t) \text{ a. a. } t \in \Omega$$

and

$$T_n(t)=\sum_{m=1}^{k(n)} \alpha_m^n \chi_{\Omega m,n}(t)$$

where $\{\Omega_{m,n}\}$ is a partition of Ω and for each n,

$$F(T_n(t), t)=\sum_{m=1}^{k(n)} \chi_{\Omega m,n}(t) \cdot F(\alpha_m^n, t)$$

is measurable. By the continuity of F, we have

$$\lim_{n\to\infty} F(T_n(t), t)=F(T(t), t)$$

and so $F(T(t), t)$ is a measurable function of $t \in \Omega$.

For the general case, let $\{\alpha_n\}$ be as in Lemma 11.

Let $A_n=\{t \in \Omega\,;\, T(t)=\alpha_n\}$, $\Omega_1=\bigcup_{n=1}^{\infty} A_n$ and $\Omega_2=\Omega \backslash \Omega_1$.

Then we have

$$=\sum_1^{\infty} \chi_{A_n}(t) \cdot F(\alpha_n, t)+\chi_{\Omega_2}(t) \lim_n F(T_n(t), t) \quad a.a. \ t \in \Omega.$$

The Lemma follows.

Let us consider the conjugate operator $F*$ of F. By Lemma 9, there is a family of convex functions $F(\alpha, t)$ with $F(\alpha)=\{F(\alpha, t)\}$. Let T be an element of $L(\mathbb{R}; S(\Omega))$. By Lemma 10, T can be regarded as an element of $S(\Omega)$; and will be denoted by $T(t)$.

For $T \in L(\mathbb{R}; S(\Omega))$, we shall define

$$F^*(T)=\bigvee_{\xi \in \mathbb{R}}(\xi \cdot T(\cdot)-F(\xi)).$$

Since there exists a dense countable set D of $\mathbb{R}$ which contains the set $\{a_n\}$ of Lemma 11, and $F^*(T)=\sup_{\xi \in D}\{\xi T(t)-F(x, t)\}$, hence we have

$$\begin{aligned} F^*(T) &= \sup_{\xi \in D}\{\xi \cdot T(t)-F(\xi, t)\} \text{(a. e.)} \\ &= \sup_{\xi \in \mathbb{R}}\{x T(t)-T(x, t)\} \\ &= F^*(T(t), t) \end{aligned}$$

where $F^*(\cdot, t)$ is the conjugate function of $F(x, t)$ with $F(x)=\{F(x, t)\}$ as in Lemma 9. We note that $\bigvee_{\zeta\in\mathbb{R}}(\xi T(\cdot)-F(x))=\sup_{\xi\in D}\{\xi T(t)-F(\xi, t)\}$ a. e. whenever $\{\alpha\in\mathbb{R}: \mu\{t\in\Omega; F(\cdot, t)$ is discontinuous at $\alpha\}>0\}$ is countable. For $\zeta\in\mathbb{R}$, considering ζ as a constant function,

$$F^*(\zeta)=\bigvee_{\xi\in\mathbb{R}}(\zeta\xi-F(\xi)).$$

Although it may happen that $F^*(\zeta)=+\infty$ a. a. $t\in\Omega$ for $\zeta\in\mathbb{R}$, we know that there exists $T_0\in S(\Omega)$ with $F^*(T_0)\in S(\Omega)$. For every $T_0\in\partial F(\alpha_0)$, $F^*(T_0)$ belongs to $S(\Omega)$. We now define F^{**}. The function

F^{**} carries $L(L(\mathbb{R}, S(\Omega)), S(\Omega))\cong L(S(\Omega), S(\Omega))$ into $P(\Omega)$. We consider F^{**} only on $\mathbb{R}$, and define

$$F^{**}(\zeta)=\bigvee_{T\in S(\Omega)}(\zeta\cdot T-F^*(T))$$

for $\zeta\in\mathbb{R}$, since $L(S(\Omega), S(\Omega))$ contains $\mathbb{R}$, considering every element $\zeta\in\mathbb{R}$ as follows: $S(\Omega)\ni\phi(t)\to\zeta\cdot\phi(t)\in S(\Omega)$. Thus, since $\bigvee_{\zeta\in R}(\zeta T(\cdot)-F(\zeta))=\sup_{\zeta\in D}\{\zeta T(t)-F(\zeta, t)\}$ a. e., we have

$$\begin{aligned} F^{**}(\zeta) &\geqq \bigvee_{\xi\in\mathbb{R}}(\zeta\cdot\xi\cdot 1-F^*(\xi,\cdot)) \\ &=\sup_{\xi\in\mathbb{R}}(\zeta\cdot\xi-F^*(\xi, t)) \\ &=F^{**}(\zeta, t) \text{ for a. a. } t\in\Omega, \end{aligned}$$

where $F^{**}(\cdot, t)$ is the conjugate function of the convex function $F^*(\zeta, t)$ for a. a. $t\in\Omega$.

On the other hand, we have

$$\begin{aligned} F^{**}(\zeta) &\leq \bigvee_{T\in S(\Omega)}(\zeta\cdot T(t)-F^*(T(t), t)) \\ &=\sup_{\xi\in\mathbb{R}}(\zeta\cdot\xi-F^*(\xi, t)) \\ &=F^{**}(\zeta, t) \end{aligned}$$

for a. a. $t\in\Omega$.

Hence, we have $F^{**}(\zeta)=F^{**}(\zeta, t)$ for a. a. $t\in\Omega$. It is easy to see that

$$F^{**}(\zeta)\leqq F(\zeta) \text{ for } \zeta\in\mathbb{R}.$$

Similarly, we can define $F^{**}(S)$ for $S\in S(\Omega)$ by

$$F^{**}(S)=\bigvee_{T\in S(\Omega)}(S\cdot T-F^*(T))$$

We now prove the following theorem, which generalizes the Fenchel-Moreau theorem:

THEOREM 2. *The equality $F^{**}(\zeta)=F(\zeta)$ hold iff the family of convex functions $F(\cdot, t)$ of Lemma 9 is lower semi-continuous at ζ for a. e. $t\in\Omega$.*

This theorem follows from the following and the original Fenchel-Moreau theorem. We shall also give a generalization of the Fenchel-Moreau theorem for $T\in S(\Omega)$. We need the following. Let F be a convex operator from $\mathbb{R}$ to $P(\Omega)$ such that $F(\alpha_0)\in S(\Omega)$ for some $\alpha_0\in\mathbb{R}$. By Lemma 9, there exists a family of convex functions $F(\alpha, t)$ for each $t \in \Omega$ with $F(\alpha) = \{F(\alpha, t)\}$. For such F, by Lemma 11 the set $\{\alpha_n\} \subset \mathbb{R}$ with

$$\mu\{t \in \Omega ; F(x, t) \text{ is discontinuous at } x = \alpha_n\} > 0$$

is countable. Hence, there exists a family of convex functions $\tilde{F}$ for each $t \in \Omega$ with

$$\tilde{F}(\alpha, t) = F(\alpha, t) \text{ (a. e.)}$$

such that $\tilde{F}(\alpha, t)$ is lower semi-continuous for $\alpha \notin \{a_n\}$ for almost all $t \in \Omega$. Such $\{\tilde{F}(\alpha, t)\}$ is uniquely determined a. e. in Ω. We call $\{\tilde{F}(\alpha, t)\}$ the *standard representation of* $F(\alpha)$. Then, we have the following Fenchel-Moreau theorem for $T \in S(\Omega)$.

THEOREM 3. *Let F be a convex operator from $\mathbb{R}$ to $P(\Omega)$ such that $F(\alpha_0) \in S(\Omega)$ for some $\alpha_0 \in \mathbb{R}$ For*

$$T(t) \in S(\Omega), \text{ we have } F^{**}(T) = F(T)$$

iff the standard representation $\{\tilde{F}(\alpha, t)\}$ of $F(\alpha)$ is lower semi-continuous at $T(t)$ for a. a. $t \in \Omega$.

Let G be an operator from $\mathbb{R}$ to $Q(\Omega)$. If $-G(a)$ is a convex operator from $\mathbb{R}$ to $P(\Omega)$, we call G a *concave* operator. We define the conjugate operator G^* of G as follows:

$$G^*(T) = -(-G(-T))^* = -\bigvee_{\xi \in \mathbb{R}} (\xi \cdot T + G(-\xi))$$
$$= \bigvee_{T \in S(\Omega)} (\xi T - G(\xi)).$$

We next consider the following programs $P(I)$ and $P(II)$.

$$P(I): \bigwedge_{\zeta \in \mathbb{R}} \{F(T(t), t) - G(T(t), t)\}$$

where $F(x, t)$ and $G(x, t)$ are the standard representations and satisfy $F^{**} = F$, $G^{**} = G$.

$$P(II): \bigvee_{T \in S(\Omega)} \{G^*(T(t), t) - F^*(T(t), t)\}.$$

THEOREM 4. *Suppose that F is a convex operator. There exists a solution T_1 in $P(I)$ with $T_1 \in S(\Omega)$ if and only if there exists a solution T_0 in $P(II)$ with $T_0 \in S(\Omega)$. In this case, we have*

$$F(T_1(t), t) - G(T_1(t), t) = G^*(T_0(t), t) - F^*(T_0(t), t)$$

for a. a. $t \in \Omega$.

PROOF. Suppose that we have $T_0 \in S(\Omega)$ with

$$\bigvee_{T \in S(\Omega)} \{G^*(T(t), t) - F^*(T(t), t)\} = G^*(T_0(t), t) - F^*(T_0(t), t) \in S(\Omega).$$

The theorem of Moreau-Rockafellar, shows that

$$\partial(f+g)(x) = \partial f(x) + \partial g(x),$$

and $\partial(G^*(\cdot, t) - F^*(\cdot, t))\ (T_0(t)) \ni 0$. Hence, there exists $\xi(t) \in \mathbb{R}$ with

$$\xi(t) \in \partial F^*(\cdot, t)(T_0(t)) \cap \partial G^*(\cdot, t)(T_0(t)) \text{ for a. a. } t \in \Omega.$$

We can choose $\xi(t)$ such that $\xi(t)$ is an element of $S(\Omega)$, considering $\xi(t)$ as a function defined on Ω. For each $t \in \Omega$, there exists $\xi \in \mathbb{R}$ with

$$\xi(\alpha - T_0(t)) \leq F^*(\alpha, t) - F^*(T_0(t), t)$$

and

$$\xi(\alpha' - T_0(t)) \geq G^*(\alpha', t) - G^*(T_0(t), t)$$

for all α and $\alpha' \in \mathbb{R}$.

Putting

$$f(\alpha, t) = \frac{F^*(\alpha, t) - F^*(T_0(t), t)}{\alpha - T_0(t)}$$

$$g(\alpha, t) = \frac{G^*(\alpha, t) - G^*(T_0(t), t)}{\alpha - T_0(t)}$$

we see easily that $f(\alpha, t)$ is increasing and $g(\alpha, t)$ is decreasing with respect to α.

Hence, putting

$$\xi(t) = \lim_{\alpha - T_0(t) \to -0} f(\alpha, t) \vee \lim_{\alpha - T_0(t) \to +0} g(\alpha, t)$$

we get a solution $\xi(t)$ of $P(I)$ that is in $S(\Omega)$. The rest of the proof is similar. Thus we complete the proof.

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