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Automorphism groups of Σ_{n+1}-invariant trilinear forms

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1. Introduction

Let Σ_{n+1} be the symmetric group on the set $\{0, 1, \cdots, n\}$ of cardinality $n+1$, $n\geqq 2$. Let $V=\langle e_1, \cdots, e_n\rangle$ be a natural n-dimensional irreducible Σ_{n+1}-module over the complex number field $\mathbf{C}$. (That is, $\{e_1, \cdots, e_n\}$ is a basis of V such that if we let $e_0=-(e_1+\cdots+e_n)$, then Σ_{n+1} acts on $\{e_0, e_1, \cdots, e_n\}$ in the standard way.) We regard Σ_{n+1} as a subgroup of $GL(V)$. We define a Σ_{n+1}-invariant symmetric trilinear form θ_n on V by

$\theta_n(e_j, e_j, e_j)=n(n-1)$, $1\leqq j\leqq n$;

$\theta_n(e_j, e_j, e_k)=-(n-1)$, $1\leqq j, k\leqq n$, $j\neq k$;

$\theta_n(e_j, e_k, e_h)=2$, $1\leqq j, k, h\leqq n$, $j\neq k\neq h\neq j$.

Now we can state our main results.

THEOREM 1. *Let* Σ_{n+1}, V, θ_n *be as above. Let* θ *be an arbitrary nonzero* Σ_{n+1}-*invariant symmetric trilinear form on* V. *Then*

$\theta=\alpha\theta_n$, $0\neq\alpha\in\mathbf{C}$

and so $Aut\theta=Aut\theta_n$, *where we define the automorphism group of* θ *to be*

$Aut\theta=\{\sigma\in GL(V):\theta(x^{\sigma}, y^{\sigma}, z^{\sigma})=\theta(x, y, z)$ *for all* $x, y, z\in V\}$.

THEOREM 2. *If* $n=2$ *or* $n\geqq 4$,

$Aut\theta_n=\langle\omega I\rangle\times\Sigma_{n+1}$,

where I *is the identity element of* $GL(V)$ *and* $\omega=(-1+\sqrt{3}i)/2$.

REMARK. The structure of $Aut\theta_3$ is described in LEMMA 2. 3.

If n is odd, our proof of THEOREM 2 is essentially an elementary analysis of the action of $Aut\theta_n$ on the set of "singular" elements of V. If n is even, we first prove that there is no singular element, which implies that $Aut\theta_n$ is finite by [6, THEOREM B]. We then apply a deep result of H. Bender [3] to complete the proof.

Symmetric bilinear and trilinear mappings

$V\times V\longrightarrow V$, $V\times V\times V\longrightarrow V$,

which are Σ_{n+1}-invariant are studied by K. HARADA [5] and by the second author [7], respectively. Our result here is analogous to that of the bilinear mapping case. This is natural, because

$V\times V\times V\longrightarrow\mathbf{C}$

can be viewed as

$V \times V \longrightarrow V^*$.

Symmetric multilinear mappings

$V \times V \times V \times V \longrightarrow V$

of degree 4, which are invariant under the standard actions of the Mathieu groups M_{11} and M_{23} with $\dim V = 10$ and 22 respectively will be studied in a subsequent paper as an application of Theorem 2. Moreover Σ_{n+1}-invariant multilinear mappings of degree 4 will also be studied in it.

For other examples of interesting trilinear forms, the reader is referred to A. Adier [1, 2], D. Frohardt [4], etc.

We conclude this seotion with the proof of Theorem 1.

Proof of Theorem 1. Let

$\beta = \theta(e_j, e_j, e_j)$,

$\gamma = \theta(e_j, e_j, e_k)$, $j \neq k$,

$\delta = \theta(e_j, e_k, e_h)$, $j \neq k \neq h \neq j$.

Since θ is Σ_{n+1}-invariant, those numbers do not depend on the choice of j, k and h. Since

$$\gamma = \theta(e_0, e_1, e_1) = \theta(-\sum_{j=1}^{n} e_j, e_1, e_1) = -\beta - (n-1)\gamma,$$

we have $\beta = -n\gamma$. Similarly, we get $(n-1)\delta = -2\gamma$ by calculating $\theta(e_0, e_1, e_2)$.

2. Proof of Theorem 2; n=odd.

Let Σ_{n+1}, V, θ_n be as in Section 1. Furthermore we use the following notation throughout the rest of this paper.

Notation 2. 1. For $X \subseteq \{0, 1, \cdots, n\}$, we let

$\Sigma_X = \{\tau \in \Sigma_{n+1} : j^\tau = j \text{ for all } j \in \{0, 1, \cdots, n\} - X\}$.

Thus $\Sigma_X \simeq \Sigma_{|X|}$.

We call a nonzero element x of V singular if $\theta_n(x, x, v) = 0$ for all $v \in V$. Now we prove a lemma which partly explains why we distinguish two cases: the cases n is odd and n is even.

Lemma 2. 2.

(i) *If n is even, there is no singular element.*

(ii) *If n is odd, the set of singular elements of V is given by*

$$\{\alpha \sum_{j \in X} e_j : X \subseteq \{1, \cdots, n\},\ |X| = \frac{n+1}{2},\ 0 \neq \alpha \in \mathbf{C}\}.$$

Proof. An element of the form described in (ii) is clearly singular. Conversely, let

$x = \zeta_1 e_1 + \cdots\cdots + \zeta_n e_n$

be a singular element. Since e_0 is not singular, x cannot be of the form ζe_0.

Therefore the ζ_j are not all equal. We may assume $\zeta_1 \neq \zeta_2$. From $\theta_n(x, x, e_j)=0$, we get

$(n+1)^2\zeta_j^2-2(n+1)\beta\zeta_j-(n+1)\gamma+2\beta^2=0$, $1 \leqq j \leqq n$, where $\beta=\zeta_1+\cdots+\zeta_n$ and $\gamma=\zeta_1^2+\cdots+\zeta_n^2$. Thus each ζ_j may be regarded as a solution to the quadratic equation (1). Since $\zeta_1 \neq \zeta_2$, each ζ_j is equal to ζ_1 or ζ_2. For each $k=1, 2$, let a_k be the number of the indices j for which $\zeta_j=\zeta_k$. Then subtracting (1) for $j=2$ from (1) for $j=1$. we get

$$(n+1)(\zeta_1+\zeta_2)=2(a_1\zeta_1^2+a_2\zeta_2).$$

Substituting this in (1) yields

$$(n+1)(\zeta_1^2+\zeta_2^2)=2(a_1\zeta_1^2+a_2\zeta_2^2).$$

Now a straightforward calculation shows that either

$$\zeta_1=0 \text{ and } a_2=(n+1)/2 \text{ or } \zeta_2=0 \text{ and } a_1=(n+1)/2.$$

We first settle the case $n=3$.

LEMMA 2. 3. *$Aut\theta_3$ is given by the semidirect product of $E=\langle \tau \in GL(V): f_j^\tau=\alpha_j f_j,\ j=1, 2, 3;\ \alpha_1\alpha_2\alpha_3=1\rangle$ by $\sum_{\{1,2,3\}}$ where $f_1=e_2+e_3,\ f_2=e_1+e_3,\ f_3=e_1+e_2$.*

PROOF. Since $\theta_3(f_1, f_2, f_3) \neq 0$, this follows immediately from LEMMA 2. 2. (ii).

REMARK. If we define a subgroup F of the above E by $\sum_{\{1,2,3\}}$.

$$F=\langle \tau \in E: f_j^\tau=\pm f_j,\ j=1, 2, 3\rangle \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_2.$$

then our original Σ_4 can be described as the semidirect product of F by $\sum_{\{1,2,3\}}$.

In the remainder of this section, we assume $n=2m-1$ is odd, $m \geqq 3$, and use the following notation.

NOTATION 2. 4.

(i) Let $\mathscr{S}$ denote the set of subsets of $\{1, \cdots, n\}$ of cardinality m.

(ii) If $(\Sigma_{j\in X}e_j)^\tau=\alpha(\Sigma_{j\in Y}e_j)$ for X and $\tau \in Aut\theta_n$, we write

$$Y=X^{(\tau)} \text{ and } \alpha=\lambda(X, \tau).$$

Note that if $\tau \in \Sigma_{n+1}-\sum_{\{1,\dots,n\}}$, then $X^{(\tau)}$ is not the same as the usual $X^\tau=\{j^\tau : j \in X\}$.

(iii) For $\mathscr{H}$ and $\tau \in Aut\theta_n$, let

$$\mathscr{H}^{(\tau)}=\{X^{(\tau)}: X \in \mathscr{H}\}.$$

(iv) Let $M=\{1, \cdots, m\}$, $N=\{m, m+1, \cdots, n\}$.

(v) Let $\mathscr{O}=\{X \in \mathscr{S}: |X \cap M|=m-1.\}$

(vi) For each $1 \leqq j \leqq m$, let

$$\mathscr{O}^j=\{X \in \mathscr{O}: \{j\}=M-X\}.$$

For each $m+1 \leqq j \leqq n$, let

$$\mathscr{O}_j=\{X \in \mathscr{O}: \{j\}=X-M\}.$$

We begin with the following lemma.

LEMMA 2. 5. *Let* $X, Y\in\mathscr{P}$ *with* $X\neq Y$ *and* $\alpha\neq 0$.

(i) *If* $|X\cap Y|\neq 1$, *then there exists a singular element* x *such that*

$$x\notin<\sum_{j\in X}e_j, \sum_{j\in Y}e_j>$$

and such that

$$(\alpha\sum_{j\in X}e_j)-(\alpha\sum_{j\in Y}e_j)+x$$

is also singular.

(ii) *If* $|X\cap Y|=1$, *there is no such* x.

PROOF. If $|X\cap Y|\neq 1$, we can choose $A\in\mathscr{P}$ so that $|A\cap Y|=m-1$, $A\not\subseteq X\cup Y$ and $A\not\supseteq X\cap Y$. Thus if we let $x=\alpha\Sigma_{j\in A}e_j$, this x has the required properties. Now assume $|X\cap Y|=1$, and let $x=\beta\Sigma_{j\in B}e_j$ be a singular element for which

$$(\alpha\sum_{j\in X}e_j)-(\alpha\sum_{j\in Y}e_j)+x$$

is also of the form $\gamma\Sigma_{j\in C}e_j$, $C\in\mathscr{P}$, $\gamma\neq 0$. Since $B\not\supseteq X\cup Y$, γ must be equal to α or $-\alpha$. Hence x is forced to be equal to $\alpha\Sigma_{j\in X}e_j$ or $-\alpha\Sigma_{j\in X}e_j$. Thus (ii) is proved.

A similar argument yields the following two lemmas.

LEMMA 2. 6. *Let* $X, Y\in\mathscr{P}$ *with* $X\neq Y$ *and* $\alpha\neq 0$.

(i) *If* $|X\cap Y|\neq m-1$, *then there exists a singular element* x *such that*

$$x\notin<\sum_{j\in X}e_j, \sum_{j\in Y}e_j>$$

and such that

$$(\alpha\sum_{j\in X}e_j)-(\alpha\sum_{j\in Y}e_j)+x$$

is also singular.

(ii) *If* $|X\cap Y|=m-1$, *there is no such* x.

LEMMA 2. 7. *Let* $X, Y\in\mathscr{P}$ *with* $X\neq Y$ *and* $0\neq\alpha\neq\pm\beta\neq 0$. *Then there is no singular element* x *such that*

$$x\notin<\sum_{j\in X}e_j, \sum_{j\in Y}e_j>$$

and such that

$$(\alpha\sum_{j\in X}e_j)-(\alpha\sum_{j\in Y}e_j)+x$$

is also singular.

Combining LEMMAS 2. 5, 2. 6 and 2. 7, we get:

LEMMA 2. 8. *Let* $X, Y\in\mathscr{P}$ *with* $|X\cap Y|=m-1$ *and let* $\tau\in Aut\theta_n$. *Then one of the following holds*:

(i) $|X^{(\tau)}\cap Y^{(\tau)}|=m=1$ *and* $\lambda(X, \tau)=\lambda(Y, \tau)$; *or*

(ii) $|X^{(\tau)}\cap Y^{(\tau)}|=1$ *and* $\lambda(X, \tau)=-\lambda(Y, \tau)$.

COROLLARY 2. 9. *Let* X, Y. $Z\in\mathscr{P}$ *with* $|X\cap Y|=|X\cap Z|=|Y\cap Z|=m-1$ *and let* $\tau\in Aut\theta_n$. *If* $|X^{(\tau)}\cap Y^{(\tau)}|=1$, *then either* $|X^{(\tau)}\cap Z^{(\tau)}|=m-1$ *and* $|Y^{(\tau)}\cap Z^{(\tau)}|=1$ *or* $|X^{(\tau)}\cap Z^{(\tau)}|=1$ *and* $|X^{(\tau)}\cap Z^{(\tau)}|=m-1$.

PROOF. The condition $|X^{(\tau)}\cap Y^{(\tau)}|=1$ implies $\lambda(X, \tau)=-\lambda(Y, \tau)$, and so $\lambda(Z, \tau)$ is equal to one of $\lambda(X, \tau)$ or $\lambda(Y, \tau)$.

Now let τ be an arbitrary element of $Aut\theta_n$. We want to show $\tau\in H=\langle\omega I\rangle\times\Sigma_{n+1}$. For this purpose, it suffices to show $H\tau H\cap H\neq\phi$.

LEMMA 2. 10. *There exist* σ, $\sigma'\in\Sigma_{n+1}$ *such that* $M^{(\sigma\tau\sigma')}=M$ *and* $\mathscr{Q}^{(\sigma\tau\sigma')}=\mathscr{Q}$

PROOF. If $|A^{(\tau)}\cap B^{(\tau)}|=m-1$ for all A, $B\in\mathscr{P}$ with $|A\cap B|=m-1$, we simply let $\sigma=\sigma'=I$. Thus assume there exist A, $B\in\mathscr{P}$ such that $|A\cap B|=m-1$ and $|A^{(\tau)}\cap B^{(\tau)}|=1$. Choose $C\in\mathscr{P}$ so that $A\cap C=B\cap C=A\cap B$. By COROLLARY 2. 9, $|A^{(\tau)}\cap C^{(\tau)}|=1$ or $|B^{(\tau)}\cap C^{(\tau)}|=1$. We may assume $|A^{(\tau)}\cap C^{(\tau)}|=1$. Now let X be an arbitrary element of $\mathscr{P}$ such that $A\cap X=A\cap B$. We want to show $|A^{(\tau)}\cap X^{(\tau)}|=1$. Suppose $|A^{(\tau)}\cap X^{(\tau)}|=m-1$. Then $|B^{(\tau)}\cap X^{(\tau)}|=|C^{(\tau)}\cap X^{(\tau)}|=1$ by COROLLARY 2. 9. But the element of $X^{(\tau)}-A^{(\tau)}$ is contained in both $B^{(\tau)}$ and $C^{(\tau)}$, and $A^{(\tau)}\cap X^{(\tau)}$ contains at least one of $A^{(\tau)}\cap B^{(\tau)}$ or $A^{(\tau)}\cap C^{(\tau)}$. This is absurd. Thus $|A^{(\tau)}\cap X^{(\tau)}|=1$. Now let k be the unique element of $A^{(\tau)}$ that is not contained in any of $X^{(\tau)}$ with $A\cap X=A\cap B$. Choose $\sigma\in\Sigma_{\{1,\cdots,n\}}$ so that $M^{\sigma}=A$ and $\{1, \cdots, m-1\}^{\sigma}=A\cap B$. Choose $\sigma''\in\Sigma_{\{1,\cdots,n\}}$ so that $(A^{(\tau)})^{\sigma''}=N$ and $k^{\sigma''}=m$. Let $\tau'=\sigma\tau\sigma''$. Then $M^{(\tau')}=N$, and $N-\bigcup_{X\in\mathscr{Q}^m}X^{(\tau')}=\{m\}$.

We separate the next point of the proof as a sublemma.

SUBLEMMA 2. 11. *If* $D\in\mathscr{Q}-\mathscr{Q}^m$ *and* $|N\cap D^{(\tau')}|=m-1$, *then* $m\in D^{(\tau')}$.

PROOF. Suppose $m\notin D^{(\tau')}$. Then $N\cap D^{(\tau')}=\{m+1, m+2, \cdots, n\}$. Choose $X\in\mathscr{Q}^m$ so that $|X\cap D|=m-1$. By Corollary 3. 9, $|X^{(\tau')}\cap D^{(\tau')}|=1$. But the element of $D^{(\tau')}-N$ is contained in $X^{(\tau')}$, and the element of $X^{(\tau')}\cap N$ is contained in $D^{(\tau')}$. This is a contradiction.

We now return to the proof of the lemma. We want to show that $|N\cap Y^{(\tau')}|=m-1$ for all $Y\in\mathscr{Q}-\mathscr{Q}^m$. By way of contradiction, suppose there exists $Y\in\mathscr{Q}-\mathscr{Q}^m$ such that $|N\cap Y^{(\tau')}|=1$. Choose $D\in\mathscr{Q}-\mathscr{Q}^m$ so that $M\cap Y=M\cap D=Y\cap D$. Since $\{Z\in\mathscr{P}: |N\cap Z|=1\}=\{X^{(\tau')}: X\in\mathscr{Q}^m\}\cup\{M\}$, $Y^{(\tau')}$ is forced to coincide with M and $|N\cap D^{(\tau')}|$ cannot be equal to 1.

Therefore $|N\cap D^{(\tau')}|=m-1$, and so $m\in D^{(\tau')}$ by the above sublemma. Also $|Y^{(\tau')}\cap D^{(\tau')}|=1$ by COROLLARY 2. 9. But both m and the element of $D^{(\tau')}-N$ is contained in $Y^{(\tau')}\cap D^{(\tau')}$, which is absurd. Thus it is shown that $|Y^{(\tau')}\cap N|=m-1$ and $m\in Y^{(\tau')}$ for all $Y\in\mathcal{Q}-\mathcal{Q}^m$ and that $|X^{(\tau')}\cap N|=1$ and $m\notin X^{(\tau')}$ for all $X\in\mathcal{Q}^m$. Hence if we let $\sigma'=\sigma''(0m)$, where $(0m)$ denotes the transposition of Σ_{n+1} that permutes 0 and m, then the conditions of the lemma are satisfied.

Now let $\tau'=\sigma\tau\sigma'$ with σ and σ' as in the lemma. Let Π be the set of those subsets of $\mathcal{Q}$ the intersection of any two distinct elements of which has cardinality $m-1$, and Π^* be the set of maximal elements of Π under inclusion. Then

$$\Pi^*=\{\mathcal{Q}^i:1\leqq j\leqq m\}\cup\{\mathcal{Q}_j: m+1\leqq j\leqq n\}.$$

On the other hand, $\lambda(X,\ \tau')=\lambda(M,\ \tau')$ for all X by Lemma 2. 8, whence $|X^{(\tau')}\cap Y^{(\tau')}|=m-1$ for all $X,\ Y\in\mathcal{Q}$ with $|X\cap Y|=m-1$. Therefore $\mathcal{H}^{(\tau')}\in\Pi^*$ for all $\mathcal{H}\in\Pi^*$. Hence there exist

$$\pi\in\Sigma_M \text{ and } \rho\in\Sigma_{\{m+1,m+2\cdots,n\}}$$

such that

$$(\mathcal{Q}^k)^{\tau'}=\mathcal{Q}^{k\pi} \text{ for all } 1\leqq k\leqq m$$

and

$$(\mathcal{Q}_k)^{\tau'}=\mathcal{Q}_{k\rho} \text{ for all } m+1\leqq k\leqq n.$$

Thus if we let $\tau''=\tau'(\pi\rho)^{-1}$, then

$$(\sum_{j\in X}e_j)^{\tau''}=\lambda(M,\ \tau')\sum_{j\in Y}e_X$$

for all $X\in\mathcal{Q}$ and for $X=M$. Since

$$V=\langle\sum_{j\in X}e_j : X\in\{M\}\cup\mathcal{Q}\rangle,$$

$\tau''=\lambda(M,\ \tau')I$. This also implies $\lambda(M,\ \tau')^3=1$, whence

$$\tau''=\sigma\tau\sigma'(\pi\rho)^{-1}\in H.$$

As is remarked immediately before LEMMA 2. 10, this completes the proof of THEOREM 2 for odd n.

2. Proof of Theorem 2; n=even.

Throughout this section, we assume n is even.

As is proved in LEMMA 2. 2.(i), there is no singular element. Therefore $Aut\theta_n$ is finite by [6, THEOREM B]. We prove THEOREM 2 by induction on n. We first settle the case $n=2$.

LEMMA 3. 1. $Ant\theta_2=\langle\omega I\rangle\times\Sigma_3$.

PROOF. Since

$$\{x\in V:\langle v:\theta_2(x,\ v,\ v)=0\rangle\neq V\}$$
$$=\{\alpha((1\pm\sqrt{3}i)e_1+2e_2):\alpha\neq 0\},$$

$Aut\theta_2$ is isomorphic to a semiderect product of $\mathbf{Z}_3\times\mathbf{Z}_3$ by $\mathbf{Z}_2$. This proves the lemma.

We now state for completeness a theorem due to H. BENDER [3], which is essential to our proof.

THEOREM. *Let H be a subgroup of even order of a finite group G, and let S be a Sylow 2-subgroup of H. Let $O(G)$ denote the maximal normal odd order subgroup of G. Assume that $N_G(S)\leqq H$, and $C_G(\tau)\leqq H$ for all elements τ of S of order 2. Then one of the following holds:*

(i) $G=H$;

(ii) *S is isomorphic to a cyclic group or a generalized quaternion group, and so S possesses a unique element of order 2; or*

(iii) *There exists a normal subgroup L of G containing $O(G)$ such that $|G/L|$ is odd, and $L/O(G)$ is isomorphic to one of $PSL(2, 2^m)$, $Sz(2^{2m-1})$ or $PSU(3, 2^{2m}/2^m)$, $m\geqq 2$. Furthermore $H=O(G)N_G(S)$, and so, in particular, $O(G)S$ is normal in H.*

Now let $G=Aut\theta_n$ and $H=\langle\omega I\rangle\times\Sigma_{n+1}$ with $n\geqq 4$. Assuming that Theorem 2 is proved for $n-2$, we shall show that G and H satisfy the assumptions of the above theorem.

LEMMA 3. 2. *The subgroup*

$C_G(e_0)=\{\sigma\in G: e_0{}^\sigma=e_0\}$

is contained in H.

PROOF. Let

$W=\langle x:\theta_n(e_0, e_0, x)=0\rangle=\langle e_j-e_k:1\leqq j, k\leqq n\rangle$. Since $C_G(e_0)$ stabilizes W, the restriction of θ to W is $C_G(e_0)$-invariant, and so, in particular, is "isomorphic" to θ_{n-1} by Theorem 1, for $C_G(e_0)\geqq\Sigma_{\{1,\dots\dots,n\}}$ and the action of $\Sigma_{\{1,\dots\dots,n\}}$ on W is natural. Since $C_{C_G(e_0)}(W)=C_G(V)=\langle I\rangle$, this means that $C_G(e_0)$ is isomorphic to a subgroup of $Aut\theta_{n-1}$. Also note that an element $\sigma\in G$ such that $x^\sigma=\omega x$ for all $x\in W$ cannot belong to $C_G(e_0)$. Hence if $n\geqq 6$, we conclude from the result of Section 2 that $C_G(e_0)$ is isomorphic to a subgroup of Σ_n. If $n=4$, let f_1, f_2, f_3 be elements of W which correspond to the f_j in LEMMA 2. 2. Since $\theta_4(f_j, f_j, e_0)\neq 0$, each of the α_j in the description of E in LEMMA 2. 2 must be equal to 1 or -1. Hence by the remark following LEMMA 2. 2, $C_G(e_0)$ is isomorphic to a subgroup of Σ_4 in this case as well. Thus $C_G(e_0)=C_H(e_0)\leqq H$ as desired.

LEMMA 3. 3. *$C_G((12))\leqq H$, where (12) denotes the transposition which permutes 1 and 2.*

PROOF. Let

$U=\langle x\in V: x^{(12)}=x\rangle=\langle e_1+e_2, e_0-e_j: j\geqq 3\rangle$.

Let

$$W=\langle x\in U : \theta_n(e_1-e_2,\ e_1-e_2,\ x)=0\rangle$$
$$=\langle e_0-e_j : j\geqq 3\rangle.$$

Since $\langle e_1-e_2\rangle=\langle x\in V : x^{(12)}=-x\rangle$, $C_G((12))$ stabilizes W. Hence an argument similar to the one used in LEMMA 3. 2 with the induction hypothesis in place of the result of Section 2 shows that $C_G((12))/C_{C_G((12))}(W)$ is isomorphic to a subgroup of $\mathbf{Z}_3\times\Sigma_{n+1}$. Thus it suffices to prove $C_{C_G((12))}(W)=\langle(12)\rangle$.

Let σ be an arbitrary element of $C_{C_G((12))}(W)$. Since σ stabilizes U, we can write

$$(e_1+e_2)^\sigma-(e_1+e_2)=\alpha(e_1+e_2)+\sum_{j\geqq 3}\beta_j(e_0-e_j).$$

From

$$\theta((e_1+e_2)^\sigma-(e_1+e_2),\ e_0-e_k,\ e_0-e_k)=0,$$

we get

$$\sum_{\substack{j\geqq 3\\ j\neq k}}\beta_j=\frac{4\alpha}{n+1},\ k\geqq 3.\ \cdots\cdots(2)$$

If we regard (2) as a simultaneous equation in β_j, the determinant of the coefficients is $(n-3)(-1)^{n-3}\neq 0$. Thus $\beta_3=\beta_4=\cdots=\beta_n$. Since

$$\Sigma_{j\geqq 3}(e_0-e_j)=(n-1)e_0+(e_1+e_2),$$

we have

$$(e_1+e_2)^\sigma-(e_1+e_2)=\delta\gamma(e_1+e_2)+\gamma e_0,$$

where

$$\gamma=(n-1)\beta_n,\ \delta=(1+((n+1)(n-3)/4)))/n-1.$$

Calculating in a similar manner with the roles of e_0 and e_3 exchanged, we get

$$(e_1+e_2)^\sigma-(e_1+e_2)=\delta\gamma(e_1+e_2)+\gamma e_3.$$

Therefore $\gamma=0$, whence $(e_1+e_2)^\sigma=e_1+e_2$. Since σ stabilizes $\langle e_1-e_2\rangle$, we also get $(e_1-e_2)^\sigma=\pm(e_1-e_2)$ by calculating

$$\theta_n((e_1+e_2)^\sigma,\ (e_1-e_2)^\sigma,\ (e_1-e_2)^\sigma).$$

Hence $\sigma\in\langle(12)\rangle$, proving the lemma.

LEMMA 3. 4. *If τ is an element of order 2 of H, $C_G(\tau)\leqq H$.*

PROOF. By taking a suitable conjugate in H, we may assume

$$\tau=(12)(34)\cdots(2k-1,\ 2k),k\leqq n/2.$$

Since $C_G(\tau)$ stabilizes

$$W=\langle x\in V : x^\tau=x\rangle,$$

$C_G(\tau)$ normalizes $P=C_{C_G(\tau)}(W)$. Since $e_0\in W$, $P\leqq H$ by Lemma 3. 2, and so

$$P=\langle(2j-1,\ 2j) : 1\leqq j\leqq k\rangle.$$

We observe that each of the elements of P conjugate to (12) in $GL(V)$ is of the form $(2j-1, 2j)$, and hence is conjugate to (12) in $N_H(P)$. Consequently

$$|N_G(P):C_{N_G(P)}((12))|=|N_H(P):C_{N_H(P)}((12))|.$$

Since $C_{N_G(P)}((12)) \leqq H$ by LEMMA 3. 3, this means $N_G(P)\leqq H$. Thus $C_G(\tau)\leqq N_G(P)\leqq H$ as desired.

Now let S be a Sylow 2-subgroup of H. Let k be the greatest integer satisfying $2^k\leqq n$. A routine calculation shows that $D_{k-1}(S)$, the k-th term of the derived series of S, is a cyclic subgroup of order 2 generated by an element σ conjugate to

$$(12)(34)\cdots(2^k-1, 2^k).$$

Hence $N_G(S)\leqq C_G(\sigma)\leqq H$. This together with Lemma 3. 4 shows that G and H satisfy the assumptions of Bender's theorem. The cases (ii) and (iii) of Bender's theorem are ruled out because of the structure of H. Hence $G=H$. This completes the proof of THEOREM 2.

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