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A proof of the Brauer's second main theorem and related results

Dedicated to Professor H. Nagao for his sixtieth birthday

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1. The second main theorem of R. Brauer is one of the most important results in the theory of modular representations of finite groups and there have been several proofs given to it. Among them Nagao's proof [4] seems to be most popular. Moreover he obtained as a by-product that the Green correspondence is, roughly speaking, compatible with the Brauer one. This result was strengthened later by Green [2]. In this paper we would like to show unified proofs to both the Brauer's second main theorem and the Nagao-Green theorem.

Throughout this paper, p will denote a prime number and G a finite group. We pick up and fix a complete discrete valuation ring R (of rank one) such that it has the residue class field, say k, of characteristic p and its quotient field is a splitting field for every subgroup of G. $Z(RG)$ denotes the center of the group ring RG and the image of $a\in RG$ by the natural map $RG\rightarrow kG$ is denoted by a^{*}. All modules considered here are operated from the right and assumed to be finitely generated. For an RG-module M and a subgroup H of G, M^{H} denotes the set of H-invariant elements of M and if $m\in M^{H}$, then $Tr_{H}^{G}(\mathrm{m})=\sum_{\sigma}m\sigma$, where σ runs through a set of coset representatives of H in G. If M is R-free, then M is called an RG-lattice and we denote by χ_{M} the character of it. Finally for a block B of RG, we denote by $\delta(B)$ its defect group.

LEMMA 1 (Green). *Let M be an RG-lattice and H a subgroup of G. Let X be an indecomposable component of M_{H} and assume that there exists $v=\sum_{\sigma}a_{\sigma}\sigma^{H}\in(RG)^{H}$ such that $Xv=X$, where $a_{\sigma}\in R$, $\sigma^{H}=Tr_{C_{H}(\sigma)}^{H}(\sigma)$ and σ runs through a set of representatives of H-conjugacy classes of G. Then there exists $\sigma\in G$ such that $a_{\sigma}^{*}\neq 0$ and X is $C_{H}(\sigma)$-projective.*

PROOF. It follows from the assumption that v actually induces an RH-automorphism of X, since X is R-free (more generally since X is a neotherian module). Let u be the inverse map of v in $End_H(X)$. Put $C_\sigma = C_H(\sigma)$ for brevity and let e be the projection from M_H to X. Noting that $\sigma eu \in End_{C_\sigma}(X)$, we have $\sum_\sigma a_\sigma Tr^H_{C_\sigma}(\sigma eu) = \sum_\sigma a_\sigma Tr^H_{C_\sigma}(\sigma) eu = veu$, which is the identity on X. This implies that some $a_\sigma Tr^H_{C_\sigma}(\sigma eu)$ is invertible in (the local ring) $End_H(X)$. Our assertion now follows by Higran's criterion on the relative projectivity.

As an immediate corollary of the above Lemma, we have

COROLLARY (Green). *Let B be a block of RG with defect group D. If M is an indecomposable RG-lattice belonging to B, then M is D-projective.*

PROOF. We let $H = G$ and v = the block idempotent of B in the above Lemma. Then M is $C_G(\sigma)$-projective for some $\sigma \in G$ such that $a^*_{\bar{\sigma}} \neq 0$ and hence Q-projective if Q is a Sylow p-subgroup of $C_G(\sigma)$. From the definition of defect group, it follows that $Q \underset{G}{\subset} D$ and therefore M is D-projective.

The next result is an easy consequence of the Theorem 1 of Watanabe [5]. However we give here a direct proof to it for the simplicity of our argument.

LEMMA 2 (Watanabe). *Let H be a subgroup of G and let B(resp. b) be a block of RG (resp. RH) with the block idempotent E (resp. e). If b^G is defined and equal to B, then there exists $w \in R[G \backslash H]^H$ such that $eE = e + (1-E)w$, where $G \backslash H$ denotes the set theoretic complement of H in G.*

PROOF. Let g be the map from RG to RH defined by:

$$g(\sigma) = \begin{cases} \sigma & \text{if} \sigma \in H \\ 0 & \text{otherwise} \end{cases}$$

Note that $g(Z(RG)) \subset Z(RH)$, so we have $E = g(E) + u$ for some $u \in R[G \backslash H]^H$ and $eE = eg(E) + eu$. We claim that $eg(E)$ is a unit in $Z(RH)e$. In fact if ω_b denotes the central k-character associated with b, then $\omega_b(e^*) = 1 = \omega_B(E^*) = \omega_b(g(E)^*) = \omega_b(e^* g(E)^*)$, which means that $eg(E)$ must be a unit in $Z(RH)e$ since $Z(RH)e$ is a local ring. Therefore there exists $r \in Z(RH)$ such that $reg(E) = e$ and hence we have $reE = e + w$ for $w = reu \in R[G \backslash H]^H$.

Multiplying both sides by E, we get $reE = eE + wE = e + w$ and thus $eE = e + (1-E)w$, as required.

We close this section with the following well-known result due to Green for later convenience.

THEOREM A (Green). *Let M be an RG-lattice which is Q-projective for some p-subgroup Q and $\sigma \in G$. Then $\chi_M(\sigma) = 0$ unless the p-part of σ is contained in some conjugate of Q.*

2. By making use of the results in the preceeding section, we first prove

THEOREM B (*Brauer's second main theorem*). *Let σ be a p-element of G and H a subgroup of G such that $H \supset C_G(\sigma)$. Let M be an RG-lattice belonging to a block B of RG and $\{b_i ; 1 \leqq i \leqq r\}$ the set of blocks of RH such that b_i^G is defined and equal to B. Then for any p-regular element τ of $C_G(\sigma)$, we have*

$$\chi_M(\sigma\tau) = \sum_{i=1}^{r} \chi_{Me_i}(\sigma\tau), \quad \textit{where } e_i \textit{ is the block idempotent of } b_i \ (1 \leqq i \leqq r).$$

PROOF. Let $b_1, \cdots, b_n$ be all the blocks of RH with block idempotents $e_1, \cdots, e_n$ respectively. Then $\chi_M(\sigma\tau) = \sum_{i=1}^{r} \chi_{Me_i}(\sigma\tau)$, We have to show $\chi_{Me_j}(\sigma\tau) = 0$ for $j \geqq r+1$. Let E be the block idempotent of B. If σ is contained in some defect group $\delta(b_j)$ of b_j, then $C_G(\delta(b_j)) \subset C_G(\sigma) \subset H$, whence it follows that b_j^G is defined. Therefore if b_j^G is not defined, σ is not contained in any conjugate of $\delta(b_j)$ and so $\chi_{Me_j}(\sigma\tau) = 0$ by Theorem A, since every indecomposable component of Me_j is $\delta(b_j)$-projective. We next assume that b_j^G is defined. Let E_j be the block idempotent of b_j^G. Then by Lemma 2, we have $E_j e_j = e_j + (1-E_j) w_j$ for some $w_j \in R[G \backslash H]^H$. Multiplying both sides by E, we get $Ee_j = E(-w_j)$. Hence by Lemma 1 any indecomposable component V of Me_j is $C_H(\gamma)$-projective for some $\gamma \in G \backslash H$. If Q is a sylow p-subgroup of $C_G(\gamma)$, then V is Q-projective. We claim that σ is not contained in any conjugate of Q. In fact if $\sigma \underset{H}{\in} Q$, then $\sigma \underset{H}{\in} C_G(\gamma)$ and $\gamma \underset{H}{\in} C_G(\sigma) \subset H$, which is a contradiction. Therefore $\chi_V(\sigma\tau) = 0$ by Theorem A and $\chi_{Me_j}(\sigma\tau) = 0$ if $b_j^G \neq B$. This completes the proof of Theorem B.

By the same argument we can prove the following

THEOREM C (Nagao-Green). *Let M be an indecomposable RG-lattice* *belonging to a block B of RG. Let V be an indecomposable component of M_H with vertex P and b a block of RH to which V belongs. If $H \supset C_G(P)$, then b^G is defined and equal to B.*

PROOF. Let E and e be the block idempotents of B and b respectively. Since $\delta(b) \supset P$, we have $C_G(\delta(b)) \subset C_G(P) \subset H$ and so b^G is defined. Let E' be the block idempotent of b^G and suppose that $E' \neq E$. By Lemma 2 we have $eE' = e + (1-E')w$ for some $w \in R[G \backslash H]^H$. As in the proof of Theorem B, $Vw = Ve = V$. Therefore V is $C_H(\sigma)$-projective for some $\sigma \in G \backslash H$. This means that $P \underset{H}{\subset} C_H(\sigma)$ and hence $\sigma \in C_G(P) \subset H$, which is a contradiction. Therefore $E' = E$ and we complete the proof of Theorem C.

3. We continue our discussion to show another application of Lemma 2. Let f be the augumentation map $kG \rightarrow k$.

LEMMA 3. *Suppose that H contains $C_G(Q)$ for some p-subgroup Q of H. Then we have $f(w)=0$ for any $w \in R[G \backslash H]^H$.*

PROOF. It suffices to show that $f(\sigma^H)=0$ unless $\sigma \in H$. However $f(\sigma^H) = [H : C_H(\sigma)]^*$, which is non-zero only if $C_H(\sigma)$ contains a Sylow p-subgroup, say P, of H. Then $Q \underset{H}{\subset} P \subset C_H(\sigma)$ and thus $\sigma \underset{H}{\in} C_G(Q) \subset H$.

Let H be a subgroup of G and b a block of RH. Recall that b is called admissible if $H \supset C_G(\delta(b))$. The third main theorem of Brauer states that if b is admissible, then b^G is principal if and only if b is so. What is characteristic of the block idempotent E_0 of the principal block of RG is that it is the only block idempotent such that $f(E_0{}^*) \neq 0$, as is easily seen. From this and Lemma 3 one direction of Brauer's third main theorem follows immediately. In fact using the same notation as in Lemma 2, assume that b is principal (and admissible). Then $f(e^*) \neq 0$, whence it follows that $f(E^*) \neq 0$ by Lemmas 2 and 3. This implies that b^G is principal. An easy proof of the other direction which has been supposed to be rather difficult will be found in the recent paper of the first author [3].

Finally we present a module theoretical version of Lemma 2. Before doing so, we recall Alperin's definition of b^G ([1]). Let H be a subgroup of G and b a block of H. We set, following Alperin, $b^G = B$ provided B is the unique block of G with $b \mid B_{H \times H}$. This notion coinsides with the Brauer's one if for example $C_G(\delta(b)) \subset H$ (Green). We hope that the following observation will have some application elsewhere, especially when module theoretical treatments are emphasized.

LEMMA 4. *Let H be a subgroup of G and b a block of H such that $b^G = B$ in the sense of Alperin. Then there exists a component of U of $B_{H\times H}$ which is isomorphic to b such that $U = RH\hat{e}$ where $\hat{e} = e+v$, e the block idempotent of b, $v \in R[G\backslash H]^H$ and $ev = ve = v$.*

PROOF. Let $B_{H\times H} = U_1 \oplus \cdots \oplus U_r \oplus U_{r+1} \oplus \cdots \oplus U_s$, where $U_1, \cdots, U_r$ are all the components of $B_{H\times H}$ which are isomorphic to b. Let $RG(1-E)_{H\times H} = U_{s+1} \oplus \cdots \oplus U_n$, where E is the block idempotent of B. Then we have $RG_{H\times H} = U_1 \oplus \cdots \oplus U_n$ and by Alperin's definition of b^G, U_i is isomorphic to b if and only if $1 \leq i \leq r$. We also have the decomposition $RG_{H\times H} = RHe \oplus RH(1-e) \oplus R[G\backslash H]$. Hence by Krull-Schmidt-Azumaya's theorem there is a $U = U_i$ for some i, $1 \leq i \leq r$, such that $U \oplus RH(1-e) \oplus R[G\backslash H] = RG_{H\times H}$. Clearly $U = RH\bar{e}$ for some $\bar{e} \in R[G]^H$ with $\bar{e}e = e\bar{e} = \bar{e}$, hence $RG_{H\times H} = RH\bar{e} \oplus RH(1-e) \oplus R[G\backslash H] = RHe \oplus RH(1-e) \oplus R[G\backslash H]$. Therefore we have $e = y_1\bar{e} + v_1$ and $\bar{e} = y_2 e + v_2$ where $y_1, y_2 \in RH$ and $v_1, v_2 \in R[G\backslash H]^H$. From the last equation $y_2 e = \bar{e} - v_2$ and from the first one we get $y_2 e = y_2 y_1 \bar{e} + y_2 v_1$. Because of the unique decomposition above we have $y_2 y_1 \bar{e} = \bar{e}$. And so $RH\bar{e} = RHy_2 y_1 \bar{e} \subset RHy_1\bar{e} \subset RH\bar{e}$, hence $RHy_1\bar{e} = RH\bar{e}$. Take $v = -v_1$ and $\hat{e} = y_1\bar{e}$. Then $\hat{e}$ and v satisfy the required property by the first equation and the uniqueness of the above decomposition.

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