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On the generalization of the theorem of Helson and Szegö

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1. Introduction and Theorem A.

T denotes the unit circle, i. e. $T=\{\zeta;|\zeta|=1\}$ while Z denotes the set of all integers. The normalized Lebesgue measure on T is denoted by m, i. e. $dm(\zeta)=\frac{d\theta}{2\pi}$ for $\zeta=e^{i\theta}$. Let χ stand for the identity function on T, i. e. $\chi(\zeta)=\zeta$. We shall use also $\chi_{k}(\zeta)=\chi(\zeta)^{k}=\zeta^{k}$ for $k\in Z$. For $p=1,2$, $L^{p}=L^{p}(T)$ is the Banach space of measurable functions f on T whose p-th power is m-integrable. The space L^{p} is equipped with the norm $\|f\|_{p}=\{\int_{T}|f(\zeta)|^{p}dm(\zeta)\}^{1/p}$. $L^{\infty}=L^{\infty}(T)$ is the space of essentially m-bounded functions f with the norm $\|f\|_{\infty}=\mathrm{ess\ sup}\,|f(\zeta)|$. $C=C(T)$ is the space of continuous functions f on T with the norm $\|f\|_{\infty}=\max_{\zeta\in T}|f(\zeta)|$.

Given a f in L^{1}, its k-th Fourier coefficient $\hat{f}(k)$ is defined by $\hat{f}(k)=\int_{T}\chi_{-k}(\zeta)f(\zeta)dm(\zeta)$ for $k\in Z$. For $p=1,2,\infty$, the Hardy space H^{p} (resp. the disc algebra A) is the closed subspace of functions f in L^{p} (resp. C) for which $\hat{f}(k)=0$ for all $k\leq-1$. A function f in H^{1} is called outer if

$$\log|\int_{T}f(\zeta)dm(\zeta)|=\int_{T}\log|f(\zeta)|dm(\zeta).$$

A function f in H^{∞} is called inner if $|f(\zeta)|=1$ a. e. on T. The subspace spanned by the functions χ_{k}, $k\in Z$ which we call trigonometric polynomials is denoted by $\mathscr{P}$. The subspace spanned by the functions χ_{k}, $k\geq 0$ which we call analytic polynomials is denoted by $\mathscr{P}_{+}$. For a natural number n, the subspace spanned by the functions χ_{k}, $k\leq-n$ is denoted by $\mathscr{P}_{-}^{n}$. We shall use also $\mathscr{P}_{-}=\mathscr{P}_{-}^{1}$. The analytic projection P_{+} from $\mathscr{P}$ onto $\mathscr{P}_{+}$ is defined by $\mathrm{P}_{+}f=\sum_{k\geq 0}\hat{f}(k)\chi_{k}$ for $f\in\mathscr{P}$. Let $\mathrm{P}_{-}=\mathrm{I}-\mathrm{P}_{+}$ where I is the identity operator on $\mathscr{P}$. For complex valued Borel functions $\alpha(\zeta)$ and $\beta(\zeta)$, we study the linear operator $\alpha\mathrm{P}_{+}+\beta\mathrm{P}_{-}$ which includes the analytic projection P_{+} and the Hilbert transform $\mathrm{H}=-i\mathrm{P}_{+}+i\mathrm{P}_{-}$. Let μ be a finite positive regular Borel measure on T whose Lebesgue decomposition is $d\mu=W\ dm+d\mu_{s}$. For a constant $\mathrm{M}>0$, the set of the finite positive regular Borel measures ν on T which satisfy

$$\int_T |(\alpha P_+ + \beta P_-)f|^2 d\nu \le M\int_T |f|^2 d\mu$$

for all $f \in \mathscr{P}_+ + \mathscr{P}^n_-$ is denoted by $R^n_\mu(\alpha P_+ + \beta P_- : M)$. We should mention that if $\{\zeta ; \alpha(\zeta) \neq \beta(\zeta)\} = \boldsymbol{T}$ and $\inf_{\zeta \in T} \max\{|\alpha(\zeta)|, |\beta(\zeta)|\} \neq 0$, then it follows that

$$\bigcup_{M>0} R^n_\mu(\alpha P_+ + \beta P_- : M) = |\alpha - \beta|^{-2} \bigcup_{M>0} R^n_\mu(H : M)$$
$$\cap \{U\ dm ; \max\{|\alpha|^2, |\beta|^2\} UW^{-1} \text{ is bounded.}\}.$$

Therefore, the set $\bigcup_{M>0} R^n_\mu(\alpha P_+ + \beta P_- : M)$ and the set $\bigcup_{M>0} R^n_\mu(H : M)$ are essentially the same in this sense. In particular, $\bigcup_{M>0} R^n_\mu(P_+ : M) = \bigcup_{M>0} R^n_\mu(H : M)$. But for each constant $M>0$, the set $R^n_\mu(H : M)$ does not seem to be able to describe the set $R^n_\mu(\alpha P_+ + \beta P_- : M)$. This is the reason why we are interested in parametrizing the set $R^n_\mu(\alpha P_+ + \beta P_- : M)$. Since $M^{-1} R^n_\mu(\alpha P_+ + \beta P_- : M) = R^n_\mu(\alpha P_+ + \beta P_- : 1)$, it is enough to parametrize the set $R^n_\mu(\alpha P_+ + \beta P_- : 1)$ which is also denoted by $R^n_\mu(\alpha P_+ + \beta P_-)$ or $R^n_\mu(\alpha, \beta)$. The set $R^1_\mu(H)$ is parametrized by R. Arocena, M. Cotlar and C. Sadosky. We should mention that their method works also to parametrize the set $R^n_\mu(H)$. They parametrized the set $R^1_\mu(H)$ as the application of the following theorem which has a lot of applications (see [1], [2], [3]).

THEOREM A. *Let μ_1 and μ_2 be finite positive regular Borel measures on $\boldsymbol{T}$ and ν be a finite complex regular Borel measure on $\boldsymbol{T}$. The following three conditions are then equivalent.*

(i) $\left|\int_T f_1 \bar{f}_2 d\nu\right| \le \{\int_T |f_1|^2 d\mu_1\}^{1/2} \{\int_T |f_2|^2 d\mu_2\}^{1/2}$

for all $f_1 \in \mathscr{P}_+$ and $f_2 \in \mathscr{P}_-$.

(ii) *There is a $k \in H^1$ such that*

$$\left|\nu(E) - \int_E k\ dm\right| \le (\mu_1(E)\mu_2(E))^{1/2}$$

for all Borel sets $E \subseteq \boldsymbol{T}$.

(iii) *There is a $k \in H^1$ such that*

$$\left|\int_T F_1 \bar{F}_2 (d\nu - k\ dm)\right| \le \{\int_T |F_1|^2 d\mu_1\}^{1/2} \{\int_T |F_2|^2 d\mu_2\}^{1/2}$$

for all $F_1, F_2 \in \mathscr{P}$

On the other hand, it is well known that H. Helson and G. Szegö characterized the celebrated positive measure μ which satisfies the condition

(a) $\int_T |Hf|^2 d\mu \le C\int_T |f|^2 d\mu$

for all $f \in \mathscr{P}$ and some constant C. This condition is equivalent to

(b) $\left|\int_T f_1 \bar{f}_2 d\mu\right| \le \rho\{\int_T |f_1|^2 d\mu\}^{1/2} \{\int_T |f_2|^2 d\mu\}^{1/2}$

for all $f_1 \in \mathscr{P}_+$, $f_2 \in \mathscr{P}_-$ and some constant $\rho < 1$.

R. Arocena, M. Cotlar and C. Sadosky proved Theorem A by the interesting method which does not use the Hardy space theory. Since the condition (i) of Theorem A is similar to the above condition (b), it seems natural to show that we can prove Theorem A by the method which is similar to the method of H. Helson and G. Szegö who used the Hardy space theory. The first purpose of this paper is to give the another proof of Theorem A which we describe in section 2. The second purpose is to apply Theorem A to parametrize the set $R_\mu^n(\alpha, \beta)$ which we describe in section 3. In section 4, we show that the results in the preceding sections imply the theorem of P. Koosis and the theorem of H. Helson, G. Szegö and D. E. Sarason. All the results obtained in this paper have the analogies to the case of the real line $\boldsymbol{R}$ instead of the unit circle $\boldsymbol{T}$. The proofs work also in this case with a slight modification.

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2. The another proof of Theorem A.

We prove that (i) implies (ii). Let the Lebesgue decompositions of measures μ_1, μ_2 and ν be $d\mu_j = W_j dm + d\mu_j^{(s)} (j=1, 2)$ and $d\nu = \phi dm + d\nu_s$. Let K_1 be the $L^2(\mu_1 + |\nu|)$ closure of $\mathscr{P}_+$ and K_2 be the $L^2(\mu_2 + |\nu|)$ closure of $\mathscr{P}_-$. By (i),

$$\left|\int_T f_1 \bar{f}_2 d\nu\right| \leq \left\{\int_T |f_1|^2 d\mu_1\right\}^{1/2} \left\{\int_T |f_2|^2 d\mu_2\right\}^{1/2}$$

for all $f_j \in K_j$ $(j=1, 2)$. It is well known that there is a Borel set $E_0 \subseteq \boldsymbol{T}$ with $m(E_0)=0$ on which $\mu_1^{(s)}, \mu_2^{(s)}$ and ν_s are concentrated, and that for all Borel sets $E \subseteq E_0$, its characteritsic function is in $K_1 \cap K_2$. Therefore,

$$|\nu_s(E)| \leq (\mu_1^{(s)}(E)\mu_2^{(s)}(E))^{1/2}$$

for all Borel sets $E \subseteq \boldsymbol{T}$. On the other hand, since the characteristic function of $\boldsymbol{T} \sim E_0$ is in $K_1 \cap \chi K_2$, it follows that

$$\left|\int_T f_1 \bar{f}_2 \phi dm\right| \leq \left\{\int_T |f_1|^2 W_1 dm\right\}^{1/2} \left\{\int_T |f_2|^2 W_2 dm\right\}^{1/2}$$

for all $f_1 \in \mathscr{P}_+$ and $f_2 \in \mathscr{P}_-$. Since $\phi \in L^1$, there is a $k_0 \in H^1$ such that log $|\phi - k_0| \in L^1$. Let N be a natural number. There are outer functions $h_1^{(N)}$, $h_2^{(N)} \in H^2$ such that for $j=1, 2$, $|h_j^{(N)}|^2 = W_j + \frac{1}{N}|\phi - k_0|$ a.e.. Therefore,

$$\left|\int_T f_1 \bar{f}_2 (\phi - k_0) dm\right| \leq \left\{\int_T |f_1|^2 |h_1^{(N)}|^2 dm\right\}^{1/2} \left\{\int_T |f_2|^2 |h_2^{(N)}|^2 dm\right\}^{1/2}$$

for all f_1 in the $L^2(|h_1^{(N)}|^2)$ closure of $\mathscr{P}_+$ and all f_2 in the $L^2(|h_2^{(N)}|^2)$ closure of $\mathscr{P}_-$. By the inner-outer factorization theorem, for all $G \in \chi H^1$,

$\|G\|_1 \leq 1$, there are functions $G_1 \in H^2$, $G_2 \in \chi H^2$ such that $G = G_1 G_2$ a. e. and $\|G_j\|_2 \leq 1$ $(j=1, 2)$. By the Beurling's theorem, $G_1(h_1^{(N)})^{-1}$ is in the $L^2(|h_1^{(N)}|^2)$ closure of $\mathscr{P}_+$ and $\overline{G_2(h_2^{(N)})^{-1}}$ is in the $L^2(|h_2^{(N)}|^2)$ closure of $\mathscr{P}_-$. Substitute these into the previous inequality, it follows that

$$\left|\int_T (h_1^{(N)} h_2^{(N)})^{-1}(\phi - k_0) G \, dm\right| \leq 1.$$

Since $\|(h_1^{(N)} h_2^{(N)})^{-1}(\phi - k_0)\|_\infty \leq N < \infty$ and $(\chi H^1)^* \cong L^\infty / H^\infty$, it follows that

$$\|(h_1^{(N)} h_2^{(N)})^{-1}(\phi - k_0) + H^\infty\| \leq 1.$$

Since the closed unit ball of H^∞ is weak-*compact in the topology of $\sigma(H^\infty, L^1/\chi H^1)$, there is a $g_N \in H^\infty$ such that

$$\|(h_1^{(N)} h_2^{(N)})^{-1}(\phi - k_0) - g_N\|_\infty \leq 1.$$

Let $k_N = k_0 + h_1^{(N)} h_2^{(N)} g_N$, then $k_N \in H^1$ and

$$|\phi - k_N| \leq |h_1^{(N)} h_2^{(N)}| \text{ a. e.}.$$

Since $|k_N| \leq |\phi| + |h_1^{(N)} h_2^{(N)}| \leq |\phi| + |\phi - k_0| + W_1 + W_2$ a. e., $\{k_N\}_{N=1}^{\infty}$ is a L^1-norm bounded sequence in H^1. Since the closed unit ball of H^1 is weak-* compact in the topology of $\sigma(H^1, C/\chi A)$, there is a subsequence $\{k_{N_j}\}_{j=1}^{\infty}$ which converges to some $k \in H^1$ in this topology. That is,

$$\lim_{j\to\infty} \int_T F k_{N_j} dm = \int_T Fk \, dm$$

for all $F \in C$. Since $|\phi - k_N| \leq |h_1^{(N)} h_2^{(N)}|$ a. e.,

$$\left|\int_T F(\phi) - k_{N_j}) dm\right| \leq \int_T |F| |h_1^{(N_j)} h_2^{(N_j)}| dm$$

for all $F \in C$. Since $\{h_1^{(N)} h_2^{(N)}\}_{N=1}^{\infty}$ is L^1-norm bounded and $|h_1^{(N)} h_2^{(N)}| \xrightarrow[N\to\infty]{} (W_1 W_2)^{1/2}$ a. e., by the Lebesgue's dominated convergence theorem, let $j \to \infty$,

$$\left|\int_T F(\phi - k) dm\right| \leq \int_T |F| (W_1 W_2)^{1/2} dm$$

for all $F \in C$. We can obtain in the usual way,

$$|\phi - k| \leq (W_1 W_2)^{1/2} \text{ a. e.}.$$

On the other hand, as we noted before,

$$|\nu_s(E)| \leq (\mu_1^{(s)}(E) \mu_2^{(s)}(E))$$

for all Borel set $E \subseteq \boldsymbol{T}$. Therefore, we get (ii).

The proof that (ii) implies (iii) is as follows. By (ii), it follows that

$$\left|\int_T \phi_1 \bar{\phi}_2 (d\nu - k \, dm)\right| \leq \left\{\int_T |\phi_1|^2 d\mu_1\right\}^{1/2} \left\{\int_T |\phi_2|^2 d\mu_2\right\}^{1/2}$$

for all step functions ϕ_1 and ϕ_2. Since for all $F \in \mathscr{P}$ there is a sequence of step functions $\{\psi_N\}_{N=1}^{\infty}$ such that $\Psi_N \xrightarrow[N\to\infty]{} F$ boundedly and a. e., by the Lebesgue's dominated convergence theorem, we get (iii). The proof that

(iii) implies (i) is clear, since $\int_T f_1\bar{f}_2 k\,dm=0$ for all $f_1\in\mathscr{P}_+$ and $f_2\in\mathscr{P}_-$. This completes the proof of Theorem A.

3. Parametrization of the set $R_\mu^n(\alpha,\beta)$.

Let $\alpha(\zeta)$ and $\beta(\zeta)$ be bounded complex Borel functions on $\boldsymbol{T}$. Let $d\mu(\zeta)=W(\zeta)dm(\zeta)+d\mu_s(\zeta)$ be a finite positive regular Borel measure on $\boldsymbol{T}$. Let n be a natural number. In this section, we apply Theorem A to parametrize the set $R_\mu^n(\alpha,\beta)$ which has already been defined in section 1. We are interested in the problem of how large $\nu\in R_\mu^n(\alpha,\beta)$ can be taken.

DEFINITION. *For a* $k\in\chi_{1-n}H^1$ *we define*

$$\Phi(k)=W\mathrm{Re}\,k-|k|^2 \text{ and } \Psi(k)=\left|\frac{\alpha-\beta}{2}\right|^2 W+\mathrm{Re}\,\bar{\alpha}\beta k$$

LEMMA 1. *If* $k\in\chi_{1-n}H^1$ *satisfies* $|W-2k|\le W$ *a. e., then it follows that* (1) $\Phi(k)\ge 0$ *a. e.* (2) $\Psi(k)W\ge\max\{|\alpha|^2,|\beta|^2\}\Phi(k)$ *a. e.* (3) $\Psi(k)\ge 0$ *a. e.*.

PROOF. It is clear that (1) is equivalent to $|W-2k|\le W$ a. e.. (2) is shown by the following equalities.

$$\Psi(k)W-|\alpha|^2\Phi(k)=\left|\frac{\alpha-\beta}{2}W-\alpha\bar{k}\right|^2 \text{ a. e..}$$

$$\Psi(k)W-|\beta|^2\Phi(k)=\left|\frac{\alpha-\beta}{2}W+\beta k\right|^2 \text{ a. e..}$$

If $k=0$, then $\Psi(k)=\Psi(0)=\left|\frac{\alpha-\beta}{2}\right|^2W\ge 0$ a. e.. If $k\ne 0$, then $|W-2k|\le W$ a. e. implies $\log W\in L^1$ since $\log|k|\in L^1$ and $|k|\le W$ a. e.. Therefore (3) holds. This completes the proof.

LEMMA 2. *Let* $d\nu=U\,dm+d\nu_s$, *then* $\nu\in R_\mu^n(\alpha,\beta)$ *if and only if there is a* $k\in\chi_{1-n}H^1$ *with* $|W-2k|\le W$ *a. e. such that*

(1) $\Psi(k)U\le\Phi(k)$ *a. e.,*

(2) $\max\{|\alpha|^2,|\beta|^2\}U\le W$ *a. e.,*

(3) $|\alpha|^2\nu_s\le\mu_s$ *and* $|\beta|^2\nu_s\le\mu_s$,

(4) $|(\mu_s-\alpha\bar{\beta}\nu_s)(E)|\le\{(\mu_s-|\alpha|^2\nu_s)(E)\}^{1/2}\{(\mu_s-|\beta|^2\nu_s)(E)\}^{1/2}$

for all Borel sets $E\subseteq\boldsymbol{T}$.

PROOF. $\nu\in R_\mu^n(\alpha,\beta)$ implies (2), (3) and that

$$\left|\int_T\chi_{n-1}f_1\bar{f}_2(d\mu-\alpha\bar{\beta}d\nu)\right|\le\left\{\int_T|f_1|^2(d\mu-|\alpha|^2d\nu)\right\}^{1/2}\left\{\int_T|f_2|^2(d\mu-|\beta|^2d\nu)\right\}^{1/2}$$

for all $f_1\in\mathscr{P}_+$ and $f_2\in\mathscr{P}_-$. By Theorem A, there is a $k\in\chi_{1-n}H^1$ such that

$$\left|\int_E(d\mu-\alpha\bar{\beta}d\nu-2k\,dm)\right|\le\left\{\int_E(d\mu-|\alpha|^2d\nu)\right\}^{1/2}\left\{\int_E(d\mu-|\beta|^2d\nu)\right\}^{1/2}$$

for all Borel sets $E\subseteq\boldsymbol{T}$. It is clear that this implies (4). By the Lebesgue's differentiation theorem, it follows that

$$|W-\alpha\bar{\beta}U-2k|\leq(W-|\alpha|^2U)^{1/2}(W-|\beta|^2U)^{1/2} \text{ a.e.}.$$

This is equivalent to (1). On the other hand,

$$|W-2k|\leq|W-\alpha\bar{\beta}U-2k|+|\alpha\bar{\beta}U|$$
$$\leq(W-|\alpha|^2U)^{1/2}(W-|\beta|^2U)^{1/2}+|\alpha\bar{\beta}U|\leq W \text{ a.e.}.$$

The converse is also true by Theorem A. This completes the proof.

LEMMA 3. *$\nu\in R_\mu^n(\alpha,\beta)$ implies that $\nu_s=0$ on $\{\xi;\alpha(\xi)\neq\beta(\xi)\}$.*

PROOF. Let $\nu\in R_\mu^n(\alpha,\beta)$, then by the direct estimation, we have $|\frac{\alpha-\beta}{4}|^2\nu\in R_\mu^n(\mathrm{H})$ since

$$\alpha\mathrm{P}_+ + \beta\mathrm{P}_- = \frac{\alpha+\beta}{2}\mathrm{I}+i\frac{\alpha-\beta}{2}\mathrm{H}.$$

Therefore by Lemma 2,

$$(\mu_s+|\frac{\alpha-\beta}{4}|^2\nu_s)(E)\leq(\mu_s-|\frac{\alpha-\beta}{4}|^2\nu_s)(E)$$

for all Borel sets $E\subseteq\boldsymbol{T}$. This implies that $|\frac{\alpha-\beta}{4}|^2\nu_s(E)=0$. Therefore, $\nu_s(E)=0$ for all Borel sets $E\subseteq\{\xi;\alpha(\xi)\neq\beta(\xi)\}$. This completes the proof.

DEFINITION. *For the bounded complex Borel functions $\alpha(\xi)$ and $\beta(\xi)$, let*

$$I_1=\{\xi;\alpha(\xi)\neq\beta(\xi)\}$$
$$I_2=\{\xi;\alpha(\xi)=\beta(\xi)\neq0\}$$
$$I_3=\{\xi;\alpha(\xi)=\beta(\xi)=0\}.$$

THEOREM 1. *Let n be a natural number. Let $\alpha(\xi)$ and $\beta(\xi)$ be bounded complex Borel functions on $\boldsymbol{T}$. Let $d\mu=W\,dm+d\mu_s$ be a finite positive regular Borel measure on $\boldsymbol{T}$. Then the finite positive regular Borel measure $d\nu=Udm+d\nu_s$ belongs to the set $R_\mu^n(\alpha,\beta)$ if and only if there is a $k\in\chi_{1-n}H^1$ with $|W-2k|\leq W$ a. e. such that the following holds.*

$$\mathrm{U}\leq\begin{cases}\dfrac{\Phi(k)}{\Psi(k)} & \text{a. e. on } (I_1\cup I_2)\cap\{\Psi(k)\neq0\}\\[2ex] \dfrac{W}{\max\{|\alpha|^2,|\beta|^2\}} & \text{a. e. on } (I_1\cup I_2)\cap\{\Psi(k)=0\}\end{cases}$$

$\nu_s=0$ on I_1 and $|\alpha|^2\nu_s(=|\beta|^2\nu_s)\leq\mu_s$ on I_2. ν has no restriction on I_3.

PROOF. If $\nu\in R_\mu^n(\alpha,\beta)$, then by Lemma 2, there is a $k\in\chi_{1-n}H^1$ with $|W-2k|\leq W$ a.e. such that $\Psi(k)U\leq\Phi(k)$ a.e.. By Lemma 1,

$$U\leq\frac{\Phi(k)}{\Psi(k)}\leq\frac{W}{\max\{|\alpha|^2,|\beta|^2\}} \text{ a.e. on } (I_1\cup I_2)\cap\{\Psi(k)\neq0\}.$$

By Lemma 2,

$$U\leq\frac{W}{\max\{|\alpha|^2,|\beta|^2\}} \text{ a.e. on } I_1\cup I_2.$$

On the other hand, by Lemma 2, $|\alpha|^2\nu_s\le\mu_s$, $|\beta|^2\nu_s\le\mu_s$ on $\boldsymbol{T}$. By Lemma 3, $\nu_s=0$ on I_1. By Lemmas 1 and 2, it follows that ν has no restriction on I_3. We can show that the converse is also true by Lemmas 1 and 2. This completes the proof.

It follows from Theorem 1 that the all elements of $R_\mu^n(\alpha,\beta)$ are absolutely continuous with respect to m for all finite positive regular Borel measures μ if and only if $\{\xi:\alpha(\xi)\neq\beta(\xi)\}=\boldsymbol{T}$. So, we are interested in parametrizing the set $R_\mu^n(\alpha,\beta)$ on that assumption.

THEOREM 2. *Let n be a natural number. Let $\alpha(\xi)$ and $\beta(\xi)$ be bounded complex Borel functions on $\boldsymbol{T}$ which satisty $\{\xi\,;\,\alpha(\xi)\neq\beta(\xi)\}=\boldsymbol{T}$ and* $\inf\max\{|\alpha(\xi)|,|\beta(\xi)|\}\neq0$. *Let μ be a finite positive regular Borel measure on $\boldsymbol{T}$ whose Lebesgue's decomposition is $d\mu=W\,dm+d\mu_s$. Then the following holds.*

$$(1)\quad R_\mu^n(\alpha,\beta)=\left\{U(\rho,k)dm\;\middle|\;\begin{array}{l}\rho\in L^\infty,\|\rho\|_\infty\le1\\ k\in\chi_{1-n}H^1,\ |W-2k|\le W\ \ a.e.\end{array}\right\}$$

$$(2)\quad R_\mu^n(\alpha,\beta)\cap\{Udm\,;\,\log U\in L^1\}$$

$$=\left\{U(\rho,k)dm\;\middle|\;\begin{array}{l}0\neq\rho\in H^\infty,\ \|\rho\|_\infty\le1\\ 0\neq k\in\chi_{1-n}H^1,\ |W-2k|\le W\ \ a.e.\\ \displaystyle\int_{\{\Psi(k)\neq0\}}\log\frac{\Phi(k)}{\Psi(k)}\,dm>-\infty\end{array}\right\}$$

where $$U(\rho,k)=\begin{cases}|\rho|\dfrac{\Phi(k)}{\Psi(k)} & a.e.\ on\ \{\Psi(k)\neq0\}\\[2ex] |\rho|\dfrac{W}{\max\{|\alpha|^2,|\beta|^2\}} & a.e.\ on\ \{\Psi(k)=0\}\end{cases}$$

PROOF. Since $\inf\max\{|\alpha(\xi)|,|\beta(\xi)|\}\neq0$, it follows that $U(\rho,k)\in L^1$ by Lemma 1. Therefore, (1) follows from Theorem 1. Next we prove (2). Let $Udm\in R_\mu^n(\alpha,\beta)$ which satisfies $\log U\in L^1$. By (1) and Lemma 1, it follows that there is a $k\in\chi_{1-n}H^1$ with $|W-2k|\le W$ a.e. and a $\rho\in L^\infty$ with $\|\rho\|_\infty\le1$ such that

$$U=|\rho|\frac{\Phi(k)}{\Psi(k)}\le|\rho|\frac{W}{\max\{|\alpha|^2,|\beta|^2\}}$$

a.e. on $\{\Psi(k)\neq0\}$ and

$$U=|\rho|\frac{W}{\max\{|\alpha|^2,|\beta|^2\}}$$

a. e. on $\{\Psi(k)=0\}$. Therefore, it follows that $\log|\rho|\in L^1$ since $\log\max\{|\alpha|,|\beta|\}\in L^1$, $\log U\in L^1$ and $W\in L^1$. Furthermore,

$$\int_{\{\Psi(k)\neq0\}}\log\frac{\Phi(k)}{\Psi(k)}\,dm\ge\int_{\{\Psi(k)\neq0\}}\log U\ dm>-\infty.$$

We show that $k\neq0$ in the following. If $k=0$, then $\Phi(k)=\Phi(0)=0$ a.e.. Ψ

$(k)=\Psi(0)=0$ a. e. follows from Lemma 2 since $\log U\in L^1$. It follows that $W=0$ a. e. since $\Psi(0)=|\frac{\alpha-\beta}{2}|^2W=0$ a. e. and $\{\zeta;\alpha(\zeta)\neq\beta(\zeta)\}=\boldsymbol{T}$. Therefore $U=0$ a. e. which is a contradiciton.

Next we show that the converse is also true. By (1), the $U(\rho, k)dm$ in the right hand is contained in $R_\mu^n(\alpha, \beta)$. So, it is enough to show that $\log U(\rho, k)\in L^1$. Since $0\neq\rho\in H^\infty$, it follows that $\log|\rho|\in L^1$. Since $0\neq k\in H^1$ with $|W-2k|\leq W$ a. e., it follows that $\log W\in L^1$. Therefore

$$\int_T \log\frac{W}{\max\{|\alpha|^2, |\beta|^2\}}dm>-\infty.$$

Since

$$\int_{\{\Psi(k)\neq 0\}}\log U(\rho, k)dm>-\infty$$

and $U(\rho, k)\in L^1$, it follows that $\log U(\rho, k)\in L^1$. This completes the proof.

We had the parametrizaiton of the set $R_\mu^n(\mathrm{P}_+)$ before, by the discussion with Dr. Y. Nakamura. The parametrization takes the more transparent form in this case. We will show in section 4 that $R_\mu^1(\mathrm{P}_+)\neq\{0\}$ if and only if $W^{-1}\in L^1$ where $d\mu=W\,dm+d\mu_s$. Therefore, the set

$$\{k\in H^1;\ |W-2k|\leq W \text{ a. e.}\}$$

can be parametrized by the Adamyan, Arov and Krein's theorem (see [5] p 179, ex. IV-18). If we combine this fact with Theorem 2, then we have the following parametrization. *If $R_\mu^1(\mathrm{P}_+)\neq\{0\}$, then it follows that*

$$R_\mu^1(\mathrm{P}_+)=\left\{|\rho|\frac{|1-G|^2(1-|s|^2)}{|1-Gs|^2}dm\ \middle|\ \begin{array}{l}\rho\in L^\infty,\ \|\rho\|_\infty\leq 1\\ s\in H^\infty,\ \|s\|_\infty\leq 1\end{array}\right\}$$

and $R_\mu^1(\mathrm{P}_+)\cap\{Udm;\ \log U\in L^1\}$

$$=\left\{|\rho|\frac{|1-G|^2(1-|s|^2)}{|1-Gs|^2}dm\ \middle|\ \begin{array}{l}0\neq\rho\in H^\infty,\ \|\rho\|_\infty\leq 1\\ 1\neq s\in H^\infty,\ \|s\|_\infty\leq 1\\ \log(1-|s|)\in L^1\end{array}\right\}$$

where G is a H^∞ function with $\|G\|_\infty\leq 1$ defined by $\frac{1+G}{1-G}=W^{-1}+i(W^{-1})^\sim$ where $(W^{-1})^\sim$ denotes the harmonic conjugate function of W^{-1}.

4. The corollaries.

In this section, we deduce the Koosis' theorem (see [10], [11]) as Corollary 1 and the Helson, Szegö and Sarason's theorem (see [6], [7]) as Corollary 2 from the results of the preceding sections in somewhat generalized form. The essential part of the proof of Corollary 1 is due to Koosis.

COROLLARY 1. *Let n be a natural number. Let $\alpha(\xi)$ and $\beta(\xi)$ be bounded complex Borel functions which satisfy $\{\xi\,;\,\alpha(\xi)\neq\beta(\xi)\}=\boldsymbol{T}$ and* $\inf \max\{|\alpha(\xi)|, |\beta(\xi)|\}\neq 0$. *Let μ be a finite positive regular Borel measure on $\boldsymbol{T}$ whose Lebesgue decomposition is $d\mu=Wdm+d\mu_s$. The following two conditions are then equivalent.*

(i) $R_\mu^n(\alpha, \beta)\neq\{0\}$.

(ii) *There is a nonzero analytic polynomial p such that $W^{-1}|p|^2\in L^1$ with* $\deg p\leq n-1$.

In this case, there is a $U\in R_\mu^n(\alpha, \beta)$ such that $\log U\in L^1$.

PROOF. We show that (i) implies (ii). Since $\{\xi\,;\,\alpha(\xi)\neq\beta(\xi)\}=\boldsymbol{T}$, it follows from Theorem 2 that (i) implies that there is a nonzero $k\in\chi_{1-n}H^1$ such that $|W-2k|\leq W$ a.e.. Since $0\neq|k|\leq W$ a.e., it follws that $\log W\in L^1$. So, there is an outer function $h\in H^2$ such that $W=|h|^2$ a.e.. Let $v=\arg k$, then it follows that $-\frac{\pi}{2}\leq v\leq\frac{\pi}{2}$ a.e. and $|k|\leq W\cos v$ a.e., since $|W-2k|\leq W$ a.e.. Let $l=\exp(\tilde{v}-iv)$ where $\tilde{v}$ denotes the harmonic conjugate function of v. Then $\mathrm{Re}\, l\in L^1$ since $-\frac{\pi}{2}\leq v\leq\frac{\pi}{2}$ a.e. (see [5] p. 161). Let $s=kl$, then it follows that $s\geq 0$ a.e.. Let $F=\chi_{n-1}h^{-2}s$, then it follows from the Smirnov's theorem that $F\in H^1$ since

$$|F|=W^{-1}s=W^{-1}|k|\,|l|\leq(\cos v)(\exp\tilde{v})=\mathrm{Re}\, l \text{ a.e..}$$

By the inner-outer factorization theorem, there is an inner funtion Q and an outer function $G\in H^2$ such that $F=QG^2$ a.e.. Let $p=hG$, then it follows that $0\neq p\in H^1$ and

$$0\leq s=\chi_{1-n}h^2F=\chi_{1-n}Qp^2=|p|^2 \text{ a.e..}$$

Therefore, $\chi_{1-n}Qp=\bar{p}$ a.e.. If we compare the Fourier coefficient on the both sides, it follows that p is in $\mathscr{P}_+$ with $\deg p\leq n-1$ and $W^{-1}|p|^2=W^{-1}s\in L^1$.

Next we show that (ii) implies (i). By (ii), we can define $F=W^{-1}|p|^2+i(W^{-1}|p|^2)^\sim$ where $(W^{-1}|p|^2)^\sim$ denotes the harmonic conjugate funciton of $W^{-1}|p|^2$. Let $k=(2F)^{-1}|p|^2$, then it follows from the Smirnov's theorem that k belongs to $\chi_{1-n}H^1$ since $\deg p\leq n-1$. For this k, $\Phi(k)=W\,\mathrm{Re}\, k-|k|^2=|k|^2\neq 0$ a.e. which implies that $\log\Phi(k)\in L^1$. Since $\Psi(k)\geq W^{-1}\max\{|\alpha|^2, |\beta|^2\}\Phi(k)$ a.e., $\log W\in L^1$ and $\log\max\{|\alpha|, |\beta|\}\in L^1$, it follows that $\log\Psi(k)\in L^1$. By Theorem 2, $U=\Psi(k)^{-1}\Phi(k)$ belongs to $R_\mu^n(\alpha, \beta)$ and $\log U\in L^1$. This completes the proof.

COROLLARY 2. *Let n be a natural number. Let α and β be complex Borel functions which satisfy $\{\xi\,;\,\alpha(\xi)\neq\beta(\xi)\}=\boldsymbol{T}$, $\alpha\bar{\beta}\in H^\infty$ and* $\max\{|\alpha|, |\beta|\}\leq 1$. *Let μ be a finite positive regular Borel measure on $\boldsymbol{T}$ whose Lebesgue decomposition is $d\mu=W\,dm+d\mu_s$. The following two conditions* *are then equivalent.*

(i) $\int_T |(\alpha P_+ + \beta P_-)f|^2 d\mu \leq \int_T |f|^2 d\mu$

for all $f \in \mathscr{P}_+ + \mathscr{P}_-^n$.

(ii) μ *is absolutely continuous with respect to m, i. e.* $\mu_s = 0$, $\log W \in L^1$ *and there is a* $g \in \chi_{1-n} H^\infty$ *such that*

$$|\exp(-i(\log W)^\sim) - g| \leq \left\{ 1 - \left|\frac{\alpha - \beta}{1 - \alpha\bar{\beta}}\right|^2 \right\}^{1/2} \text{ a. e.}$$

where $(\log W)^\sim$ *denotes the harmonic conjugate function of* $\log W$.

PROOF. By Theorem 1, (i) is equivalent to that μ is absolutely continuous and

$$\left\{ \left|\frac{\alpha - \beta}{2}\right|^2 W + \mathrm{Re}\,\bar{\alpha}\beta k \right\} W \leq W \mathrm{Re}\, k - |k|^2 \text{ a. e.}.$$

It follows that

$$|(1 - \alpha\bar{\beta}) W - 2k|^2 \leq (1 - |\alpha|^2)(1 - |\beta|^2) W^2 \text{ a. e.}.$$

Since $\max\{|\alpha|, |\beta|\} \leq 1$ and $\alpha\bar{\beta} \in H^\infty$ it follows that $\mathrm{Re}(1 - \alpha\bar{\beta}) \geq 0$ a. e. which implies that $1 - \alpha\bar{\beta}$ is an outer function (see [8] p. 117). Let $\phi = \exp(\log W + i(\log W)^\sim)$ and $g = 2k(1 - \alpha\bar{\beta})^{-1}\phi^{-1}$ then it follows that $g \in \chi_{1-n} H^\infty$ and

$$\left|\frac{|\phi|}{\phi} - g\right| \leq \left\{ 1 - \left|\frac{\alpha - \beta}{1 - \alpha\bar{\beta}}\right|^2 \right\}^{1/2} \text{ a. e.}.$$

This implies (ii), and the converse is also true. This completes the proof.

References

[1] AROCENA, R. and COTLAR, M.: A generalized Herglotz-Bochner theorem and L^2-weighted inequalities with finite measures. Proc. Conference in honor of Prof. A. Zygmund, Chicago. (1981) 258–269.

[2] AROCENA, R., COTLAR, M. and SADOSKY, C.: Weighted inequalities in L^2 and lifting properties. Adv. Math. supplementary studies 7A, special volume dedicated to L. Schwartz. (1981) 95–128.

[3] COTLAR, M. and SADOSKY, C.: On the Helson-Szegö theorem and a related class of modified Toeplitz kernels. Proc. Symp. Pure Math. A. M. S. 35 (1979) 383–407.

[4] DYM, H. and MCKEAN, H. P.: Gaussian processes, function theory and the inverse spectral problem. Academic Press, New York, (1976).

[5] GARNETT, J. B.: Bounded analytic functions. Academic Press, New York, (1981).

[6] HELSON, H. and SARASON, D. E.: Past and future. Math. Scand. 21 (1967) 5–16.

[7] HELSON, H. and SZEGÖ, G.: A problem in prediction theory. Annali di Mat. Pura et Applicata 4, 51 (1960) 107–138.

[8] KOOSIS, P.: Introduction to H^p spaces. London Math. Society, Lecture notes series 40, Cambridge Univ. Press. (1980).

[9] KOOSIS, P.: Weighted quadratic means of Hilbert transforms. Duke Math. J. 38 (1971) 609–634.

[10] KOOSIS, P.: Moyennes quadratiques pondérées de transformées de Hilbert et fonctions de type exponential. C. R. Acad. Sci. Paris 276 (1973) 1201-1204.

[11] KOOSIS, P.: Moyennes quadratiques pondérées de fonctions périodiques et de leurs conjuguées harmoniques. C. R. Acad. Sci. Paris 291 (1980) 255-257.

[12] SARASON, D. E.: Function theory on the unit circle. Department of Math., Virginia Poly. Inst. and State Univ., Blacksburg, (1978).

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