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A generalization of monodiffric Volterra integral equations

By Shih Tong TU

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1. Introduction

Various different types of discrete Volterra integral equations have been discussed by Deeter [2], Duffin and Duris [3], Fenyes and Kosik [4], and Tu [6, 7]. In [4], Fenyes and Kosik have solved discrete Volterra equations of the type

$$nf_n+\sum_{k=0}^{n} f_k g_{n-k}=h_n$$

by the method of operational calculus. By using the convolution product for discrete function theory, Duffin and Duris [3] discussed a solution of the discrete Volterra type

$$u(z)=f(z)+\lambda\int_0^z k(z-t):u(t)\ dt,\ \text{where } \lambda \text{ is a constant.} \qquad (1.1)$$

On the other hand, Deeter [2] gave a different approach to the equation (1.1) by using some further results of operational calculus. Our aim in this paper is to define the convolution product of p-monodiffric functions and to prove some properties of p-monodiffric functions. We then find the general solutions of the generalized monodiffric Volterra type integral equations (1.1). When $p=1$, our results reduce to the classical results of p-monodiffric functions which have been developed by Berzsenyi [1] and Tu [6].

2. Definitions and Notations

Most of the definitions and notations given here are taken from reference [7]. Let $\boldsymbol{C}$ be the complex plane,
$D=\{z\in\boldsymbol{C}\mid z=x+iy\}$ where $x, y\in\{pj\mid j=0,1,2,\cdots,\ 0<p\leqq 1\}$
and $f: D\rightarrow\boldsymbol{C}$.

DEFINITION 1. The p monodiffric residue of f at z is the value

$$M_pf(z)=(i-1)f(z)+f(z+ip)-if(z+p). \qquad (2.1)$$

DEFINITION 2. The function f is said to be p monodiffric at z if $M_pf(z)=0$. The function f is said to be p monodiffric in D if it is p monodiffric at any point in D (denoted by $f\in M_p(D)$).

DEFINITION 3. The p monodiffric derivative f' of f is defined by

$$f'(z)=\frac{1}{2p}[(i-1)f(z)+f(z+p)-if(z+ip)]. \qquad (2.2)$$

We also use the symbols $\frac{df}{dz}$ or $D_z f$ to represent f'. It is easy to see that $f'(z)$ can be formulated in the following forms:

$$f'(z)=\frac{f(z+p)-f(z)}{p} \text{ or } f'(z)=\frac{1}{ip}[f(z+ip)-f(z)], \qquad (2.3)$$

if $f\in M_p(D)$ at z.

DEFINITION 4. The line integral of f from z to $z+$hp is defined by

$$\int_Z^{Z+hp} f(t)\ dt=hpf(z) \qquad \text{if } h=1 \text{ or } i$$

$$=-\int_{Z+hp}^{Z} f(t)\ dt \qquad \text{if } h=-1 \text{ or } -i. \qquad (2.4)$$

More generally, if $\Omega=\{a=z_0, z_1, \ldots, z_n=b\}$ is a discrete curve in D, then the line integral of f from a to b along Ω is defined by

$$\int_\Omega f(t)\mathrm{dt}=\int_a^b f(t)\mathrm{dt}=\sum_{k=1}^{n}\int_{Z_{k-1}}^{Z_k} f(t)\mathrm{dt}. \qquad (2.5)$$

For the properties of the line integral, the reader may refer to reference [7].

3. The Convolution Product

In order to involve two monodiffric functions, Berzsenyi [1] defined the "double dot" convolution line integral and $*$-convolution product. We now extend them to p-monodiffric functions.

DEFINITION 5. The convolution line integral of f and g from z to $z+hp$ is defined by

$$\int_Z^{Z+hp} f(t):g(t)\ \mathrm{dt}=f(z+hp)\ [g(z+hp)-g(z)] \text{ if } h=1 \text{ or } i$$

$$=-\int_{Z+hp}^{Z} f(t):g(t)\ \mathrm{dt} \quad \text{if } h=-1 \text{ or } -i. \qquad (3.1)$$

More generally, the convolution line integral of f and g from a to b along Ω is defined by

$$\int_\Omega f(t):g(t)\ \mathrm{dt}=\int_a^b f(t):g(t)\ \mathrm{dt}=\sum_{k=1}^{n}\int_{Z_{k-1}}^{Z_k} f(t):g(t)\ \mathrm{dt} \qquad (3.2)$$

It is also easy to show that the convolution line integral of f and g is independent of path in D for every a, $b\in D$. We begin with the following lemma.

LEMMA 1. *Let* $B_p f(z)=(i-1)f(z)+f(z-ip)-if(z-p)$. *Then the convolution line integral along the discrete closed curve* $C(z)=\langle z, z+p, z+p+ip, z+ip, z\rangle$ *is given by*

$$\int_{C(Z)} f(t):g(t)\ dt=[g(z+p)-g(z)]B_p f(z+p+ip)$$
$$+[f(z+p+ip)-f(z+ip)]M_p g(z).$$

PROOF. It follows directly from the definition 5.

In [5], the function f is said to be p-comonodiffric at z if $B_p f(z)=0$.

THEOREM 3.1. *Suppose that f is p-comonodiffric and g is p-monodiffric in D. Let $a, b\in D$, then the integral $\int_a^b f(t):g(t)\ dt$ is independent of the discrete curve in D connecting a to b.*

PROOF: Apply Lemma 1.

For the properties of the convolution line integral we have

THEOREM 3.2.

(1) $\int_c (f+g)(t):h(t)dt=\int_c f(t):h(t)dt+\int_c g(t):h(t)\ dt$

(2) $\int_c f(t):(g+h)(t)dt=\int_c f(t):g(t)dt+\int_c f(t):h(t)dt$

(3) $\int_c kf(t):g(t)dt=k\int_c f(t):g(t)dt=\int_c f(t):kg(t)dt$

where $f, g, h\in M_p(D)$ and k is a constant.

Now, we define a convolution product as follows:

DEFINITION 6. The $*$-product of p-monodiffric function is defined by

$$(f*g)(z)=\int_0^Z f(z-t):g(t)dt. \tag{3.3}$$

Throughout this section, we shall confine ourselves to the function $f: Z^+\times Z^+\to \boldsymbol{C}$ where $Z^+\times Z^+=\{(m, n)\mid m, n=0, 1, \cdots\}$. By making obvious modification, the results of this paper may be extended to the larger domain D.

Similar to the results in [1] we have the following properties for the $*$-product of p-monodiffric functions.

THEOREM 3.3. *Let $f, g, h\in M_p(Z^+\times Z^+)$ and suppose k is a constant. Then*

(a) $f*g\in M_p(Z^+\times Z^+)$

(b) $(f+g)*h=(f*h)+(g*h)$

(c) $f*(g+h)=(f*g)+(f*h)$

(d) $(kf)*g=k(f*g)=f*(kg)$.

PROOF. Since $M_p(f*g)(z)=(i-1)(f*g)(z)+(f*g)(z+ip)-i(f*g)(z+p)$

$$=\int_0^Z M_p f(z-t):g(t)\ dt+f(0)M_p g(z)=0.$$

Thus (a) is proved.

The proofs of (b), (c) and (d) are easy.

For the commutativity and associativity of the convolution products we have

THEOROM 3.4. *Let $f, g\in M_p(Z^+\times Z^+)$ and suppose that $f(0)=g(0)=0$.*

Then $f*g=g*f$.

PROOF. According to the Definition 2, it is sufficient to prove that $(f*g)(z)=(g*f)(z)$ for every z along the positive x-axis. Along the positive x-axis, let $C(z)=\langle 0, p, 2p, \cdots, kp\rangle$ be the path of integration where k is a positive integer and $0<p\leqq 1$. Then

$$(g*f)(kp)=\sum_{j=1}^{k}\int_{(j-1)p}^{jp} g(kp-t):f(t)\ \mathrm{dt}=\sum_{j=1}^{k} g(kp-jp)[f(jp)-f(jp-p)]$$

$$(f*g)(kp)=\sum_{j=1}^{k} f(kp-jp)[g(jp)-g(jp-p)]$$

Thus, $(g*f)(kp)-f(kp)g(0)=(f*g)(kp)-f(0)g(kp)$.

Since $f(0)=g(0)=0$, this concludes the proof.

THEOREM 3.5. *Suppose* f, g *and* $h\in M_p(Z^+\times Z^+)$ *and* $g(0)=0$ *or* $(f*h)(z)=0$, *then* $(f*g)*h=f*(g*h)$.

PROOF. Let $C(z)=\langle 0, p, 2p, \cdots, jp\rangle$ be the path of integration, where j is a positive integer and $0<p\leqq 1$.

$$\begin{aligned}[(f*g)*h](jp)&=\int_0^{jp}(f*g)(jp-t):h(t)\mathrm{dt}\\ &=\sum_{k=1}^{j}(f*g)(jp-kp)[h(kp)-h(kp-p)]\\ &=\sum_{k=1}^{j-1}(f*g)(jp-kp)[h(kp)-h(kp-p)].\end{aligned}$$

Since $(f*g)(jp-kp)=\sum_{m=1}^{j-k} f(jp-kp-mp)[g(mp)-g(mp-p)]$,

take $k+m=n+1$, then we have

$$[(f*g)*h](jp)=\sum_{k=1}^{j-1}\sum_{n=k}^{j-1} f(jp-np-p)\,[g(np-kp+p)-g(np-kp)]\,[h(kp)-h(kp-p)].$$

On the other hand, we find that

$$\begin{aligned}[f*(g*h)](jp)&=\sum_{k=1}^{j} f(jp-kp)\sum_{m=1}^{k-1}[g(kp-mp)-g(kp-mp-p)]\\ &\qquad[h(mp)-h(mp-p)]+g(0)\sum_{k=1}^{j}\mathrm{f(jp-kp)[h(kp)-h(kp-p)]}\\ &=\sum_{k=1}^{j-1}\sum_{n=k}^{j-1} f(jp-np-p)[g(np-kp+p)-g(np-kp)]\\ &\qquad[h(kp)-h(kp-p)]+g(0)(f*h)(jp).\end{aligned}$$

Therefore, it yields

$$[(f*g)*h](z)=[f*(g*h)](z)-g(0)(f*h)(z).$$

4. Generalized Monodiffric Volterra Integral Equations

In this section we shall extend an earlier result [6] about the general solutions to the monodiffric Volterra integral equations

$$u(z)=f(z)+\lambda\int_0^Z k(z-t):u(t)\ \mathrm{dt}. \tag{4.1}$$

If $f(z)$ and $K(z)$ are p monodiffric in $Z^+ \times Z^+$ the integral equation (4.1) is called a generalized monodiffric Volterra integral equation.

LEMMA 2. *Let $f(z)$ and $K(z)$ be p monodiffric in $Z^+ \times Z^+$. Suppose there exist a solution $u(z)$ such that $u(z)=f(z)+\lambda\int_0^z K(z-t):u(t)\,dt$ and $1-\lambda K(0)\neq 0$, then $u(z)$ is p monodiffric in $Z^+ \times Z^+$.*

PROOF. Since $M_p u(z)=(i-1)u(z)+u(z+ip)-iu(z+p)$

$$=M_p f(z)+\lambda\Big[\int_0^z M_p K(z-t):u(t)\mathrm{dt}+\int_z^{z+ip} K(z+ip-t):u(t)\mathrm{dt}$$

$$-i\int_z^{z+p} K(z+p-t):u(t)\mathrm{dt}\Big]$$

$$=\lambda K(0)[u(z+ip)-u(z)-iu(z+p)+iu(z)]=\lambda K(0)M_p u(z)$$

we have $M_p u(z)[1-\lambda K(0)]=0$.

Thus, the Lemma is proved.

THEOREM 4.1. *Let $f(z)$ and $K(z)$ be p monodiffric in $Z^+ \times Z^+$. If $1-\lambda K(0)\neq 0$ then there exists a unique p monodiffric function $u(z)$ in $Z^+ \times Z^+$ such that*

$$u(z)=f(z)+\lambda\int_0^z K(z-t):u(t)\,dt \text{ with } u(0)=f(0). \tag{4.2}$$

Moreover, the solution of (4.2) can be calculated by the following stepping formula:

$$u(z+hp)=u(z)+\frac{1}{1-\lambda K(0)}[f(z+hp)-f(z)$$

$$+\lambda hp\int_0^z K'(z-t):u(t)\,dt] \tag{4.3}$$

for $h=1$ or i.

PROOF. Since $u(z+hp)-u(z)$

$$=f(z+hp)-f(z)+\lambda\int_0^z hpK'(z-t):u(t)\ \mathrm{dt}+\lambda K(0)[u(z+hp)-u(z)],$$

we obtain (4.3).

Now, it remains to prove that the values which we get from (4.3) satisfy the equation (4.2). It suffices to show that (4.2) has a solution for the points on the positive x-axis. From (4.3) we get

$$u(p)=\frac{1}{1-\lambda K(0)}[f(p)-\lambda K(0)f(0)].$$

On the other hand, $u(p)$ can be obtained from (4.2). In fact

$$u(p)=f(p)+\lambda\int_0^p K(p-t):u(t)\mathrm{dt}$$

$$=f(p)+\lambda K(0)u(p)-\lambda K(0)u(0)$$

$$u(p)=\frac{1}{1-\lambda K(0)}[f(p)-\lambda K(0)f(0)].$$

Therefore, (4.2) has a solution for $z=p$. By induction, we suppose that

(4. 2) has a solution for $z=(m-1)p$, i. e.,

$$u[(m-1)p]=f[(m-1)p]+\lambda\int_0^{(m-1)p}K[(m-1)p-t]:u(t)\ \mathrm{dt}$$
$$=f[(m-1)p]+\lambda\{K(0)u[(m-1)p]+pK'(0)u[(m-2)p]+\cdots$$
$$+pK'[(m-3)p]u(p)-K[(m-2)p]u(0)\}.$$

Since $1-\lambda K(0)\neq 0$, we get

$$u[(m-1)p]=\frac{1}{1-\lambda K(0)}\{f[(m-1)p]$$
$$+\lambda p\{\sum_{j=0}^{m-3}K'(jp)u[(m-j-2)p]\}-\lambda pK[(m-2)p]u(0)\}. \tag{4.4}$$

We claim that (4. 2) has a solution for $z=mp$

$$u(mp)=f(mp)+\lambda\int_0^{mp}K(mp-t):u(t)\ \mathrm{dt}$$

i. e.,

$$u(mp)=\frac{1}{1-\lambda K(0)}\{f(mp)+\lambda p\sum_{j=0}^{m-2}K'(jp)u[(m-j-1)p]$$
$$-\lambda pK[(m-1)p]u(0)\}. \tag{4.5}$$

From the stepping formula, we have

$$u(mp)=u[(m-1)p]+\frac{1}{1-\lambda K(0)}\{f(mp)-f[(m-1)p]$$
$$+\lambda p\int_0^{(m-1)p}K'[(m-1)p-t]:u(t)\mathrm{dt}\}$$
$$=u[(m-1)p]+\frac{1}{1-\lambda K(0)}\{f(mp)-f[(m-1)p]$$
$$-\lambda p\sum_{j=0}^{m-3}K'(jp)u[(m-j-2)p]+\lambda pK[(m-2)p]u(0)$$
$$+\lambda p\sum_{j=0}^{m-2}K'(jp)u[(m-j-1)p]-\lambda pK[(m-1)p]u(0)\}. \tag{4.6}$$

Substituting (4. 4) into (4. 6), we obtain (4.5). Thus we proved that (4. 2) has a solution for the points on the positive x-axis. Due to the Definition 2, a function $u(z)\in M_p(Z^+\times Z^+)$ is uniquely determined by its values on the positive x-axis. Therefore the theorem is proved.

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Department of Mathematics
Chung Yuan Christian University
Chung-Li, Taiwan, Republic of China