Hokkaido Mathematical Journal, Vol. 14(1985), p. 149-154

On the number of irreducible characters in a finite group II

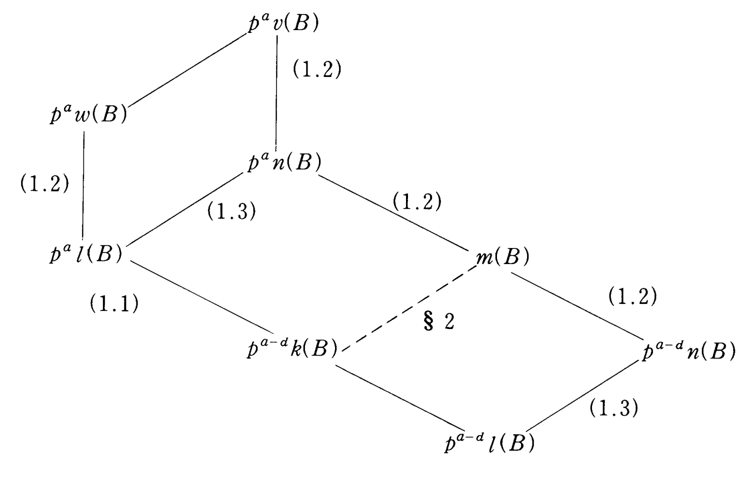
Dedicated to Professor Hirosi Nagao on his 60th birthday

By Tomoyuki WADA
(Received November 27, 1984)

§1 Introduction.

Let F be an algebraically closed field of characteristic p, and G be a finite group with a Sylow p-subgroup P. Let B be a block ideal of the group algebra FG which can be regarded as an indecomposable direct summand of FG as an $F(G\times G)$-module. We denote by $k(B)$ and $l(B)$ the number of irreducible ordinary and modular characters in B, respectively. In [8] the author introduced two invariants $m(B)$ and $n(B)$ associated with B that is the number of indecomposable direct summands of $B_{\Delta(P)}$ and $B_{P\times P}$, where Δ is the diagonal map from G to $G\times G$. We obtained some relations among four invariants $k(B)$, $l(B)$, $m(B)$ and $n(B)$, and it turned out that relation between $m(B)$ and $n(B)$ has a strong resemblance to that between $k(B)$ and $l(B)$. Furthermore, in [9] we proved that $l(B)\leqq n(B)$ and investigate the structure of B when equality holds. In this paper we will show that $|P:D|k(B)\leqq m(B)$ if a defect group D of B is contained in the center of P.

Let us set $|P|=p^{a}$, $|D|=p^{d}$ and $\dim_{F}B=p^{2a-d}v(B)$, where $v(B)=u(B)^{2}w(B)$ is the p'-number mentioned in [2] and [8]. Then our results can be written as the following diagram,



In the above diagram (*), we mean that the upper term is greater than or equal to the lower one, and (1.1) was proved in (5D) of [1] and Proposition 2 of [6], and we will mention it in (2.7). (1.2) was proved in Propositions (2C)–(2E) of [8], and (1.3) was proved in Theorem 2 of [9].

§2 Relation between k(B) and m(B).

Let Irr(B) and IBr(B) be the set of all irreducible ordinary and Brauer characters in B, respectively.

(2.1) (Proposition(2B), [8]). $m(B)=\sum_{\chi\in\mathrm{Irr}(B)}(\chi_P, \chi_P)$,

$n(B)=\sum_{\chi\in\mathrm{Irr}(B)}(\chi_P, 1_P)^2$.

Let σ be a p-element of G. By $B(\sigma)$ we denote the direct sum of block ideals b of $C_G(\sigma)$ such that $b^G=B$. Let $\{d^{\sigma}_{\chi\phi^{\sigma}_i}\}$ be the generalized decomposition number with respect to σ. Then it is well-known that $\sum_{\chi\in\mathrm{Irr}(B)} d^{\sigma}_{\chi\phi^{\sigma}_i}\overline{d^{\sigma}_{\chi\phi^{\sigma}_j}}=c^{\sigma}_{ij}$. Then the following holds.

(2.2) $\sum_{\chi\in\mathrm{Irr}(B)}|\chi(\sigma)|^2=\dim_F B(\sigma)$.

Proof. Since $\chi(\sigma)=\sum_{\phi^{\sigma}_i\in B(\sigma)} d^{\sigma}_{\chi\phi^{\sigma}_i}\phi^{\sigma}_i(1)$ by Brauer's second main theorem, we have that

$$\begin{aligned}\sum_{\chi\in\mathrm{Irr}(B)}|\chi(\sigma)|^2&=\sum_{\chi}\sum_{\phi^{\sigma}_i,\phi^{\sigma}_j} d^{\sigma}_{\chi\phi^{\sigma}_i}\overline{d^{\sigma}_{\chi\phi^{\sigma}_j}}\phi^{\sigma}_i(1)\phi^{\sigma}_j(1)\\&=\sum_{i,j}c^{\sigma}_{ij}\phi^{\sigma}_i(1)\phi^{\sigma}_j(1)\\&=\sum_{\phi^{\sigma}_i\in B(\sigma)}\phi^{\sigma}_i(1)\Phi^{\sigma}_i(1)\\&=\dim_F B(\sigma),\end{aligned}$$

where Φ^{σ}_i is the principal indecomposable character associated with ϕ^{σ}_i.

Set $D^G=\{\sigma^x\mid\sigma\in D, x\in G\}$. As is well-known, when σ is a p-element and D is a defect group of B, then $\chi(\sigma)=0$ for all $\chi\in\mathrm{Irr}(B)$ if σ is not contained in D^G (see Green [2] and Feit[3], Lemma IV, 2.4). Then the following holds.

(2.3) $m(B)=1/|P|\sum_{\sigma\in D^G\cap P}\dim_F B(\sigma)$,

$n(B)=1/|P|^2\sum_{\sigma\in D^G\cap P}|\sigma^G\cap P|\dim_F B(\sigma)$.

Proof. From (2.1), (2.2) we have that

$$m(B)=\sum_{\chi\in\mathrm{Irr}(B)}(\chi_P, \chi_P)$$

$$=\sum_{\chi}1/|P|\sum_{\sigma\in P}|\chi(\sigma)|^2$$
$$=1/|P|\sum_{\sigma\in D^G\cap P}\dim_F B(\sigma).$$

Furthermore, by the orthogonality relation of blocks (see Feit [3], Lemma IV, 6.3) we have that

$$n(B)=\sum_{\chi\in\mathrm{Irr}(B)}(\chi_P,1_P)^2$$
$$=\sum_{\chi}1/|P|^2\sum_{\sigma,\tau\in P}\chi(\sigma)\overline{\chi(\tau)}$$
$$=1/|P|^2\sum_{\sigma,\tau\in P}\sum_{\chi}\chi(\sigma)\overline{\chi(\tau)}$$
$$=1/|P|^2\sum_{\sigma\in P}\sum_{\chi}|\sigma^G\cap P||\chi(\sigma)|^2$$
$$=1/|P|^2\sum_{\sigma\in D^G\cap P}|\sigma^G\cap P|\dim_F B(\sigma).$$

(2.4) *It holds that* $m(B)\leqq p^a n(B)$ *and equality holds if and only if* $\sigma^G\cap P=\{\sigma\}$ *for all* $\sigma\in D$.

PROOF. It is easy observation from (2.3). See also Proposition(2E) and Theorem(3B) of [8].

By (2.1) and the diagram $(*)$ it generally holds that $k(B)\leqq m(B)$ and $p^{a-2d}k(B)\leqq m(B)$. But our purpose here is to show the following more precise relation.

(2.5) THEOREM. *Let* B *be a block of* G *with defect group* D. *Suppose* $D\leqq Z(P)$, *then* $p^{a-d}k(B)\leqq m(B)$.

PROOF. Let S be a complete set of representatives of the conjugate classes in G consisting of elements in D. Then Lemmas IV, 6.5, 6.6 in [3] imply that $k(B)=\sum_{\sigma\in S}l(B(\sigma))$, where $l(B(\sigma))=\sum_{b^G=B}l(b)$. By our assumption, D is abelian and $C_G(\sigma)\leqq P$ for all $\sigma\in D$, hence if b is a block of $C_G(\sigma)$ such that $b^G=B$, then D is a defect group of b (see Feit [3], Lemma V, 6.1), and hence

$$\dim_F b=p^{2a-d}v(b)\geqq p^{2a-d}l(b).$$

Therefore, we have from (2.2) that

$$m(B)=1/|P|\sum_{\sigma\in D^G\cap P}\dim_F B(\sigma)$$
$$=1/|P|\sum_{\sigma}\sum_{b}p^{2a-d}v(b)$$
$$\geqq p^{a-d}\sum_{\sigma}\sum_{b}l(b)$$
$$\geqq p^{a-d}\sum_{\sigma\in S}l(B(\sigma))$$
$$=p^{a-d}k(B).$$

(2. 6) EXAMPLE. We shall give examples which show that our assumption $D \leqq Z(P)$ is necessary in (2. 5).

(1) Non-solvable case.

Let $G=S_5$ and $p=2$. Let B be the block of G with 2-defect 1. Then Irr $(B)=\{\chi_1, \chi_2\}$ and $\chi_i(1)=4$ for $i=1, 2$. A defect group D of B is of order 2 and it is not normal in any Sylow 2-subgroup P of G. We have easily that $p^{a-d}k(B)=8$, and $m(B)=6$.

(2) Solvable case.

Let G be the dihedral group of order 24 and $p=2$. Then G has the unique non-principal 2-block B of 2-defect 2. Hence $\mathrm{Irr}(B)=\{\chi_i \mid 1\leqq i\leqq 4\}$ and $\chi_i(1)=2$ for $1\leqq i\leqq 4$. Since D is a defect group of an element of order 3, D is cyclic group of order 4 and $D \lhd G$. But D is not contained in the center of any Sylow 2-subgroup P. And, we have also that $p^{a-d}k(B)=8$, $m(B)=6$. This example has been informed from Mr. Murai, and see section 3.

On the equality in Theorem (2. 5) we have the following.

(2. 7). *In the diagram* $(*)$, $k(B)=p^d l(B)$ *if and only if* $k(B)=p^d$ *and* $l(B)=1$.

PROOF. Let $k_i(B)$ be the number of irreducible ordinary characters in B of height i. Then Olsson has proved that

$$k_0(B)+\sum_{i>0} k_i(B)p^{i+1} \leqq p^d l(B)$$

(see Proposition 2, [6]). Hence, if $k(B)=p^d l(B)$, then $k(B)=k_0(B)$. Furthermore, Olsson has shown that if $k(B)=k_0(B)$, then $k(B)\leqq p^d\sqrt{l(B)}$ (see Proposition 15, [6]). Therefore, we have that $l(B)=1$, and hence $k(B)=p^d$.

(2. 8) REMARK. Note that Okuyama and Tsushima proved that $k(B)=p^d l(B)$ implies that D is abelian and the inertial index is 1 which is the converse of the well-known theorem of Brauer (Proposition 1 and Theorem 3, [5]). In the diagram $(*)$, (2. 7) implies that $k(B)=p^d l(B)$ if and only if $k(B)=p^d w(B)$.

(2. 9) *In the diagram* $(*)$, *it generally holds that* $k(B)\leqq p^d n(B)$. *On the equality, the following are equivalent*;

(1) $k(B)=p^d n(B)$,

(2) $k(B)=p^d v(B)$.

PROOF. (2)→(1). Obvious. (1)→(2). If $k(B)=p^d n(B)$, then $k(B)=p^d l(B)$. Therefore, by (2. 7) $l(B)=1$ and $k(B)=p^d$. This implies that $n(B)=1$. Then $v(B)=1$ (see Proposition (2E), 2) in [8]).

(2.10) THEOREM. *Let B be a block of G with defect group D. Suppose $D \leqq Z(P)$, then the following are equivalent ;*

(1) $p^{a-d}k(B)=m(B)$,

(2) $k(B)=p^d v(B)$.

Furthermore, in this case it holds that $[G, D] \leqq \mathrm{Ker}\ B$.

PROOF. (2)→(1). Obvious. (1)→(2). It follows from the proof of Theorem (2.5) that if $p^{a-d}k(B)=m(B)$, then $\sigma^G \cap P=\{\sigma\}$ for all $\sigma \in D$. By (2.4) this implies that $m(B)=p^a n(B)$. Hence we have that $k(B)=p^d n(B)$. Then (2.9) yields that $k(B)=p^d v(B)$, and since $m(B)=p^a v(B)$, it follows that $[G, D] \leqq \mathrm{Ker}\ B$ by Theorem (3B), 2) in [8].

§ 3 Correction.

Mr. Masafumi Murai has kindly pointed out that the argument on the Green correspondence in step 3 of the proof of Theorem (4B) in my paper [8] is incorrect, and informed me of the following counter example to Theorem (4B). The author thanks Mr. Murai for his valuable suggestion.

EXAMPLE. Let G be a dihedral group of order $2^n r$, where $n \geqq 3$ and $r>1$ is odd. If $p=2$, then G has a non-princinal block B. Since $\chi(1)=2$ for all $\chi \in \mathrm{Irr}(B)$, we have that $k(B)=p^d v(B)$. On the other hand, as any non-identity element of odd order of G has a cyclic defect group D of order 2^{n-1}, D is a defect group of B and $D \lhd G$. But it does not hold that $[G, D] \leqq \mathrm{Ker}\ B$.

Now, we shall state here that, under some stronger condition, Theorem (4B) remains true.

THEOREM. *Let B be a block with defect group D and defect d. We assume that $D \lhd P$ for a Sylow p-subgroup P of G. If $k(B)=p^d v(B)$ (i. e. $\chi(l)=|P:D|$ for all $\chi \in \mathrm{Irr}(B)$), then the following holds.*

1) *D is abelian,*

2) *$G=PC_G(D)\mathrm{Ker}\ B$, in particular $[G, D \cap Z(P)] \leqq \mathrm{Ker}\ B$, and hence if $D \leqq Z(P)$, then $[G, D] \leqq \mathrm{Ker}\ B$.*

PROOF. We may assume that Ker $B=1$ by induction on $|G|$. Our assumption $k(B)=p^d v(B)$ implies that $l(B)=v(B)=1$, and hence by Theorem (4A) in [8] we have that $D \lhd G$.

1) Since every $\chi \in \mathrm{Irr}(B)$ is of height 0, it follows from the theorem of Reynolds ([7]) that D is abelian.

2) It suffices to show that $|G: C_G(D)|$ is a power of p. Let $C=G_G(D)$ and b be a block of C covered by B. Let $T(b)$ be the inertial group of b and $\tilde{B}$ be a block of $T(b)$ covering b. Then we have that $b^{T(b)}=\tilde{B}$, $\tilde{B}^G=$

B and hence b, $\tilde{B}$ have a defect group D. It is well known that there is a 1-1 correspondence between $\mathrm{Irr}(\tilde{B})$ and $\mathrm{Irr}(B)$ by the map sending ζ *to* ζ^G (Theorem V. 2.5, [3]). Let $\zeta \in \mathrm{Irr}(\tilde{B})$ and then $\zeta^G = \chi \in \mathrm{Irr}(B)$. Since $C \lhd G$, it follows from the theorem of Clifford that $\zeta_C = e \sum_{i=1}^{t} \sigma_i$ for some integer e, where $t = |T(b) : I_G(\sigma_1)|$ and $\sigma_i \in \mathrm{Irr}(b)$.

Hence $(*)$ $\quad |P : D| = \chi(1)$

$$= |G : T(b)|\, \zeta(1)$$
$$= |G : T(b)|\ |T(b) : I_G(\sigma_1)|\, e\, \sigma_1(1).$$

As the inertial index $e(B) = |T(b) : C|$ is prime to p (Lemma V. 5.2, [3]) and e divides $e(B)$, we have that

$$|T(b) : I_G(\sigma_1)|\ \ e = 1.$$

(In fact, $e(B)=1$, also 1) directly follows from [5] in our case.) Therefore ζ_C is irreducible for all $\zeta \in \mathrm{Irr}(\tilde{B})$. Since $D \leqq Z(C)$, $[T(b), D] \leqq \mathrm{Ker}\ \tilde{B}$. This implies that $T(b) = C_{T(b)}(D)\ \mathrm{Ker}\ \tilde{B}$. But, as $\mathrm{Ker}\ \tilde{B} \lhd T(b)$ and it is a p'-group, and $D \lhd G$, we have that $\mathrm{Ker}\ \tilde{B} \leqq C_{T(b)}(D)$. Hence $T(b) = C$, and then $|G : C|$ is a power of p by $(*)$. This completes the proof of Theorem.

Corollary (4C), 2) is still true, but Corollary (4D), 1) is not true under the condition $D \lhd P$. However, for instance, if $D \leqq Z(P)$, then the assertion holds.

References

[1] R. BRAUER: On blocks and sections in finite groups II. Amer. J. Math., 90 (1968), 895–925.

[2] R. BRAUER: Notes on representations of finite groups I. J. London Math. Soc., (2), 13 (1976), 162–166.

[3] W. FEIT: The Representation Theory of Finite Groups. North-Holland, New York, 1982.

[4] J. A. GREEN: Blocks of modular representations. Math. Zeit., 79 (1962), 100–115.

[5] T. OKUYAMA and Y. TSUSHIMA: Local properties of p-block algebras of finite groups. Osaka J. Math., 20 (1983), 33–41.

[6] J. B. OLSSON: Inequalities for block theoretic invariants. Springer Lect. Notes in Math., 903 (1981), 270–284.

[7] W. F. REYNOLDS: Blocks and normal subgroups. Nagoya Math. Jour., 22 (1963), 15–32.

[8] T. WADA: Blocks with a normal defect group. Hokkaido Math. Jour., 10 (1981), 319–332.

[9] T. WADA: On the number of irreducible characters in a finite group. Hokkaido Math. Jour., 12 (1983), 74–82.

Tokyo University of
Agriculture and Technology