Hokkaido Mathematical Journal Vol. 14(1985) p. 365-397

Global rigidity of compact classical Lie groups

Dedicated to Professor Nobuo Shimada on his 60-th birthday

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(Received December 17, 1984)

Introduction.

The subject we treat in the present paper is the problem of global rigidity of compact classical Lie groups as Riemannian manifolds imbedded isometrically in the spaces of matrices.

Let $\boldsymbol{F}$ be one of the fields $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$, where we mean by $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$ the field of real numbers, the field of complex numbers and the field of quaternions. We denote by $G(n, \boldsymbol{F})$ the compact classical Lie group $SO(n)$, $U(n)$ or $Sp(n)$ according as $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{Q}$. Let $M(n, \boldsymbol{F})$ be the space of all $n\times n$ matrices over $\boldsymbol{F}$. Then there can be defined a euclidean inner product in $M(n, \boldsymbol{F})$ invariant under left and right multiplications of matrices contained in $G(n, \boldsymbol{F})$. With this euclidean inner product $M(n, \boldsymbol{F})$ may be regarded as a real euclidean space of dimension $n^2\cdot\dim_{\boldsymbol{R}}\boldsymbol{F}$. Then the induced metric on the submanifold $G=G(n, \boldsymbol{F})$ in $M(n, \boldsymbol{F})$ defines a Riemannian metric on G invariant under left and right actions of G on itself. The focus of this paper is the problem of global rigidity of the inclusion map of the Riemannian manifold $G=G(n, \boldsymbol{F})$ into $M(n, \boldsymbol{F})$, which is an isometric imbedding.

Lef $\boldsymbol{f}$ be an isometric immersion of a Riemannain manifold M into the N-dimensional euclidean space $\boldsymbol{R}^N$. In his paper [9], N. Tanaka showed that there is a linear differential operator L associated with $\boldsymbol{f}$ whose kernel is naturally isomorphic with the space of infinitesimal isometric deformations of $\boldsymbol{f}$. He introduced the notion of elliptic isometric immersions and then established the global rigidity theorem for elliptic isometric immersions: Assume that an isometric immersion $\boldsymbol{f}$: $M\rightarrow\boldsymbol{R}^N$ satisfies the following conditions: i) M is compact; ii) $\boldsymbol{f}$ is elliptic; iii) $\boldsymbol{f}$ is globally infinitesimally rigid, i. e., $\dim Ker\ L=\frac{1}{2}N(N+1)$. Then if two immersions $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ of M into $\boldsymbol{R}^N$ lie both near to $\boldsymbol{f}$ with respect to the C^3-topology, and if they induce the same Riemannian metric, then there exists a unique euclidean transformation a of $\boldsymbol{R}^N$ such that $\boldsymbol{f}_2=a\boldsymbol{f}_1$.

In the present paper we prove the following fact: Assume that G is one of the compact classical Lie groups $SO(n)(n\geqq 5)$, $U(n)(n\geqq 3)$ and $Sp(n)$ $(n\geqq 1)$. Then the inclusion map $\boldsymbol{f}$: $G\rightarrow M(n,\boldsymbol{F})$ is elliptic and globally infinitesimally rigid, i. e., dim Ker $L=\frac{1}{2}N(N+1)$, where $N=n^2\cdot\dim_{\boldsymbol{R}}\mathbf{F}$ (see § 1 and §§ 3−4). Consequently by applying the global rigidity theorem stated above, we obtain actual global rigidity theorems for $\boldsymbol{f}$ in the sense of Tanaka.

Now we explain the contents of this paper. In § 1 we consider the second fundamental forms of $\boldsymbol{f}$ and prove that $\boldsymbol{f}$ is elliptic if G is one of $SO(n)(n\geqq 5)$, $U(n)(n\geqq 3)$ and $Sp(n)(n\geqq 1)$. In § 2 we review some facts on the theory of representations of G that are needed in the subsequent sections. In §§ 3−4, we investigate the space Ker L. In its nature L is compatible with the left and right actions of G on itself. We determine the space Ker L by carrying out the following two steps: 1) *Decompose the space of 1-forms on G under the left and right actions of G*; 2) *Determine all irreducible components of this decomposition that are contained in Ker L.*

The first step has been treated in [5]. An actual decomposition can be given by utilizing the decomposition of the representative ring on G. The second step is carried out by using some facts on the theory of (finite dimensinal) representations of G. We note that there is an important and remarkable symmetry between the left and right actions of G on the space Ker L. After carrying out the above two steps, we obtain the following result: For each G such that $\boldsymbol{f}$ is elliptic, it holds that dim Ker $L=\frac{1}{2}N(N+1)$.

Finally in § 4. 5 we discuss the cases $G=SO(4)$ and $U(2)$. In both cases we know that the space Ker L are of infinte dimension.

Throughout this paper we assume the differentiability of class C^∞.

§ 1. The canonical isometric imbeddings of $SO(n)$, $U(n)$ and $Sp(n)$.

1.1. **The field $\boldsymbol{Q}$ of quaternions.** In this paragraph we briefly review the field $\boldsymbol{Q}$ of quaternions. The field $\boldsymbol{Q}$ is an associative division algebra of dimension 4 over the field $\boldsymbol{R}$ of real numbers with a basis composed of 4 elements 1, e_1, e_2, e_3 whose multiplication table is given by

$$1\cdot e_i=e_i\cdot 1=e_i,\ e_i^2=-1\ (i=1,\ 2,\ 3);$$
$$e_{\sigma(1)}\cdot e_{\sigma(2)}=-e_{\sigma(2)}\cdot e_{\sigma(1)}=e_{\sigma(3)},$$

where σ is an even permutation of the set $\{1, 2, 3\}$. Any quaternion q may be written in the form $q=a_0+\sum_{i=1}^{3}a_ie_i$ with a_0, a_1, a_2, $a_3\in\boldsymbol{R}$. Addition and multiplication are defined by the usual distributivity laws and the above tables.

Let $q=a_0+\sum_{i=1}^{3}a_ie_i\in\boldsymbol{Q}$. We define the conjugate $\bar{q}$ of q and the real part $\mathrm{Re}(q)$ of q as follows: $\bar{q}=a_0-\sum_{i=1}^{3}a_ie_i$, $\mathrm{Re}(q)=a_0$. Then we have $q\cdot\bar{q}=\bar{q}\cdot q=\sum_{i=0}^{3}a_i^2$ and $\mathrm{Re}(q)=\frac{1}{2}(q+\bar{q})$. Hence if $q\neq 0$ then the inverse q^{-1} of q is given by $q^{-1}=(\sum_{i=0}^{3}a_i^2)\ \bar{q}$.

The following formulas can be easily verified:

$$\overline{aq+bq'}=a\bar{q}+b\bar{q}',\ \overline{qq'}=\bar{q}'\bar{q},\ \bar{\bar{q}}=q\,;$$

$$\mathrm{Re}(aq+bq')=a\ \mathrm{Re}(q)+b\ \mathrm{Re}(q'),$$

$$\mathrm{Re}(qq')=\mathrm{Re}(q'q),\ \mathrm{Re}(\bar{q})=\mathrm{Re}(q),$$

where $a,\ b\in\boldsymbol{R}$ and $q,\ q'\in\boldsymbol{Q}$.

The field $\boldsymbol{R}$ of real numbers may be regarded as a subfield of $\boldsymbol{Q}$ in a natural way. Moreover the field $\boldsymbol{C}$ of complex numbers may be regarded as a subfield of $\boldsymbol{Q}$. Let $\boldsymbol{C}_1$ be the set of all quaternions of the form $a_0+a_1e_1$ with $a_0,\ a_1\in\boldsymbol{R}$. It is easy to see that $\boldsymbol{C}_1$ is a subfield of $\boldsymbol{Q}$. Then if we assign to each complex number $a_0+a_1\sqrt{-1}$ with $a_0,\ a_1\in\boldsymbol{R}$ the quaternion $a_0+a_1e_1\in\boldsymbol{C}_1$, we obtain an isomorphism of the field $\boldsymbol{C}$ to the subfield $\boldsymbol{C}_1$ in $\boldsymbol{Q}$. We identify the fields $\boldsymbol{C}$ and $\boldsymbol{C}_1$ by this isomorphism. Then the conjugate $\bar{q}$ and the real part $\mathrm{Re}(q)$ of $q\in\boldsymbol{C}$ as a quaternion are just equal to the conjugate and the real part of q in the usual sense.

1.2. The space of matrices. Let $\boldsymbol{F}$ be one of the fields $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$. As we have remarked in the previous paragraph, $\boldsymbol{F}$ may be regarded as a subfield of $\boldsymbol{Q}$. The conjugate and real part of an element $q\in\boldsymbol{F}$ imply those of q considered as an element of $\boldsymbol{Q}$.

Let $M(n,\boldsymbol{F})$ be the space of $n\times n$ matrices over $\boldsymbol{F}$. In the usual way, addition, multiplication and scalar multiplication are defined in $M(n,\boldsymbol{F})$. Thus $M(n,\boldsymbol{F})$ is an associative algebra of dimension n^2 over $\boldsymbol{F}$ and is regarded as a Lie algebra by the usual bracket operation: $[X,\ Y]=XY-YX$ $(X,\ Y\in M(n,\boldsymbol{F}))$.

Now let us set $G(n,\boldsymbol{R})=SO(n)$, $G(n,\boldsymbol{C})=U(n)$ and $G(n,\boldsymbol{Q})=Sp(n)$. Actually $G=G\ (n,\boldsymbol{F})$ is the subset of $M(n,\boldsymbol{F})$ composed of all matrices $X\in M(n,\boldsymbol{F})$ satisfying ${}^t\bar{X}X=X{}^t\bar{X}=I_n$, where I_n denotes the unit matrix in $M(n,\boldsymbol{F})$. (If $\boldsymbol{F}=\boldsymbol{R}$, we have to add further the equation $\det X=1$.) As is well known, $G=G(n,\boldsymbol{F})$ is a connected and compact Lie group.

Let $X,\ Y\in M(n,\boldsymbol{F})$. We define their product $(X,\ Y)\in\boldsymbol{R}$ by setting

$$(X, Y)=\mathrm{Re}(\mathrm{trace}({}^t\bar{X}Y)).$$

Then we have:

PROPOSITION 1.1. *The product* $(,)$ *is a euclidean inner product in* $M(n, \boldsymbol{F})$ *invariant under the left and right multiplication of elements of* $G=G(n, \boldsymbol{F})$, *i. e.,*

(1°) $(X, Y)=(Y, X),\ (X, aY+bZ)=a(X, Y)+b(X, Z)$;

(2°) $(X, X)>0$ *if* $X\neq 0$;

(3°) $(g\cdot X, g\cdot Y)=(X\cdot g, Y\cdot g)=(X, Y)$,

where $X, Y, Z\in M(n, \boldsymbol{F})$, $a, b\in\boldsymbol{R}$ *and* $g\in G$.

We first prove:

LEMMA 1.2. *Let* $X, Y\in M(n, \boldsymbol{F})$. *Then*:

(1) $\overline{{}^t(XY)}={}^t\bar{Y}\cdot{}^t\bar{X}$.

(2) $\mathrm{Re}(\mathrm{trace}(XY))=\mathrm{Re}(\mathrm{trace}(YX))$.

PROOF. Write $X=(x_{ij})$ and $Y=(y_{ij})$. Then the ij-component u_{ij} of the product XY is represented by $u_{ij}=\sum_{k=1}^{n}x_{ik}y_{kj}$. Hence the ij-component of the left hand side of (1) is given by $\bar{u}_{ji}=\sum_{k=1}^{n}\overline{x_{ij}y_{ki}}=\sum_{k=1}^{n}\bar{y}_{ki}\bar{x}_{jk}$, which is nothing but the ij-component of the right hand side of (1). The equality (2) is proved by: $\mathrm{Re}(\mathrm{trace}(XY))=\sum_{i,j}\mathrm{Re}(x_{ij}y_{ji})=\sum_{i,j}\mathrm{Re}(y_{ji}x_{ij})=\mathrm{Re}(\mathrm{trace}(YX))$.

Q. E. D.

PROOF of PROPOSITION 1.1. (1°) From (1) of Lemma 1.2, we can deduce that: $(X, Y)=\mathrm{Re}(\mathrm{trace}({}^t\bar{X}Y))=\mathrm{Re}(\mathrm{trace}(\overline{{}^t({}^t\bar{X}Y)}))=\mathrm{Re}(\mathrm{trace}({}^t\bar{Y}X))=(Y, X)$. The $\boldsymbol{R}$-linearity of $(,)$ can be easily verified.

(2°) Since $(X, X)=\sum_{i,j}\bar{x}_{ji}x_{ji}$, we know that if $X\neq 0$, then it holds that $(X, X)>0$.

(3°) By (1) and (2) of Lemma 1.2, we have:

$$\begin{aligned}(g\cdot X, g\cdot Y)&=\mathrm{Re}(\mathrm{trace}(\overline{{}^tg\cdot X}\cdot g\cdot Y))=\mathrm{Re}(\mathrm{trace}({}^t\bar{X}\cdot{}^t\bar{g}\cdot g\cdot Y))\\&=\mathrm{Re}(\mathrm{trace}({}^t\bar{X}Y))\qquad=(X, Y);\\(X\cdot g, Y\cdot g)&=\mathrm{Re}(\mathrm{trace}(\overline{{}^tX\cdot g)}\cdot Y\cdot g))=\mathrm{Re}(\mathrm{trace}({}^t\bar{g}\cdot{}^t\bar{X}\cdot Y\cdot g))\\&=\mathrm{Re}(\mathrm{trace}(g\cdot{}^t\bar{g}\cdot{}^t\bar{X}\cdot Y))=\mathrm{Re}(\mathrm{trace}({}^t\bar{X}Y))\qquad=(X, Y).\end{aligned}$$

Q. E. D.

The space $M(n,\boldsymbol{F})$ endowed with the euclidean inner product (,) can be considered as the $n^2\cdot\dim_{\boldsymbol{R}}\boldsymbol{F}$-dimensional real euclidean space. Since the group $G=G(n,\boldsymbol{F})$ is a submanifold of $M(n,\boldsymbol{F})$, there is induced a Riemannian metric ν on G. With this induced metric ν, G is considered as a Riemannian manifold. Let $\boldsymbol{f}$ be the inclusion map of G into $M(n,\boldsymbol{F})$. Then $\boldsymbol{f}$ is an isometric imbedding of the Riemannian manifold $\{G,\nu\}$ into the euclidean space $M(n,\boldsymbol{F})$.

Let $g\in G$. We define $\boldsymbol{R}$-linear endomorphisms l_g and r_g of $M(n,\boldsymbol{F})$ by setting $l_gX=g\cdot X$, $r_gX=X\cdot g$; $X\in M(n,\boldsymbol{F})$. By Proposition 1. 1, we know that both l_g and r_g are isometries of $M(n,\boldsymbol{F})$ with respect to (,). If we set $\mathrm{Ad}(g)=l_g\cdot r_g^{-1}$, then $\mathrm{Ad}(g)$ is also an isometry of $M(n,\boldsymbol{F})$. Since $l_gG=G$, $r_g G=G$ and $\mathrm{Ad}(g)G=G$, they induce isometric diffeomorphisms of G with respect to ν, which are also denoted by l_g, r_g and $\mathrm{Ad}(g)$.

1. 3. Elliptic immersions (see [6], [9]). In this paragraph we recall the definition of elliptic immersions.

Let M be a Riemannian manifold. We denote by $T=T(M)$ the tangent bundle of M and by $T^*=T^*(M)$ the dual of T. Let $\boldsymbol{f}$: $M\rightarrow\boldsymbol{R}^m$ be an isometric immersion of M into $\boldsymbol{R}^m$. We denote by $N=N(M)$ the normal bundle of $\boldsymbol{f}$, which is regarded as a subbundle of the trivial bundle $M\times\boldsymbol{R}^m$. Let ∇ be the covariant differentiation associated with the Riemannian metric of M. Then it is known that for any x, $y\in T_p$, the second derivative $\nabla_x\nabla_y\boldsymbol{f}$ of $\boldsymbol{f}$ takes its value in the normal space N_p. Let $v\in N_p$. We define a symmetric bilinear form $\theta(v)$ on T_p by:

$$\theta(v)(x,y)=(v,\nabla_x\nabla_y\boldsymbol{f}),\ x,\ y\in T_p.$$

$\theta(v)$ is usually called the second fundamental form of $\boldsymbol{f}$ corresponding to v. By definition an isometric immersion $\boldsymbol{f}$ is said to be *non-degenerate* if and only if the bundle homomorphism θ: $N\ni v\mapsto\theta(v)\in S^2T^*$ is injective.

Let $\boldsymbol{f}$: $M\rightarrow\boldsymbol{R}^m$ be a non-degenerate isometric immersion. Then the image $\mathfrak{n}=\theta(N)$ forms a subbundle of S^2T^*, which is called the bundle of second fundamental forms of $\boldsymbol{f}$. By definition a non-degenerate isometric immersion $\boldsymbol{f}$ is said to be *elliptic* if for each $p\in M$, every non-zero element $\theta(v)\in\mathfrak{n}_p$ has at least two eigenvalues of the same sign.

1. 4. The second fundamental forms of $\boldsymbol{f}$: $G\rightarrow M(n,\boldsymbol{F})$. We assert now some facts concerning second fundamental forms of the inclusion map $\boldsymbol{f}$: $G\rightarrow M(n,\boldsymbol{F})$.

Let $\mathfrak{g}$ denote the tangent vector space of G at the identity e, i. e., $\mathfrak{g}=T_e(G)$. As usual $\mathfrak{g}$ may be considered as a subspace of $M(n,\boldsymbol{F})$. Let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{g}$ in $M(n,\boldsymbol{F})$ with respect to (,). Explicitly they are expressed as follows:

$$\mathfrak{g}=\{X\in M(n,\boldsymbol{F})\mid {}^t\bar{X}=-X\},$$

$$\mathfrak{p}=\{X\in M(n,\boldsymbol{F})\mid {}^t\bar{X}=X\}.$$

It can be easily observed that:

$$\mathrm{Ad}(g)\mathfrak{g}=\mathfrak{g},\ \mathrm{Ad}(g)\mathfrak{p}=\mathfrak{p}\ \textit{for}\ g\in G;$$

$$[\mathfrak{g},\ \mathfrak{g}]\subset\mathfrak{g},\ [\mathfrak{g},\ \mathfrak{p}]\subset\mathfrak{p}, [\mathfrak{p},\ \mathfrak{p}]\subset\mathfrak{g}.$$

Let $g\in G$. Since l_g and r_g are both isometries of $M(n,\boldsymbol{F})$, we have: $T_g=l_g\mathfrak{g}=r_g\mathfrak{g}$, $N_g=l_g\mathfrak{p}=r_g\mathfrak{p}$.

LEMMA 1.3. *Let $g\in G$ and X, $Y\in\mathfrak{g}$. Then*:

(1) $\nabla_{l_gX}\nabla_{l_gY}\boldsymbol{f}=\frac{1}{2}l_g(XY+YX)$.

(2) $\nabla_{r_gX}\nabla_{r_gY}\boldsymbol{f}=\frac{1}{2}r_g(XY+YX)$.

PROOF. Let Z be an arbitrary element of $\mathfrak{g}$. We denote by $\tilde{Z}$ the left invariant vector field on G such that $\tilde{Z}_e=Z$. It is well known that $\tilde{Z}$ is autoparallel, i. e., $\Delta_{\tilde{Z}}\tilde{Z}=0$ and the 1-parameter transformation $\mathrm{Exp}(t\tilde{Z})(t\in\boldsymbol{R})$ of G generated by $\tilde{Z}$ is given by $\mathrm{Exp}(tZ)(g)=g\cdot\exp(tZ)$, where $\exp(tZ)=\sum_{n=0}^{\infty}\frac{t^n}{n!}Z^n\ (\in G)$. Hence we have:

$$\begin{aligned}\nabla_{l_gZ}\nabla_{l_gZ}\boldsymbol{f}&=\nabla_{\tilde{Z}_g}\nabla_{\tilde{Z}_g}\boldsymbol{f}=\nabla_{\tilde{Z}_g}(\nabla_{\tilde{Z}}\boldsymbol{f})-\nabla_{(\nabla_{\tilde{Z}}\tilde{Z})_g}\boldsymbol{f}\\&=(\tilde{Z}^2\boldsymbol{f})_g=\left|\frac{d^2}{dt^2}\boldsymbol{f}(g\cdot\exp(tZ))\right|_{t=0}\\&=l_gZ^2.\end{aligned}$$

Therefore putting $Z=X$, Y and $X+Y$ into the above equality, we get

$$\nabla_{l_gX}\nabla_{l_gY}\boldsymbol{f}+\nabla_{l_gY}\nabla_{l_gX}\boldsymbol{f}=l_g(XY+YX).$$

Since $\nabla_{l_gX}\nabla_{l_gY}\boldsymbol{f}=\nabla_{l_gY}\nabla_{l_gX}\boldsymbol{f}$, we obtain the equality (1).

We next show the equality (2). Note that $r_gX=l_g\mathrm{Ad}(g^{-1})X$ and $r_gY=l_g\mathrm{Ad}(g^{-1})Y$. Then from the equality (1) we get:

$$\begin{aligned}\nabla_{r_gX}\nabla_{r_gY}\boldsymbol{f}&=\frac{1}{2}l_g(\mathrm{Ad}(g^{-1})X\cdot\mathrm{Ad}(g^{-1})Y+\mathrm{Ad}(g^{-1})Y\cdot\mathrm{Ad}(g^{-1})X)\\&=\frac{1}{2}r_g(XY+YX).\end{aligned}$$

Q. E. D.

Let g, $h\in G$ and $A\in N_h$. Then by Lemma 1. 3, we can show that $l_g^*\theta(A)=\theta(l_g^{-1}A)$, $r_g^*\theta(A)=\theta(r_g^{-1}A)$. Consequently we have the following

PROPOSITION 1.4. *The image* $\mathfrak{n}=\theta(N)$ *forms a subbundle of* S^2T^* *invariant under the left and right actions of* G, *i. e.,* $l_g^*\mathfrak{n}=\mathfrak{n}$ *and* $r_g^*\mathfrak{n}=\mathfrak{n}$ *for all* $g\in G$. *Therefore*

(1) $\boldsymbol{f}$ *is non-degenerate if and only if the map* $\theta: \mathfrak{p}\to S^2\mathfrak{g}^*$ *is injective.*

(2) *Assume that* $\boldsymbol{f}$ *is non-degenerate. Then* $\boldsymbol{f}$ *is elliptic if and only if every non-zero element* $\theta(A)\in\mathfrak{n}_e$ $(A\in\mathfrak{p})$ *has at least two eigenvalues of the same sign.*

We now state the main theorem of this section.

THEOREM 1.5. *Let* $\boldsymbol{f}: G\to M(n,\boldsymbol{F})$ *be the inclusion map of* $G=G(n,\boldsymbol{F})$ *into* $M(n,\boldsymbol{F})$. *Then*:

(1°) $\boldsymbol{f}$ *is non-degenerate if* G *is either* $SO(n)(n\geqq 3)$, $U(n)(n\geqq 1)$ *or* $Sp(n)(n\geqq 1)$.

(2°) $\boldsymbol{f}$ *is elliptic if* G *is either* $SO(n)(n\geqq 5)$, $U(n)(n\geqq 3)$ *or* $Sp(n)$ $(n\geqq 1)$.

Let $A\in\mathfrak{p}$. We define a symmetric endomorphism $\hat{\theta}(A)$ of $\mathfrak{g}$ with respect to (,) by setting

$$(\hat{\theta}(A)X,\ Y)=-\theta(A)(X,\ Y) \text{ for } X,\ Y\in\mathfrak{g}.$$

Then we have

LEMMA 1.6. $\hat{\theta}(A)X=\frac{1}{2}(AX+XA),\ X\in\mathfrak{g}.$

PROOF. First note that $AX+XA\in\mathfrak{g}$. By Lemma 1.2, we obtain:

$$\begin{aligned}(A,\ XY+YX)&=\mathrm{Re}(\mathrm{trace}({}^t\bar{A}(XY+YX)))\\&=\mathrm{Re}(\mathrm{trace}(AXY+AYX))\\&=\mathrm{Re}(\mathrm{trace}((AX+XA)Y)\\&=-\mathrm{Re}(\mathrm{trace}({}^t\overline{(AX+XA)}Y))\\&=-(AX+XA,\ Y).\end{aligned}$$

Therefore we have the desired equality.

Q. E. D.

Let k and l be integers such that $1\leqq k\leqq n$ and $1\leqq l\leqq n$. We denote by E_{kl} the matrix in $M(n,\boldsymbol{R})$ $(\subset M(n,\boldsymbol{F}))$ whose ij-component is given by $\delta_{ik}\delta_{jl}$, where δ means the Kronecker's delta. We then set $X_{kl}=E_{kl}-E_{lk}$ $(1\leqq k<l\leqq n)$ and $Y_{kl}=E_{kl}+E_{lk}$ $(1\leqq k\leqq l\leqq n)$.

LEMMA 1.7. *Let* A_0 *be a real diagonal matrix in* $M(n,\boldsymbol{F})$, *i. e.,* $A_0=$

$\sum_{i=1}^{n} a_i E_{ii}$ with $a_i \in \boldsymbol{R}$. Then the eigenvalues of the symmetric endomorphism $\hat{\theta}$ (A_0) of $\mathfrak{g}$ is given as follows:

G	*eigenvalues*	*multiplicity*
$SO(n)$	$\frac{1}{2}(a_k+a_l)$ $(1 \leqq k < l \leqq n)$	1
$U(n)$	$a_k (1 \leqq k \leqq n)$	1
	$\frac{1}{2}(a_k+a_l)$ $(1 \leqq k < l \leqq n)$	2
$Sp(n)$	$a_k (1 \leqq k \leqq n)$	3
	$\frac{1}{2}(a_k+a_l)$ $(1 \leqq k < l \leqq n)$	4

PROOF. We choose a basis of $\mathfrak{g}$ composed of the following matrices:

G	basis
$SO(n)$	$X_{kl}(1 \leqq k < q \leqq n)$
$U(n)$	$X_{kl}(1 \leqq k < l \leqq n)$
	$e_1 Y_{kl}(1 \leqq k \leqq l \leqq n)$
$Sp(n)$	$X_{kl}(1 \leqq k < l \leqq n)$
	$e_s Y_{kl}(1 \leqq k \leqq l \leqq n,\ 1 \leqq s \leqq 3)$

Then by a direct calculation, we have

$$\hat{\theta}(A_0) X_{kl} = \frac{1}{2}(a_k + a_l) X_{kl} \qquad (1 \leqq k < l < n);$$

$$\hat{\theta}(A_0)(e_s Y_{kl}) = \frac{1}{2}(a_k + a_l) e_s Y_{kl} \qquad (1 \leqq k \leqq l \leqq n,\ 1 \leqq s \leqq 3).$$

Hence we obtain the lemma.

Q. E. D.

PROOF of THEOREM 1. 5. Owing to Proposition 1. 4, we have only to discuss at the identity $e \in G$. We first note that for each $A \in \mathfrak{p}$, there are an element $g \in G$ and a real diagonal matrix A_0 such that $A = \mathrm{Ad}(g) A_0$. In the case where $G = SO(n)$ or $U(n)$, this assertion is a well known fact. Similarly in the case Where $G = Sp(n)$, we can show the assertion by applying the general theory of orthogonal symmetric Lie algebras. The details are left to the reader.

(1°) Suppose that there is a non-zero $A \in \mathfrak{p}$ such that $\theta(A) = 0$. Let A_0 be a real diagonal matrix conjugate to A under $\mathrm{Ad}(G)$. Then we have θ $(A_0) = 0$ and hence $\hat{\theta}(A_0) = 0$. It cannot happen for a non-zero A_0 except the case where $G = SO(2)$ (see Lemma 1. 7). This proves the assertion (1°).

(2°) We prove that for a non-zero A $\in \mathfrak{p}$, $\hat{\theta}(A)$ has at least two eigenvalues of the same sign. Let A_0 be a real diagonal matrix conjugate to A under $\mathrm{Ad}(G)$. Then both the eigenvalues of $\hat{\theta}(A)$ and $\hat{\theta}(A_0)$ coincide. Therefore we may assume that $A=A_0=\sum_{i=1}^{n} a_iE_{ii}$. We can further assume that $a_1\neq 0$, because $A\neq 0$.

(i) $G=Sp(n)(n\geqq 1)$. By Lemma 1. 7, we know that $\hat{\theta}(A_0)$ has at least three non-zero eigenvalues a_1, a_1, a_1. This proves the assertion.

(ii) $G=U(n)(n\geqq 3)$. If $a_1+a_i\neq 0$ for some $i(\neq 1)$, then by Lemma 1. 7, we know that $\hat{\theta}(A_0)$ has at least two non-zero eigenvalues $\frac{1}{2}(a_1+a_i)$, $\frac{1}{2}(a_1+a_i)$ of the same sign. On the contrary if $a_1+a_i=0$ for all $i(\neq 1)$, then it holds that $a_i=-a_1\neq 0$. Thus $\hat{\theta}(A_0)$ has at least two eigenvalues $a_2,\dots,a_n$ of the same sign. (We are assuming that $n\geqq 3$.)

(iii) $G=SO(n)(n\geqq 5)$. Let us set $k=\#\{i\mid a_1+a_i=0\}$. First assume that $k\geqq 3$. Without loss of generality we can assume that $a_1+a_2=a_1+a_3=a_1+a_4=0$. Then we have $a_2+a_3=a_2+a_4=a_3+a_4=-2a_1\neq 0$. Hence $\hat{\theta}(A_0)$ has at least three non-zero eigenvalues $\frac{1}{2}(a_2+a_3)$, $\frac{1}{2}(a_2+a_4)$, $\frac{1}{2}(a_3+a_4)$ of the same sign. Next assume that $k=2$ and $a_1+a_2=a_1+a_3=0$. Then we have $a_1+a_i\neq 0(4\leqq i\leqq n)$ and $a_2+a_3\neq 0$. Since $n\geqq 5$, we have rank $\hat{\theta}(A_0)\geqq 3$. Hence $\hat{\theta}(A_0)$ has at least two non-zero eigenvalues of the same sign. Finally we assume that $k\leqq 1$ and $a_1+a_i\neq 0(3\leqq i\leqq n)$. Since $n\geqq 5$ it holds that rank $\hat{\theta}(A_0)\geqq 3$. Hence $\hat{\theta}(A_0)$ has at least two non-zero eigenvalues of the same sign.

Q. E. D.

REMARK. In each case where $G=SO(3)$, $SO(4)$ or $U(2)$, we can prove that $\boldsymbol{f}$ is not elliptic. In fact, let us choose a real diagonal matrix $A_0=\sum a_i E_{ii}$ as follows: (i) $G=SO(3)$, $a_1=a_2=-a_3=-a_4$ $(\neq 0)$; (ii) $G=SO(4)$, $a_1=a_2=-a_3=-a_4$ $(\neq 0)$; (iii) $G=U(2)$, $a_1=-a_2$ $(\neq 0)$. Then we have rank $\hat{\theta}(A_0)=2$ and the non-zero eigenvalues of $\hat{\theta}(A_0)$ are of the form $\pm a$ $(\in \boldsymbol{R})$. In the case where $G=U(1)$, $\boldsymbol{f}$ is not elliptic, because dim $G=1$.

§ 2. Results from the representation theory.

In this section we review some facts from the representation theory. We denote by r the rank of G. Then we have $r=[\frac{n}{2}]$ if $G=SO(n)$ and $r=n$ if $G=U(n)$ or $Sp(n)$.

2. 1. Root systems. Let $\{\lambda_i\mid 1\leqq i\leqq r\}$ be the vectors in $\mathfrak{g}$ defined by:

(i) $G=SO(n)$: $\lambda_i=\frac{1}{2}(E_{2i-1,2i}-E_{2i,2i-1})$ $(1\leqq i\leqq r)$.

(ii) $G=U(n)$ or $Sp(n)$: $\lambda_i=e_1E_{i,i}$ $(1\leqq i\leqq r)$.

Then it is easily verified that the vectors $\{\lambda_i | 1 \leqq i \leqq r\}$ are linearly independent and that $[\lambda_i, \lambda_j]=0$, $(\lambda_i, \lambda_j)=c\delta_{ij}$, where $c=\frac{1}{2}$ if $G=SO(n)$ and $c=1$ if $G=U(n)$ or $Sp(n)$. We can also verify that the subspace $\mathfrak{t}$ of $\mathfrak{g}$ generated by the vectors $\{\lambda_i | 1 \leqq i \leqq r\}$ is a maximal abelian subalgebra, i. e., $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let us introduce and fix a linear order "$<$" in $\mathfrak{t}$ satisfying $\lambda_1 > \lambda_2 > \dots > \lambda_r > 0$.

Let $M(n, \boldsymbol{F})^c$ be the complexification of $M(n, \boldsymbol{F})$. (In what follows for a real vector space W, we denote by W^c the complexification of W, i. e., $W^c = W + \sqrt{-1}\, W$.) Addition and multiplication and the bracket operation are naturally defined in $M(n, \boldsymbol{F})^c$. The inner product (,) is naturally extended to a non-degenerate symmetric complex bilinear form of $M(n, \boldsymbol{F})^c$.

By definition, a vector $\alpha \in \mathfrak{t}$ is called a *root* if there is a non-zero vector $Z \in \mathfrak{g}^c$ satisfying $[H, Z] = \sqrt{-1}(\alpha, H)Z$ for each $H \in \mathfrak{t}$. Let Δ be the set of non-zero roots and Π the set of simple roots. (A positive root is said to be *simple* if it cannot be written as a sum of any two positive roots.)

For each $\alpha \in \Delta$, we denote by $\mathfrak{g}_\alpha$ the root subspace of $\mathfrak{g}^c$ corresponding to α, i. e.,

$$\mathfrak{g}_\alpha = \{Z \in \mathfrak{g}^c | [H, Z] = \sqrt{-1}(\alpha, H)Z \text{ for each } H \in \mathfrak{t}\}.$$

Then it is well known that $\dim_c \mathfrak{g}_\alpha = 1$ and $\mathfrak{g}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ *(direct sum).*

We exhibit in the following table the set of non-zero roots Δ and a system of vectors $\{Z_\alpha (\in \mathfrak{g}_\alpha) | \alpha \in \Delta\}$ satisfying $(Z_\alpha, Z_{-\alpha})=1$; $[Z_\alpha, Z_{-\alpha}] = \sqrt{-1}\,\alpha$ and the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_s\}$:

G	α	Z_α	Π
$SO(2r)$	$\pm(\lambda_i - \lambda_j)$ $(1 \leqq i < j \leqq r)$	$\frac{1}{2\sqrt{2}}(U_{ij}^{\pm} + V_{ij}^{\pm})$	$\alpha_i = \lambda_i - \lambda_{i+1} (1 \leqq i \leqq r-1)$
	$\pm(\lambda_i + \lambda_j)$ $(1 \leqq i < j \leqq r)$	$\frac{1}{2\sqrt{2}}(U_{ij}^{\pm} - V_{ij}^{\pm})$	$\alpha = \lambda_{r-1} + \lambda_r$
$SO(2r+1)$	$\pm(\lambda_i - \lambda_j)$ $(1 \leqq i < j \leqq r)$	$\frac{1}{2\sqrt{2}}(U_{ij}^{\pm} + V_{ij}^{\pm})$	$\alpha_i = \lambda_i - \lambda_{i+1} (1 \leqq i \leqq r-1)$
	$\pm(\lambda_i + \lambda_j)$ $(1 \leqq i < j \leqq r)$	$\frac{1}{2\sqrt{2}}(U_{ij}^{\pm} - V_{ij}^{\pm})$	$\alpha_r = \lambda_r$
	$\pm\lambda_i$ $(1 \leqq i \leqq r)$	$\frac{1}{2} U_{i, r+1}$	

$U(n)$	$\pm(\lambda_i-\lambda_j)$ $(1\leqq i<j\leqq n)$	$\frac{1}{2}(X_{ij}\mp\sqrt{-1}\,e_1 Y_{ij})$	$\alpha_i=\lambda_i-\lambda_{i+1}(1\leqq i\leqq n-1)$
$Sp(n)$	$\pm(\lambda_i-\lambda_j)$ $(1\leqq i<j\leqq n)$	$\frac{1}{2}(X_{ij}\mp\sqrt{-1}\,e_1 Y_{ij})$	$\alpha_i=\lambda_i-\lambda_{i+1}(1\leqq i\leqq n-1)$
	$\pm(\lambda_i+\lambda_j)$ $(1\leqq i<j\leqq n)$	$\frac{1}{2}(e_2\mp\sqrt{-1}\,e_3)Y_{ij}$	$\alpha_n=2\lambda_n$
	$\pm 2\lambda_i$ $(1\leqq i\leqq n)$	$\frac{1}{2\sqrt{2}}(e_2\mp\sqrt{-1}\,e_3)Y_{ii}$	

In the above table we put:

$$U_{ij}{}^{\pm}=X_{2i-1,2j-1}\pm\sqrt{-1}\,X_{2i,2j-1}$$

$$V_{ij}{}^{\pm}=X_{2i,2j}\mp\sqrt{-1}\,X_{2i-1,2j}.$$

Now let us set

$$\theta(X,\ Y)=\frac{1}{2}(XY+YX)\ \textit{for}\ X,\ Y\in\mathfrak{g}.$$

Then it is clear that θ is a $\mathfrak{p}$-valued symmetric bilinear form on $\mathfrak{g}$. In a natural way we extend θ to a $\mathfrak{p}^c$-valued symmetric bilinear form on $\mathfrak{g}^c$, which is also denoted by θ. Then for each $X,\ Y,\ Z\in\mathfrak{g}^c$, we have

$$[Z,\ \theta(X,\ Y)]=\theta([Z,X],\ Y)+\theta(X,\ [Z,\ Y])$$

Let $H\in\mathfrak{t}^c$. We denote by $S(H)$ the set given by $S(H)=\{i\,|\,(H,\lambda_i)\neq 0\}$ and put $s(H)=\#S(H)$. Then we have

PROPOSITION 2.1.

(1°) $\theta(\lambda_i,\lambda_j)=0$ *if* $i\neq j$.

(2°) $\theta(\lambda_i,Z_\alpha)=0$ *if* $i\notin S(\alpha)$.

(3°) $\theta(\alpha,Z_\alpha)=0$ *if* $s(\alpha)=2$.

(4°) $\theta(Z_{-\alpha},Z_\alpha)=\frac{1}{(\alpha,\alpha)}\theta(\alpha,\alpha)$ *if* $s(\alpha)=2$.

(5°) $\theta(Z_\alpha,Z_\alpha)=0$ *if* $s(\alpha)=2$.

(6°) $\theta(Z_\alpha,Z_\beta)=0$ *if* $\alpha+\beta\notin\Delta\cup\{0\}$, $S(\alpha)\neq S(\beta)$.

PROOF. (1°) Obvious.

(2°) Choose j such that $j\in S(\alpha)$. Then $i\neq j$ and hence from (1°) we get

$$0=[Z_\alpha, \theta(\lambda_i, \lambda_j)]=\theta([Z_\alpha, \lambda_i], \lambda_j)+\theta(\lambda_i, [Z_\alpha, \lambda_j]).$$

Since $(\alpha, \lambda_i)=0$, we have $[Z_\alpha, \lambda_i]=-\sqrt{-1}(\alpha, \lambda_i)Z_\alpha=0$. Hence:

$$0=\theta(\lambda_i, [Z_\alpha, \lambda_j])=-\sqrt{-1}(\alpha, \lambda_j)\theta(\lambda_i, Z_\alpha).$$

This implies that $\theta(\lambda_i, Z_\alpha)=0$.

(3°) Assume that $S(\alpha)=\{i, j\}(i\neq j)$. Then it is the case of $\alpha=\pm(\lambda_i-\lambda_j)$ or $\alpha=\pm(\lambda_i+\lambda_j)$. Since $i\neq j$, we have

$$0=[Z_\alpha, \theta(\lambda_i, \lambda_j)]=-\sqrt{-1}\,\theta((\alpha, \lambda_i)\lambda_j+(\alpha, \lambda_j)\lambda_i, Z_\alpha).$$

On the other hand since $(\alpha, \lambda_i)\lambda_j+(\alpha, \lambda_j)\lambda_i=\pm c\ \alpha$, it follows that $\theta(\alpha, Z_\alpha)=0$.

(4°) Since $s(\alpha)=2$, we obtain by (3°)

$$0=[Z_{-\alpha}, \theta(\alpha, Z_\alpha)]=\theta([Z_{-\alpha}, \alpha], Z_\alpha)+\theta(\alpha, [Z_{-\alpha}, Z_\alpha])$$
$$=\sqrt{-1}(\alpha, \alpha)\theta(Z_{-\alpha}, Z_\alpha)-\sqrt{-1}\,\theta(\alpha, \alpha).$$

Hence we have $\theta(Z_{-\alpha}, Z_\alpha)=\frac{1}{(\alpha, \alpha)}\theta(\alpha, \alpha)$.

(5°) Since $s(\alpha)=2$, we obtain by (3°)

$$0=[Z_\alpha, \theta(\alpha, Z_\alpha)]=\theta([Z_\alpha, \alpha], Z_\alpha)+\theta(\alpha, [Z_\alpha, Z_\alpha])$$
$$=\sqrt{-1}(\alpha, \alpha)\theta(Z_\alpha, Z_\alpha).$$

Hence we have $\theta(Z_\alpha, Z_\alpha)=0$.

(6°) Assume that $i\in S(\alpha)$, $i\notin S(\beta)$. Then since $(\beta, \lambda_i)=0$, we get from (2°)

$$0=[Z_\alpha, \theta(\lambda_i, Z_\beta)]=\theta([Z_\alpha, \lambda_i], Z_\beta)+\theta(\lambda_i, [Z_\alpha, Z_\beta])$$
$$=-\sqrt{-1}(\alpha, \lambda_i)\theta(Z_\alpha, Z_\beta).$$

Hence we have $\theta(Z_\alpha, Z_\beta)=0$.

Q. E. D.

2. 2. Dominant integral forms. Let $\Gamma(G)$ be the subset of $\mathfrak{t}$ defined by $\Gamma(G)=\{H\in\mathfrak{t}\,|\exp(2\pi H)=e\}$.

By definition a vector $\lambda\in\mathfrak{t}$ is said to be an *integral form* if $(\lambda, H)\in\boldsymbol{Z}$ for each $H\in\Gamma(G)$. An integral form λ is said to be *dominant* if $(\lambda, \alpha_i)\geqq 0$ for each $\alpha_i\in\Pi$.

Let $D(G)$ be the set of all dominant integral forms. Then $D(G)$ is composed of all $\Lambda=\sum m_i\lambda_i$ $(m_i\in\boldsymbol{Z})$ satisfying:

(i) $G=SO(2r)$: $m_1\geqq m_2\geqq \dots \geqq m_{r-1}\geqq|m_r|$;

(ii) $G=SO(2r+1): m_1 \geqq m_2 \geqq \ldots \geqq m_{r-1} \geqq m_r \geqq 0$;

(iii) $G=U(n): m_1 \geqq m_2 \geqq \ldots \geqq m_{n-1} \geqq m_n$;

(iv) $G=Sp(n): m_1 \geqq m_2 \geqq \ldots \geqq m_{n-1} \geqq m_n \geqq 0$.

Let $\rho: G \to \mathrm{GL}(V_\rho)$ be an irreducible representation of G. (A *representation* $\tau: G \to \mathrm{GL}(U)$ always means a "complex continuous" representation, i. e., U is a finite dimensional complex vector space, $\mathrm{GL}(U)$ is the group of complex linear automorphisms of U and τ is a continuous homomorphism of G into $\mathrm{GL}(U)$. We denote by the same letter τ the derivative of τ.)

By definition a vector $\lambda \in \mathfrak{t}$ is said to be a *weight* of ρ (or V_ρ) if there is a non-zero vector $v \in V_\rho$ satisfying $\rho(H)v=\sqrt{-1}(\lambda, H)v$ for each $H \in \mathfrak{t}$. For each weight λ of ρ, we denote by $(V_\rho)_\lambda$ the subspace of V_ρ defined by

$$(V_\rho)_\lambda=\{v \in V_\rho \mid \rho(H)v=\sqrt{-1}(\lambda, H)v \text{ for each } H \in \mathfrak{t}\}.$$

As is well known, each weight of ρ is an integral form and the highest weight $\Lambda(\rho)$ of ρ is a dominant integral form. Let $\mathscr{D}(G)$ be the set of all equivalence classes of all irreducible representations of G. It is well known that the map which assigns the highest weight $\Lambda([\rho])$ to each equivalence class $[\rho] \in \mathscr{D}(G)$ gives a one-to-one correspondence between the sets $\mathscr{D}(G)$ and $D(G)$.

PROPOSITION 2.2. *There exists an isometric linear automorphism κ of* $\mathfrak{t}$ *such that:*

(i) *κ preserves the set of simple roots, i. e., $\kappa\Pi=\Pi$.*

(ii) *For each $[\rho] \in \mathscr{D}(G)$, κ sends the highest weight $\Lambda([\rho])$ onto the highest weight $\Lambda([\rho^*])$ of $[\rho^*]$, i. e., $\kappa\Lambda([\rho])=\Lambda([\rho^*])$, where $\rho^*: G \to \mathrm{GL}(V_\rho^*)$ stands for the dual representation of ρ defined by $\rho^*(g)={}^t\rho(g^{-1})$ $(g \in G)$.*

Explicitly κ is represented as follows:

G	κ
$U(n)$	$\kappa(\lambda_i)=-\lambda_{n-i+1}$ $(1 \leqq i \leqq n)$
$SO(2r)$ (r : *odd*)	$\kappa(\lambda_i)=\lambda_i$ $(1 \leqq i \leqq r-1)$; $\kappa(\lambda_r)=-\lambda_r$
otherwise	$\kappa(\lambda_i)=\lambda_i$ $(1 \leqq i \leqq r)$

PROOF. Such κ having the properties (i) and (ii) is given by the -1 multiple of the unique element κ_1 in the Weyl group $W(G)$ such that $\kappa_1\Pi=-\Pi$ (see [4]). Hence in order to show the proposition it suffices to prove that $\kappa\Pi=\Pi$ and $-\kappa \in W(G)$. Here the fact $\kappa\Pi=\Pi$ can be easily observed. Let

W be the group of permutations of the set $\Omega=\{\lambda_1,\dots,\lambda_r,-\lambda_1,\dots,-\lambda_r\}$ composed of all σ such that $\sigma(-\lambda_i)=-\sigma(\lambda_i)(1\leqq i\leqq r)$. It is well known that by restricting the action of $W(G)$ on $\mathfrak{t}$ to the set Ω, $W(G)$ may be regarded as a subgroup of W. Let $\sigma\in W$. We denote by $I(\sigma)$ the number of λ_i's$(1\leqq i\leqq r)$ such that $\sigma(\lambda_i)<0$. Then the Weyl group can be characterized as follows:

(i) $G=U(n)$: $W(G)=\{\sigma\in W\mid I(\sigma)=0\}$.

(ii) $G=SO(2r)$: $W(G)=\{\sigma\in W\mid I(\sigma): \textit{even}\}$.

(iii) $G=SO(2r+1)$ or $Sp(n)$: $W(G)=W$.

Hence we know that $-\kappa\in W(G)$.

Q. E. D.

2.3. The tensor product $V_\rho\otimes\mathfrak{g}^c$. Let $\tau: G\to GL(U)$ be a representation of G and let $\sigma: G\to GL(V_\sigma)$ be an irreducible representation of G. We denote by $\mathrm{Hom}_G(V_\sigma, U)$ the vector space of all complex linear maps f of V_σ into U satisfying $f(\sigma(g)v)=\tau(g)f(v)$ for $g\in G$, $v\in V_\sigma$. Then the integer $\dim_c \mathrm{Hom}_G(V_\sigma, U)$ indicates the maximum number of linearly independent $\tau(G)$-invariant irreducible submodules in U that are isomorphic to V_σ as G-modules.

Let $\rho: G\to GL(V_\rho)$ be an irreducible representation of G. We define a representation $\rho\otimes\mathrm{Ad}: G\to GL(V_\rho\otimes\mathfrak{g}^c)$ by $g\mapsto\rho(g)\otimes\mathrm{Ad}(g)$. Now let us define an integer $a([\rho],[\sigma])$ by setting $a([\rho],[\sigma])=\dim_c\mathrm{Hom}_G(V_\sigma, V_\rho\otimes\mathfrak{g}^{c*})$. If the highest weights of $[\rho]$ and $[\sigma]$ are respectively given by Λ and M, we also denote by $a(\Lambda,\mathrm{M})$ the integer $a([\rho],[\sigma])$.

PROPOSITION 2.3. (see [5]). *Let* Λ, $\mathrm{M}\in D(G)$. *Then the integer* $a(\Lambda,\mathrm{M})$ *is given as follows*:

(1) *The case* $\mathrm{M}=\Lambda$: $a(\Lambda,\Lambda)=n-\#\{\alpha_i\mid(\Lambda,\alpha_i)=0\}$.

(2) *The case* $\mathrm{M}=\Lambda+\alpha$ *for some* $\alpha\in\Delta$:

$$a(\Lambda,\Lambda+\alpha)=\begin{cases}0 & \textit{if the pair } \{\Lambda,\alpha\} \textit{ is contained in the following table;}\\ 1 & \textit{otherwise.}\end{cases}$$

(3) *The case* $\mathrm{M}\neq\Lambda$, $\Lambda+\alpha$ *for any* $\alpha\in\Delta$: $a(\Lambda,\mathrm{M})=0$.

G	α	Λ
$SO(2r+1)$	$\pm\lambda_i(1\leqq i\leqq r-1)$	$(\Lambda,\lambda_r)=0$
$Sp(n)$	$\pm(\lambda_i+\lambda_{i+1})(1\leqq i\leqq n-1)$	$(\Lambda,\lambda_i-\lambda_{i+1})=0$

PROOF. Let $X\in\mathfrak{g}^c$. We define an element $X^*\in\mathfrak{g}^{c*}$ by setting $X^*(Y)=$

(X, Y) for $Y\in\mathfrak{g}^c$. Then we can easily observe that $(\mathrm{Ad}(g)X)^*=\mathrm{Ad}^*(g)X^*$ for $g\in G$, $X\in\mathfrak{g}^c$. Since $(,)$ is non-degenerate, the map $\mathfrak{g}^c\in X\rightarrow X^*\in\mathfrak{g}^{c*}$ gives an isomorphism between $\mathfrak{g}^c$ and $\mathfrak{g}^{c*}$. Hence we know that the representations Ad and Ad* are equivalent. Therefore except the case where $G=U(n)$, we obtain the proposition by Theorem 2.1 in [5]. In the case where $G=U(n)$, $\mathfrak{g}$ is a direct sum of its center $\mathfrak{z}$ composed of all constant multiples of the identity matrix and the simple ideal $\mathfrak{s}$ composed of all $X\in\mathfrak{g}$ such that trace $X=0$. As is well known, $\mathfrak{s}$ is the Lie algebra of $SU(n)$, the subgroup of $U(n)$ composed of all $g\in U(n)$ such that det $g=1$. Since $\mathfrak{g}=\mathfrak{z}+\mathfrak{s}$ (*direct sum*), we have

$$V_\rho\otimes\mathfrak{g}^{c*}=V_\rho\otimes\mathfrak{z}^{c*}+V_\rho\otimes\mathfrak{s}^{c*}(\textit{direct sum}).$$

Further we know that both $V_\rho\otimes\mathfrak{z}^{c*}$ and $V_\rho\otimes\mathfrak{s}^{c*}$ are invariant under the action $\rho\otimes\mathrm{Ad}$ of G, and $V_\rho\otimes\mathfrak{z}^{c*}$ is irreducible and isomorphic to V_ρ. Thus we have $\dim_c \mathrm{Hom}_G(V_\sigma, V_\rho\otimes\mathfrak{g}^{c*})=\delta_{[\rho],[\sigma]}+\dim_c \mathrm{Hom}_G(V_\sigma, V_\rho\otimes\mathfrak{s}^{c*})$. The integer $\dim_c \mathrm{Hom}_G(V_\sigma, V_\rho\otimes\mathfrak{s}^{c*})$ can be determined by a method similar to that used in the proof of Theoren 2.1 in [5]. The details are left to the reader.

Q. E. D.

§ 3. Tensor fields on G and the differential operator L.

In this section we recall the definition of the differential operator L associated with the inclusion map f and investigate the kernel of L.

3. 1. The differential operator L. For a positive integer k we denote by $\otimes^k T^*$ the k-th tensor product of $T^*=T^*(G)$ and denote by S^kT^* the k-th symmetric tensor product of T^*.

We first define a differential operator $D:\Gamma(T^*)\rightarrow\Gamma(S^2T^*)$ by setting

$$(D\phi)_g(X, Y)=(\nabla_X\phi)(Y)+(\nabla_Y\phi)(X),$$

where $\phi\in\Gamma(T^*)$, $g\in G$ and $X, Y\in T_g$.

Let $S^2T^*/\mathfrak{n}$ denote the quotient bundle of S^2T^* by the bundle $\mathfrak{n}$ of second fundamental forms associated with f and let π denote the natural projection $\pi: S^2T^*\rightarrow S^2T^*/\mathfrak{n}$. The differential operator L is then defined as the composite $\pi\cdot D$, i. e., $L=\pi\cdot D:\Gamma(T^*)\rightarrow\Gamma(S^2T^*/\mathfrak{n})$.

Let *Ker* L denote the kernel of the differential operator L, i. e., the subspace of $\Gamma(T^*)$ composed of all ϕ such that $L\phi=0$. Evidently we have

$$Ker\ L=\{\phi\in\Gamma(T^*)\,|\,D\phi\in\Gamma(\mathfrak{n})\}.$$

Let us define the actions L and R of G on $\Gamma(\otimes^kT^*)$ by $L(g)\phi=$

$(l_{g^{-1}})^{*}\phi$; $R(g)\phi=(r_g)^{*}\phi$ ($\phi\in\Gamma(\otimes^{k}T^{*})$, $g\in G$). Then we can easily show that $L(g)\cdot R(h)=R(h)\cdot L(g)$ for g, $h\in G$.

PROPOSITION 3.1. *The space Ker L is invariant under the actions L and R of G, i. e.,*

$$L(g)(Ker\ L)\subset Ker\ L;\ R(g)(Ker\ L)\subset Ker\ L \quad for\ g\in G.$$

PROOF. We first note that $L(g)\ \Gamma(\mathfrak{n})\subset\Gamma(\mathfrak{n})$ and $R(g)\ \Gamma(\mathfrak{n})\subset\Gamma(\mathfrak{n})$. These facts follow from Proposition 1.4. Further note that since $l_{g^{-1}}$ and r_g are isometries of G, we have $L(g)\cdot D=D\cdot L(g)$ and $R(g)\cdot D=D\cdot R(g)$.

We now assume that $\phi\in Ker\ L$. Then we have $D\phi\in\Gamma(\mathfrak{n})$. Hence we get $D\mathrm{L}(g)\phi=L(g)D\phi\in\Gamma(\mathfrak{n})$ and $DR(g)\phi=R(g)D\phi\in\Gamma(\mathfrak{n})$. Therefore we have $L(g)\phi$ and $R(g)\phi\in Ker\ L$, proving the prposition.

Q. E. D.

3.2. Tensor fields on *G*. In what follows all the objects are considered in the complex category. All the real vector spaces are complexified and $\boldsymbol{R}$-linear maps are extended to $\boldsymbol{C}$-linear maps in a natural way.

Let $C^{\infty}(G)$ be the algebra of complex valued C^{∞} functions on G. We define a norm $\|\ \ \|$ in $C^{\infty}(G)$ by

$$\|f\|=\max_{x\in G}|f(x)|,\ f\in C^{\infty}(G).$$

We introduce a topology in $C^{\infty}(G)$ determined by this norm. Two actions L and R of G on $C^{\infty}(G)$ are defined by $L(g)f=(l_{g^{-1}})^{*}f$; $R(g)f=(r_2)^{*}f$, $f\in C^{\infty}(G)$, $g\in G$. We note that by the action L (or R), G acts continuously on each finite dimensional invariant subspace of $C^{\infty}(G)$.

Let $\mathfrak{o}(G)$ be the subspace of $C^{\infty}(G)$ composed of all $f\in C^{\infty}(G)$ such that the $L(G)$-orbit passing through f is contained in a finite dimensional subspace of $C^{\infty}(G)$. Clearly we know that $\mathfrak{o}(G)$ is $R(G)$-invariant.

Let $\rho: G\rightarrow\mathrm{GL}(V_{\rho})$ be an irreducible representation of G. We denote by $\mathfrak{o}_{[\rho]}(G)$ the sum of irreducible $L(G)$-submodules of $C^{\infty}(G)$ isomorphic to V_{ρ} as G-modules. Then we can easily see that $\mathfrak{o}_{[\rho]}(G)$ is an $L(G)$-submodule of $\mathfrak{o}(G)$ and is also $R(G)$-invariant.

PROPOSITION 3.2 (Theorem of Peter-Weyl).

(1) *The $L(G)$-submodule $\mathfrak{o}(G)$ is dense in $C^{\infty}(G)$ w. r. t. the topology determined by the norm $\|\ \ \|$ and*

$$\mathfrak{o}(G)=\sum_{[\rho]\in\mathscr{D}(G)}\mathfrak{o}_{[\rho]}(G)\ (direct\ sum).$$

(2) *For each $[\rho]\in\mathscr{D}(G)$, define a $\boldsymbol{C}$-linear map ϕ_{ρ}: $V_{\rho}\otimes V_{\rho}^{*}\rightarrow C^{\infty}(G)$ by*

$$\Phi_\rho(v\otimes\xi)(g)=\xi(\rho(g^{-1})v),\ v\in V_\rho,\ \xi\in V_\rho^* \text{ and } g\in G.$$

Then Φ_ρ *gives an isomorphism between* $V_\rho\otimes V_\rho^*$ *and* $\mathfrak{o}_{[\rho]}(G)$ *and*

$$L(g)\Phi_\rho(v\otimes\xi)=\Phi_\rho(\rho(g)v\otimes\xi),$$

$$R(g)\Phi_\rho(v\otimes\xi)=\Phi_\rho(v\otimes\rho^*(g)\xi).$$

Let $\phi\in\Gamma(\otimes^k T^{*c})$. We define an element $\hat{\phi}\in C^\infty(G)\otimes(\otimes^k\mathfrak{g}^c)^*$ by setting

$$\hat{\phi}(X_1,\dots,X_k)(g)=\phi_g(l_gX_1,\dots,l_gX_k),$$

where $X_1,\dots,X_k\in\mathfrak{g}^c$ and $g\in G$. Then the assignment $\phi\mapsto\hat{\phi}$ gives an isomorphism between $\Gamma(\otimes^k T^*)$ and $C^\infty(G)\otimes(\otimes^k\mathfrak{g}^c)^*$. We identify them by this isomorphism. Then the following lemma is easy to see.

LEMMA 3.3. *Let* $\phi\in\Gamma(\otimes^k T^{*c})$. *Then*:

$$(L(g)\phi)^\wedge(X_1,\dots,X_k)(a)=\hat{\phi}(X_1,\dots,X_k)(g^{-1}a);$$

$$(R(g)\phi)^\wedge(X_1,\dots,X_k)(a)=\hat{\phi}(Ad(g^{-1})X_1,\dots,Ad(g^{-1})X_k)(ag),$$

where $X_1,\dots,X_k\in\mathfrak{g}^c$, g, $a\in G$.

We now define a norm $\|\ \|^{(k)}$ in $\Gamma(\otimes^k T^{*c})$ by

$$\|\phi\|^{(k)}=\max_{\{i_1,\dots,i_k\}}|\hat{\phi}(E_{i_1},\dots,E_{i_k})|,$$

where $\phi\in\Gamma(\otimes^k T^{*c})$ and $\{E_i\}$ denotes an orthonormal basis of $\mathfrak{g}$ *w. r. t.* the inner product $(,)$. We introduce a topology in $\Gamma(\otimes^k T^{*c})$ determined by this norm. Then it can easily seen that by the action L(or), G acts continuously on each finite dimensional invariant subspace of $\Gamma(\otimes^k T^{*c})$.

Let us set $\mathfrak{o}(\otimes^k T^{*c})=\mathfrak{o}(G)\otimes(\otimes^k\mathfrak{g}^c)^*$ and set for each $[\rho]\in\mathscr{D}(G)$, $\mathfrak{o}_{[\rho]}(\otimes^k T^{*c})=\mathfrak{o}_{[\rho]}(G)\otimes(\otimes^k\mathfrak{g}^c)^*$. Then by Lemma 3.3, we know that $\mathfrak{o}(\otimes^k T^{*c})$ and $\mathfrak{o}_{[\rho]}(\otimes^k T^{*c})$ are invariant under the actions L and R of G. Moreover we know that $\mathfrak{o}(\otimes^k T^{*c})$ is the set of all $\phi\in\Gamma(\otimes^k T^{*c})$ such that the $L(G)$-orbit passing through ϕ is contained in a finite dimensional subspace of $\Gamma(\otimes^k T^{*c})$ and that for each $[\rho]\in\mathscr{D}(G)$, $\mathfrak{o}_{[\rho]}(\otimes^k T^{*c})$ is the sum of irreducible $L(G)$-submodules of $\Gamma(\otimes^k T^{*c})$ isomorphic to V_ρ as G-modules.

PROPOSITION 3.4. (1) *The subspace* $\mathfrak{o}(\otimes^k T^{*c})$ *is dense in* $\Gamma(\otimes^k T^{*c})$ *w. r. t. the topology determined by the norm* $\|\ \|^{(k)}$ *and*

$$\mathfrak{o}(\otimes^k T^{*c})=\sum_{[\rho]\in\mathscr{D}(G)}\mathfrak{o}_{[\rho]}(\otimes^k T^{*c})\,(\textit{direct sum}).$$

(2) *For each* $[\rho]\in\mathscr{D}(G)$, *set* $\Phi_\rho^{(k)}=\Phi_\rho\otimes(\otimes^k\mathrm{Ad}^*)$. *Then* $\Phi_\rho^{(k)}$ *gives an isomorphism between* $V_\rho\otimes V_\rho^*\otimes(\otimes^k\mathfrak{g}^{c*})$ *and* $\mathfrak{v}_{[\rho]}(\otimes^k\mathrm{T}^{*c})$ and:

$$L(g)\Phi_\rho^{(k)}(v\otimes\xi)=\Phi_\rho^{(k)}(\rho(g)v\otimes\xi);$$

$$R(g)\Phi_\rho^{(k)}(v\otimes\xi)=\Phi_\rho^{(k)}(v\otimes\rho_k^*(g)\xi),$$

where $v\in V_\rho$, $\xi\in V_\rho^*\otimes(\otimes^k\mathfrak{g}^{c*})$, $g\in G$ *and* $\rho_k^*=\rho^*\otimes(\otimes^k\mathrm{Ad}^*)$.

3.3. The space *Ker L*. Let $\rho: G\to\mathrm{GL}(V_\rho)$ be an irreducible representation of G. We define a $\boldsymbol{C}$-linear map $E_\rho: V_\rho^*\otimes\mathfrak{g}^{c*}\to V_\rho^*\otimes S^2\mathfrak{g}^{c*}$ by setting

$$E_\rho\xi(X,Y)=\rho^*(X)(\xi(Y))+\rho^*(Y)(\xi(X)),\ \xi\in V_\rho^*\otimes\mathfrak{g}^{c*},\ X,\ Y\in\mathfrak{g}^c.$$

Then we have the following

LEMMA 3.5. (1°) $\rho_2^*(g)\cdot E_\rho=E_\rho\cdot\rho_1^*(g)$ *for each* $g\in G$.

(2°) *Let* $v\in V_\rho$ *and* $\xi\in V_\rho^*\otimes\mathfrak{g}^{c*}$. *Then*:

$$D\Phi_\rho^{(1)}(v\otimes\xi)=\Phi_\rho^{(2)}(v\otimes E_\rho\xi).$$

PROOF (1°) The assertion follows from a direct calculation.

(2°) Let us set $\phi=\Phi_\rho^{(1)}(v\otimes\xi)$. Let $X\otimes\mathfrak{g}$ be an arbitrary element. Then for each $g\in G$, we heve

$$\begin{aligned}(D\phi)^\wedge(X,X)(g)&=(D\phi)_g(l_gX,l_gX)=(D\phi)_g(\tilde{X}_g,\tilde{X}_g)\\&=2(\nabla_{\tilde{X}_g}\phi)(\tilde{X}_g)\\&=2\{\tilde{X}_g(\phi(\tilde{X}))-\phi((\nabla_{\tilde{X}}\tilde{X})_g)\\&=2\ \tilde{X}_g(\hat{\phi}(X)).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\tilde{X}_g(\hat{\phi}(X))&=\frac{\mathrm{d}}{\mathrm{dt}}\hat{\phi}(X)(g\ \exp tX)|_{t=0}\\&=\frac{\mathrm{d}}{\mathrm{dt}}\xi(X)(\rho((g\ \exp tX)^{-1})v)|_{t=0}\\&=\rho^*(X)(\xi(X))(\rho(g^{-1})v)\\&=\frac{1}{2}\Phi_\rho^{(2)}(v\otimes E_\rho\xi)(X,X)(g).\end{aligned}$$

Hence we have:

$$(D\phi)^\wedge(X,X)=\Phi_\rho^{(2)}(v\otimes E_\rho\xi)(X,X).$$

Since X is an arbitrary element in $\mathfrak{g}$, we obtain the equality $(D\phi)^\wedge = \Phi_\rho^{(2)}(v \otimes E_\rho\xi)^\wedge$ and hence $D\phi = \Phi_\rho^{(2)}(v \otimes E_\rho\xi)$. This proves the assertion.

Q. E. D.

Let $\rho : G \to GL(V_\rho)$ be an irreducible representation of G. We denote by $L_{[\rho]}$ the restriction of the differential operator L to $\mathfrak{v}_{[\rho]}(T^{*c})$ and by *Ker* $L_{[\rho]}$ the kernel of $L_{[\rho]}$. Then we have

$$Ker\ L_{[\rho]} = \{\phi \in \mathfrak{v}_{[\rho]}(T^{*c}) \mid D\phi \in \Gamma(\mathfrak{n}^c)\}.$$

We now define a subspace $W(\rho)$ of $V_\rho^* \otimes \mathfrak{g}^{c*}$ by

$$W(\rho) = \{\xi \in V_\rho^* \otimes \mathfrak{g}^{c*} \mid E_\rho\xi \in V_\rho^* \otimes \mathfrak{n}_e^c\}.$$

Then we have

PROPOSITION 3. 6.

(1) $(Ker\ L)^c \cap \mathfrak{v}(T^{*c}) = \sum_{[\rho]\in\mathscr{D}(G)} Ker\ L_{[\rho]}$ *(direct sum).*

(2) *For each* $[\rho] \in \mathscr{D}(G)$, $W(\rho)$ *is invariant under the action* ρ_1^* *of* G *and* :

$$Ker\ L_{[\rho]} = \Phi_\rho^{(1)}(V_\rho \otimes W(\rho)).$$

PROOF. We first note that under the identification $\Gamma(\otimes^2 T^{*c}) \ni \phi \mapsto \hat{\phi} \in C^\infty(G) \otimes (\otimes^2 \mathfrak{g}^c)^*$, $\Gamma(\mathfrak{n}^c)$ is identified with $C^\infty(G) \otimes \mathfrak{n}_e^c$. Hence we have $\Gamma(\mathfrak{n}^c) \cap \mathfrak{v}(\otimes^2 T^{*c}) = \mathfrak{v}(G) \otimes \mathfrak{n}_e^c$; $\Gamma(\mathfrak{n}^c) \cap \mathfrak{v}_{[\rho]}(\otimes^2 T^{*c}) = \mathfrak{v}_{[\rho]}(G) \otimes \mathfrak{n}_e^c$ for each $[\rho] \in \mathscr{D}(G)$. Consequently we have :

$$\Gamma(\mathfrak{n}^c) \cap \mathfrak{v}(\otimes^2 T^{*c}) = \sum_{[\rho]\in\mathscr{D}(G)} \Gamma(\mathfrak{n}^c) \cap \mathfrak{v}_{[\rho]}(\otimes^2 T^{*c}) \quad \textit{(direct sum)}.$$

Next we note that since $L(g)\cdot D = D\cdot L(g)$ for $g \in G$, it follows that $D(\mathfrak{v}(T^{*c})) \subset \mathfrak{v}(\otimes^2 T^{*c})$ and $D(\mathfrak{v}_{[\rho]}(T^{*c})) \subset \mathfrak{v}_{[\rho]}(\otimes^2 T^{*c})$ for each $[\rho] \in \mathscr{D}(G)$. Then the assertion (1) can be easily verified.

We now show the assertion (2). Let $g \in G$. Then by (1°) of Lemma 3. 5, we obtain that

$$E_\rho(\rho_1^*(g) W(\rho)) = \rho_2^*(g)(E_\rho(W(\rho))) \subset \rho_2^*(g)(V_\rho^* \otimes \mathfrak{n}_e^c).$$

On the other hand since $\otimes^2 \mathrm{Ad}^*(g)(\mathfrak{n}_e^c) = \mathfrak{n}_e^c$(see Proposition 1. 4), we have $\rho_2^*(g)(V_\rho^* \otimes \mathfrak{n}_e^c) \subset V_\rho^* \otimes \mathfrak{n}_e^c$. Hence we have $E_\rho(\rho_1^*(g) W(\rho)) \subset V_\rho^* \otimes \mathfrak{n}_e^c$, proving that $\rho_1^*(g) W(\rho) \subset W(\rho)$. Let $\{v_i\}$ be a basis of V_ρ and let $\sum v_i \otimes \xi_i$ $(\xi_i \in V_\rho^* \otimes \mathfrak{g}^{c*})$ be an arbitrary element of $V_\rho \otimes V_\rho^* \otimes \mathfrak{g}^{c*}$. Then by (2°) of Lemma 3. 5, we have

$$D\Phi_\rho^{(1)}(\sum v_i \otimes \xi_i) = \Phi_\rho^{(2)}(\sum v_i \otimes E_\rho\xi_i).$$

Hence if $\Phi_{\rho}^{(1)}(\sum v_i\otimes \xi_i)\in Ker\ L_{[\rho]}$, Then we have $E_{\rho}\xi_i\in V_{\rho}^{*}\otimes \mathfrak{n}_e^c$ for each ξ_i. This implies that $\xi_i\in W(\rho)$. Consequently we know that $Ker\ L_{[\rho]}\subset\Phi_{\rho}^{(1)}(V_{\rho}\otimes W(\rho))$. On the contrary it is obvious that $Ker\ L_{[\rho]}\supset\Phi_{\rho}^{(1)}(V_{\rho}\otimes W(\rho))$. Thus we have $Ker\ L_{[\rho]}=\Phi_{\rho}^{(1)}(V_{\rho}\otimes W(\rho))$.

Q. E. D.

Let $\sigma: G\rightarrow \mathrm{GL}(V_{\sigma})$ be an irreducible represntation. We define an integer $c([\rho], [\sigma])$ by

$$c([\rho], [\sigma])=\dim_c \mathrm{Hom}_G(V_{\sigma}, W(\rho)).$$

If the highest weights of $[\rho]$ and $[\sigma]$ are respectively represented by Λ and M, then we also denote by $c(\Lambda, \mathrm{M})$ the integer $c([\rho], [\sigma])$.

PROPOSITION 3.7. (1°) *Let* $[\rho]$ *and* $[\sigma]\in\mathscr{D}(G)$ *Then*:

(a) $c([\rho], [\sigma])\leqq a([\rho^*], [\sigma])$. *In particular if* $[\rho^*]\neq[\sigma]$, *then* $c([\rho], [\sigma])\leqq 1$.

(b) *Let* $[\rho]\in\mathscr{D}(G)$. *Then*:

(2°)(i) $\dim W(\rho)=\sum_{[\rho]\in\mathscr{D}(G)} c([\rho], [\sigma]) \dim V_{\rho}$.

(ii) $\dim Ker\ L_{[\rho]}=\sum_{[\rho],\ [\sigma]\in\mathscr{D}(G)} c([\rho], [\sigma]) \dim V_{\rho}\cdot\dim V_{\sigma}$.

PROOF. (1°) (a) Since $W(\rho)$ is a $\rho_1(G)$-submodule of $V_{\rho}^{*}\otimes\mathfrak{g}^{c*}$, we have

$$c([\rho], [\sigma])=\dim_c \mathrm{Hom}_G(V_{\sigma}, W(\rho))\leqq\dim_c \mathrm{Hom}_G(V_{\sigma}, V_{\rho}^{*}\otimes\mathfrak{g}^{c*})$$
$$=a([\rho], [\sigma]).$$

Thus if $[\rho^*]\neq[\sigma]$, then we have $a([\rho^*], [\sigma])=1$ (see Propsition 2.3). This proves the assertion.

(b) Let us define an involutive diffeomorphism ε of G by $\varepsilon(g)=g^{-1}$. Then we have: $\varepsilon(e)=e$; $\varepsilon\cdot l_{g^{-1}}=r_g\cdot\varepsilon$, $\varepsilon\cdot r_g=l_{g^{-1}}\cdot\varepsilon\ (g\in G)$. Consequently we have $\varepsilon_*(X)=-X\ (X\in\mathfrak{g})$; $\varepsilon^*\cdot L(g)=R(g)\cdot\varepsilon^*$, $\varepsilon^*\cdot R(g)=L(g)\cdot\varepsilon^*$ $(g\in G)$. Since ε is an isometry of G, we have $\varepsilon^*\cdot D=D\cdot\varepsilon^*$ (see the definition of D). Moreover we have

LEMMA 3.8. *The bundle* $\mathfrak{n}$ *of second fundamental forms is invariant by* ε^*, *i. e.* $\varepsilon^*\mathfrak{n}=\mathfrak{n}$.

PROOF. We have to show that it holds that $\varepsilon^*\mathfrak{n}_g=\mathfrak{n}_{g^{-1}}$ for each $g\in G$. Let $A\in\mathfrak{p}$, X, $Y\in\mathfrak{g}$. Then:

$$(\varepsilon^*\theta(l_gA))(l_{g^{-1}}X, l_{g^{-1}}Y)=\theta(l_gA)(\varepsilon_*l_{g^{-1}}X, \varepsilon_*l_{g^{-1}}Y)$$

$$=\theta(l_gA)(-r_gX,\ -r_gY)=(l_gA,\ \theta(r_gX,\ r_gY))$$
$$=(l_gA,\ r_g\theta(X,\ Y)) \qquad =(r_{g^{-1}}A,\ l_{g^{-1}}\theta(X,\ Y))$$
$$=\theta(r_{g^{-1}}A)(l_{g^{-1}}X,\ l_{g^{-1}}Y).$$

Hence we have $\varepsilon^*\theta(l_gA)=\theta(r_{g^{-1}}A)\in\mathfrak{n}_{g^{-1}}$. This proves the lemma.

Q. E. D.

Let W_σ (resp. $W\rho$) be the sum of $\rho_1(G)$-submodules of $W(\rho)$ (resp. $W(\sigma)$) that are isomorphic to V_σ (resp. V_ρ) as G-modules. Then W_σ is isomorphic to the direct sum of $c([\rho],\ [\sigma])$-copies of V_σ and W_ρ is isomorphic to the direct sum of $c([\sigma],\ [\rho])$-copies of V_ρ.

Now let us set $U=\Phi_\rho^{(1)}(V_\rho\otimes W_\sigma)$. Then it is easy to see that U is invariant under the actions L and R of G and U is isomorphic to the direct sum of $c([\rho],\ [\sigma])$.$\cdot$dim V_ρ-copies of V_σ under the action R (see Proposition 3.4). Thus ε^*U is invariant under the actions L and R of G. Under the action L, ε^*U is isomorphic to the direct sum of $c([\rho],\ [\sigma])\cdot$dim V_ρ-copies of V_σ (and hence $\varepsilon^*U\subset\mathfrak{o}_{[\sigma]}(T^{*c})$) and under the action R, ε^*U is isomorphic to the direct sum of $c([\rho],\ [\sigma])\cdot$dim V_σ-copies of V_ρ. On the other hand since $D(U)\subset\Gamma(\mathfrak{n}^c)$, we obtain by Lemma 3.8 that $D(\varepsilon^*U)=\varepsilon^*D(U)\subset\Gamma(\mathfrak{n}^c)$. Therefore we have $\varepsilon^*U\subset Ker\ L_{[\sigma]}$. Hence we know that $\varepsilon^*U\subset\Phi_\rho^{(1)}(V_\sigma\otimes W_\rho)$. Thus we have $c([\rho],\ [\sigma])\leqq c([\sigma],\ [\rho])$. If we permute $[\rho]$ and $[\sigma]$, we get $c([\sigma],\ [\rho])\leqq c([\rho],\ [\sigma])$. This implies that the equality $c([\rho],\ [\sigma])=c([\sigma],\ [\rho])$.

The assertion (2°) (i) and (ii) can be obtained by the very definitions.

Q. E. D.

§4. Determination of the space *Ker* $\dot{L}$.

In this section we consider the case where the inclusion map f is elliptic. We assume that G is either $SO(n)(n\geqq5)$, $U(n)(n\geqq3)$ or $Sp(n)(n\geqq1)$.

4.1. Extreme vectors η. Let $\rho: G\to\mathrm{GL}(V_\rho)$ and $\sigma: G\to\mathrm{GL}(V_\sigma)$ be two irreducible representations of G. We denote by Λ, M the highest weights of $[\rho]$ and $[\sigma]$ respectively and denote by Λ^*, M^* the highest weights of $[\rho^*]$ and $[\sigma^*]$ respectively.

Now let us assume that $c([\rho],\ [\sigma])\neq0$. Then we have $a([\rho^*],\ [\sigma])=a(\Lambda^*,\ \mathrm{M})\neq0$ (see Proposition 3.7). Therefore by Proposition 2.3, we know that the following two cases are possible:

(A): $\mathrm{M}=\Lambda^*+\alpha_0$ for some $\alpha_0\in\Delta$.
(B): $\mathrm{M}=\Lambda^*$.

Let $\mathscr{S}$ be the set of all pairs $\{\Lambda, \mathrm{M}\}\in D(G)\times D(G)$ such that $c(\Lambda, \mathrm{M})\neq 0$. Then the set $\mathscr{S}$ is divided into two subsets according as $\mathrm{M}\neq\Lambda^*$ ($\mathscr{S}_A$) or $\mathrm{M}=\Lambda^*$ ($\mathscr{S}_B$). Further the set $\mathscr{S}_A$ is divided into two subsets according as $\mathrm{M}>\Lambda^*$ ($\mathscr{S}_{A^+}$) or $\mathrm{M}<\Lambda^*$ ($\mathscr{S}_{A^-}$). We remark that the set $\mathscr{S}_{A^-}$ can be easily obtained by the set $\mathscr{S}_{A^+}$. In fact let $\{\Lambda, \mathrm{M}\}\in\mathscr{S}_{A^-}$. Then there exists a negative root α_0 such that $\mathrm{M}=\Lambda^*+\alpha_0$. Owing to the symmetry $c(\Lambda, \mathrm{M})=c(\mathrm{M}, \Lambda)$ (see Proposition 3. 7), we have $\{\mathrm{M}, \Lambda\}\in\mathscr{S}_A$. Hence there exists a non-zero root β such that $\Lambda=\mathrm{M}^*+\beta$. We now prove $\beta>0$. If we apply κ to both sides of the equality $\mathrm{M}=\Lambda^*+\alpha_0$, we obtain that $\mathrm{M}^*=\Lambda+\kappa(\alpha_0)$ (see Proposition 2. 2). Hence we have $\beta=-\kappa(\alpha_0)$. Since $\kappa\Pi=\Pi$, κ preserves the set of negative roots. Thus we have $\kappa(\alpha_0)<0$ and hence $\beta>0$. Therefore we have $\{\mathrm{M}, \Lambda\}\in\mathscr{S}_{A^+}$. Conversely we can see that if $\{\Lambda, \mathrm{M}\}\in\mathscr{S}_{A^+}$ then $\{\mathrm{M}, \Lambda\}\in\mathscr{S}_{A^-}$. Therefore we have: $\mathscr{S}_{A^-}=\{\{\mathrm{M}, \Lambda\}\mid\{\Lambda, \mathrm{M}\}\in\mathscr{S}_{A^+}\}$.

In the following we determine the sets $\mathscr{S}_{A^+}$ and $\mathscr{S}_B$.

Let $c([\rho], [\sigma])\neq 0$. Let W_σ be the sum of $\rho_1(G)$-submodules of $W(\rho)$ ($\subset V_\rho^*\otimes\mathfrak{g}^{c*}$) isomorphic to V_σ as G-modules. Since $c([\rho], [\sigma])\neq 0$, we have $W_\sigma\neq 0$. Further if $[\rho^*]\neq[\sigma]$, we have $c([\rho], [\sigma])=1$ (see Proposition 3. 7) and hence W_σ is irreducible.

Now let us choose and fix a non-zero extreme vector η in W_σ. (For a representation $\tau: G\rightarrow GL(U)$, a vector $v\in U$ is called *extreme* if it holds $\tau(Z)v=0$ for each $\beta\in\Delta$, $\beta>0$ and $Z\in\mathfrak{g}_\beta$)) If $[\rho^*]\neq[\sigma]$, then W_σ is irreducible and hence η is uniquely determined up to a constant multiple.

We now write down η explicitly. Let $X\in\mathfrak{g}^c$. We define an element $X^*\in\mathfrak{g}^{c*}$ by $X^*(Y)=(X, Y)$ ($Y\in\mathfrak{g}^c$). We then have $\mathrm{Ad}^*(g)X^*=(\mathrm{Ad}(g)X)^*$ for $g\in G$. Hence: $\mathrm{Ad}^*(Y)X^*=(\mathrm{Ad}^*(Y)X)^*=[Y, X]^*$ for $X, Y\in\mathfrak{g}^c$.

Let $\{Z_\alpha\ (\in\mathfrak{g}_\alpha)\mid\alpha\in\Delta\}$ be a system of vectors in $\mathfrak{g}^c$ satisfying: $(Z_\alpha, Z_{-\alpha})=1$ and $[Z_\alpha, Z_{-\alpha}]=\sqrt{-1}\,\alpha$ for each $\alpha\in\Delta$. Let v_{Λ^*} be a non-zero vector in $(V_\rho^*)_{\Lambda^*}$. Then by a similar method used in the proof of Proposition 2. 4 in [5], we have

(A^+) The case where $\{\Lambda, \mathrm{M}\}\in\mathscr{S}_{A^+}$:

$$\eta=v_{\Lambda^*}\otimes Z_{\alpha_0}^*+\sum_{\substack{\alpha\in\Lambda\\ \alpha>\alpha_0}}v_{\mathrm{M}-\alpha}\otimes Z_\alpha^*\ ;\ v_{\mathrm{M}-\alpha}\in(V_\rho^*)_{\mathrm{M}-\alpha}.$$

(B) The case where $\{\Lambda, \mathrm{M}\}\in\mathscr{S}_B$:

$$\eta=v_{\Lambda^*}\otimes H^*+\sum_{\substack{\alpha\in\Delta\\ \alpha>0}}v_{\mathrm{M}-\alpha}\otimes Z_\alpha^*\ ;\ v_{\mathrm{M}-\alpha}\in(V_\rho^*)_{\mathrm{M}-\alpha},\ (0\neq)\ H\in\mathfrak{t}^c.$$

LEMMA 4. 1. *Let β be a positive root.*

(i) *If* $\{\Lambda, M\}\in\mathcal{S}_{A^+}$, *then*:

$$\rho^*(Z_\beta)\ v_{\Lambda^*-\beta}=n_\beta v_{\Lambda^*},$$

where n_β *denotes the constant determined by*: $[Z_{\alpha_0}, Z_\beta]=n_\beta Z_{\alpha_0+\beta}$ *if* $\alpha_0+\beta\in\Delta$; $n_\beta=0$ *if* $\alpha_0+\beta\notin\Delta$.

(ii) *If* $\{\Lambda, \mathrm{M}\}\in\mathcal{S}_B$, *then*:

$$\rho^*(Z_\beta)\ v_{\Lambda^*-\beta}=\sqrt{-1}\ (\beta, H)\ v_{\Lambda^*}.$$

PROOF. Applying $\rho_1^*(Z_\beta)$ to η, we get:

(i) The case where $\{\Lambda, \mathrm{M}\}\in\mathcal{S}_{A^+}$:

$$0=v_{\Lambda^*}\otimes[Z_\beta, Z_{\alpha_0}]^*+\ \dots\ +\rho^*(Z_\beta)v_{\Lambda^*-\beta}\otimes Z^*_{\alpha_0+\beta}+\ \dots\ .$$

(ii) The case where $\{\Lambda, \mathrm{M}\}\in\mathcal{S}_B$:

$$0=v_{\Lambda^*}\otimes[Z_\beta, H]^*+\ \dots\ +\rho^*(Z_\beta)v_{\Lambda^*-\beta}\otimes Z^*_\beta+\ \dots\ .$$

Hence we have (i) $\rho^*(Z_\beta)\ v_{\Lambda^*-\beta}=n_\beta v_{\Lambda^*}$ if $\{\Lambda,\mathrm{M}\}\in\mathcal{S}_{A^+}$ and (ii) $\rho^*(Z_\beta)\ v_{\Lambda^*-\beta}=\sqrt{-1}(\beta, H)\ v_{\Lambda^*}$ if $\{\Lambda,\mathrm{M}\}\in\mathcal{S}_B$.

Q. E. D.

The following lemma plays an important role in the subsequent discussions.

LEMMA 4.2.

(1°) $E_\rho\eta(\lambda_i, \lambda_j)=0$ *if* $i\neq j$.

(2°) $E_\rho\eta(\lambda_i, Z_\alpha)=0$ *if* $i\notin S(\alpha)$.

(3°) $E_\rho\eta(\alpha, Z_\alpha)=0$ *if* $s(\alpha)=2$.

(4°) $E_\rho\eta(Z_{-\alpha}, Z_\alpha)=\dfrac{1}{(\alpha, \alpha)}E_\rho\eta(\alpha, \alpha)$ *if* $s(\alpha)=2$.

(5°) $E_\rho\eta(Z_\alpha, Z_\alpha)=0$ *if* $s(\alpha)=2$.

(6°) $E_\rho(Z_\alpha, Z_\beta)=0$ *if* $\alpha+\beta\notin\Delta\cup\{0\}$, $S(\alpha)\neq S(\beta)$.

PROOF. Since $\eta\in W(\rho)$, we have $E_\rho\eta\in V^*_\rho\otimes\mathfrak{n}^e$. Hence there exist vectors $v_i^*\in V^*_\rho$ and $A_i\in\mathfrak{p}^c$ satisfying $E_\rho\eta(X, Y)=\sum(A_i, \theta(X, Y))\ v_i^*$ for all $X, Y\in\mathfrak{g}^c$. Thus the lemma follows from Proposition 2.1.

Q. E. D.

4.2. The set $\mathcal{S}_{A^+}$. We first prepare the following two lemmas.

LEMMA 4.3. $S(\Lambda^*)\subset S(\alpha_0)$. *Consequently* $s(\Lambda^*)\leqq 2$.

PROOF. Let $i\notin S(\alpha_0)$. Then by (2°) of Lemma 4.2, we have $E_\rho\eta(\lambda_1,$

$Z_{-\alpha_0})=0$. On the other hand a direct calculation shows that

$$E_\rho\eta(\lambda_i, Z_{-\alpha_0})=\rho^*(\lambda_i)(\eta(Z_{-\alpha_0}))+\rho^*(Z_{-\alpha_0})(\eta(\lambda_i))$$
$$=\sqrt{-1}(\Lambda^*, \lambda_i)v_{\Lambda^*}.$$

Since $v_{\Lambda^*}\neq 0$, it follows that $(\Lambda^*, \lambda_i)=0$, i. e., $i\notin S(\Lambda^*)$. Hence we have $S(\Lambda^*)\subset S(\alpha_0)$. Thus we get $s(\Lambda^*)\leqq s(\alpha_0)\leqq 2$ (see § 2. 1).

Q. E. D.

LEMMA 4. 4. *If* $s(\alpha_0)=2$, *then* $(\Lambda^*, \alpha_0)=0$.

PROOF. By (3°) of Lemma 4. 2, we have $E_\rho\eta(\alpha_0, Z_{-\alpha_0})=0$. On the other hand we have:

$$E_\rho\eta(\alpha_0, Z_{-\alpha_0})=\rho^*(\alpha_0)(\eta(Z_{-\alpha_0}))+\rho^*(Z_{-\alpha_0})(\eta(\alpha_0))$$
$$=\sqrt{-1}(\Lambda^*, \alpha_0)v_{\Lambda^*}.$$

Hence we have $(\Lambda^*, \alpha_0)=0$.

Q. E. D.

By Lemma 4. 3, we know that $s(\Lambda^*)=0$, 1 or 2.

I. *The case where* $s(\Lambda^*)=2$. In this case we have $s(\alpha_0)=2$ and $S(\Lambda^*)=S(\alpha_0)$.

LEMMA 4. 5. *Under the above assumptions the following three cases are possible*:

(*a*) $\Lambda^*=m(\lambda_1+\lambda_2)(m>0)$; $\alpha_0=\lambda_1-\lambda_2$.
(*b*) $\Lambda^*=m(\lambda_1-\lambda_r)(m>0)$; $\alpha_0=\lambda_1-\lambda_r$.
(*c*) $\Lambda^*=-m(\lambda_{r-1}+\lambda_r)(m>0)$; $\alpha_0=\lambda_{r-1}-\lambda_r$.

PROOF. Since Λ^* is a dominant integral form with $s(\Lambda^*)=2$, we have

(a') $\Lambda^*=m_1\lambda_1+m_2\lambda_2(m_1\geqq m_2>0)$; $\alpha_0=\lambda_1\pm\lambda_2$.
(b') $\Lambda^*=m_1\lambda_1+m_r\lambda_r(m_1>0>m_r)$; $\alpha_0=\lambda_1\pm\lambda_r$.
(c') $\Lambda^*=m_{r-1}\lambda_{r-1}+m_r\lambda_r(0>m_{r-1}\geqq m_r)$; $\alpha_0=\lambda_{r-1}\pm\lambda_r$.

Since $(\Lambda^*, \alpha_0)=0$ (see Lemma 4. 4), we obtain the three cases stated in the lemma.

Q. E. D.

Case (a). We first assume that $r\geqq 3$ and set $\beta=\lambda_2-\lambda_r(\in\Delta)$. Then we have $\alpha_0+\beta=\lambda_1-\lambda_r\in\Delta$ and hence $v_{\Lambda^*-\beta}\neq 0$ (see Lemma 4. 1). Since $s(\alpha_0+\beta)=2$, we obtain by (3°) of Lemma 4. 2 that $E_\rho\eta(\alpha_0+\beta, Z_{-(\alpha_0+\beta)})=0$. On the other hand we have

$$E_\rho\eta(\alpha_0+\beta, Z_{-(\alpha_0+\beta)})=\rho^*(\alpha_0+\beta)(\eta(Z_{-(\alpha_0+\beta)}))$$
$$+\rho^*(Z_{-(\alpha_0+\beta)})(\eta(\alpha_0+\beta))$$
$$=\sqrt{-1}\,(\Lambda^*-\beta, \alpha_0+\beta)\ v_{\Lambda^*-\beta}.$$

Since $v_{\Lambda^*-\beta}\neq 0$, we have

$$(\Lambda^*-\beta, \alpha_0+\beta)=(m\ \lambda_1+(m-1)\lambda_2+\lambda_r, \lambda_1-\lambda_r)$$
$$=(m-1)c=0.$$

This implies that $m=1$.

We next examine the case where $r\leqq 2$.

$G=SO(5)$: We set $\beta=\lambda_2(\in\Delta)$. Then we have $\alpha_0+\beta=\lambda_1$ and hence $v_{\Lambda^*-\beta}\neq 0$ (see Lemma 4.1). From (2°) of Lemma 4.2 and from a direct calculation, we get

$$0=E_\rho\eta(\lambda_2, Z_{-(\alpha_0+\beta)})=\sqrt{-1}\ (\Lambda^*-\beta, \lambda_2)\ v_{\Lambda^*-\beta}.$$

Hence:

$$(\Lambda^*-\beta, \lambda_2)=(m\lambda_1+(m-1)\lambda_2, \lambda_2)=(m-1)c=0.$$

This implies that $m=1$.

$G=Sp(2)$: We set $\beta=\lambda_1+\lambda_2(\in\Delta)$. Then in an analogous way as in the case $G=SO(5)$, we can prove that $m=1$.

$G=Sp(1)$: This is not the case.

Thus we have obtained the following result in *Case* (a) :

(1) $G(\neq Sp(1))$: $\Lambda^*=\lambda_1+\lambda_2$; $\mathrm{M}=2\lambda_1$; $\alpha_0=\lambda_1-\lambda_2$.

Case (b). This case can happen only if $G=SO(4)$ (see § 2.1) and hence it is excluded from our considerations.

Case (c). This case can happen only if $G=U(n)$ $(n\geqq 3)$. By the same manner as in *Case* (a), we obtain:

(2) $G=U(n)$ $(n\geqq 3)$: $\Lambda^*=-(\lambda_{r-1}+\lambda_r)$; $\mathrm{M}=-2\lambda_r$; $\alpha_0=\lambda_{r-1}-\lambda_r$.

II. *The case where* $s(\Lambda^*)=1$. We first prove that $s(\alpha_0)=1$. In fact if $s(\alpha_0)=2$, then it follows from Lemma 4.4 that $(\Lambda^*, \alpha_0)=0$. However it is impossible because $S(\Lambda^*)\subset S(\alpha_0)$ (see Lemma 4.3). Accordingly we have $s(\alpha_0)=1$. We note that a non-zero root α_0 with $s(\alpha_0)=1$ exist only if $G=SO(2r+1)$ or $Sp(r)$ (see § 2.1). Then the following two cases are possible:

(d) $\Lambda^*=m\lambda_1(m>0)$; $\alpha=k\lambda_1$;

(e) $\Lambda^*=-m\lambda_r(m>0)$; $\alpha=k\lambda_r$,

where $k=1$ if $G=SO(2r+1)$ and $k=2$ if $G=Sp(r)$.

Case (d). We first consider the case where $G=SO(2r+1)$. Then since $(\Lambda^*, \lambda_r)=(m\lambda_1, \lambda_r)=0$, the pair $\{\Lambda^*, \alpha_0\}$ is contained in the table in Proposition 2. 3. Hence we have $a(\Lambda^*, \mathrm{M})=0$ and hence $c(\Lambda, \mathrm{M})=0$ (see Proposition 3. 7).

We next consider the case where $G=Sp(r)(r\geqq 2)$. We set $\beta=\lambda_1-\lambda_2$. Then we have $\alpha_0>\beta$, $\alpha_0+\beta=3\lambda_1-\lambda_2\notin\Delta\cup\{0\}$ and $\mathrm{S}(\alpha_0)\neq\mathrm{S}(\beta)$. Hence from (6°) of Lemma 4. 2 and from a direct calculation, we get

$$0=E_\rho\eta(Z_{-\alpha_0}, Z_{-\beta})=\rho^*(Z_{-\beta})\ v_{\Lambda^*}.$$

This implies that $\Lambda^*-\beta$ is not a weight of V_ρ^*. Therefore we have

$$(\Lambda^*, \beta)=(m\lambda_1, \lambda_1-\lambda_2)=mc=0.$$

Hence we get $m=0$, contradicting the assumption $s(\Lambda^*)=1$.

Finally we conider the case where $G=Sp(1)$. We note that each element of $\mathfrak{n}_e^c$ is given by a constant multiple of the inner product (,) (see § 1). On the other hand, from a direct calculation we get $E_\rho\eta(\lambda_1, \lambda_1)=0$. Hence we have $E_\rho\eta=0$. Thus we have

$$\begin{aligned}0&=E_\rho\eta(Z_{-\alpha_0}, Z_{-\alpha_0})=2\rho^*(Z_{-\alpha_0})(\eta(Z_{-\alpha_0}))\\&=2\rho^*(Z_{-\alpha_0})\ v_{\Lambda^*}.\end{aligned}$$

This implies that $\Lambda^*-\alpha_0$ is not a weight of V_ρ^*. Therefore we have

$$(\Lambda^*, \alpha_0)=(m\lambda_1, 2\lambda_1)=2\ mc=0.$$

Hence we have $m=0$, contradiction the assumption $s(\Lambda^*)=1$.

Case (e). In neither the case $G=SO(2r+1)$ nor $G=Sp(n)$, Λ^* is not a dominant integral form.

III. *The case where* $s(\Lambda^*)=0$, *i. e.*, $\Lambda=\Lambda^*=0$. Since ρ is a trivial representation, we have $V_\rho^*\otimes\mathfrak{g}^{c*}\cong\mathfrak{g}^{c*}\cong\mathfrak{g}^c$. Hence we know that α_0 is equal to the highest root of $\mathfrak{g}^c$. Therefore we have:

(3) $G(=all)$: $\Lambda^*=0$; $\mathrm{M}=\alpha_0=$ *the highest root of* $\mathfrak{g}^c$.

4. 3. The set $\mathscr{S}_{\mathrm{B}}$. As we have seen, each extreme vector η can be written in the form

$$\eta=v_{\Lambda^*}\otimes H^*+\sum_{\substack{\alpha\in\Delta\\ \alpha>0}} v_{\Lambda^*-\alpha}\otimes Z_\alpha^*,$$

where $v_{\Lambda^*-\alpha}\in(V_\rho^*)_{\Lambda^*-\alpha}$ and $(0\neq)\ H\in\mathfrak{t}^c$. Here we remark that η is uniquely determined by H (see the proof of Proposition 2. 4. in [5]).

We now prepare the following three lemmas.

LEMMA 4.6. *$S(\Lambda^*)\subset S(H)$. Consequently $s(\Lambda^*)\leqq s(H)$.*

PROOF. Let $i\notin S(H)$. We choose j such that $j\in S(H)$. Then by (1°) of Lemma 4.2, we have $E_\rho\eta(\lambda_i, \lambda_j)=0$. On the other hand, a direct calculation shows that

$$E_\rho\eta(\lambda_i, \lambda_j)=\sqrt{-1}(\lambda_j, H)(\Lambda^*, \lambda_i)\ v_{\Lambda^*}.$$

Since $(\lambda_j, H)\neq 0$, we have $(\Lambda^*, \lambda_i)=0$, i. e., $i\notin S(\Lambda^*)$. This implies that $S(\Lambda^*)\subset S(H)$.

Q. E. D.

LEMMA 4.7. *If $\Lambda^*\neq 0$, then $s(H)\leqq 2$.*

PROOF. Suppose that $s(H)\geqq 3$. Let j, k and l be three arbitrary distinct integers contained in $S(H)$. Then by (1°) of Lemma 4.2, we have

$$E_\rho\eta(\lambda_j, \lambda_k)=E_\rho\eta(\lambda_k, \lambda_l)=E_\rho\eta(\lambda_l, \lambda_j)=0.$$

Writing $\Lambda^*=\sum m_i\lambda_i$ and $H=\sum h_i\lambda_i$, we get:

$$m_jh_k+m_kh_j=0\,;$$
$$m_kh_l+m_lh_k=0\,;$$
$$m_lh_j+m_jh_l=0.$$

Since $h_j\neq 0$, $h_k\neq 0$, $h_l\neq 0$, we obtain by the above equalities that $m_j=m_k=m_l=0$. Since j, k and l are selected arbitrarily from $S(H)$, we have $s(\Lambda^*)=0$, i. e., $\Lambda^*=0$ (see Lemma 4.6).

Q. E. D.

LEMMA 4.8. *Let β satisfy $\beta>0$ and $s(\beta)=2$. Then it holds either $(\beta, H)=0$ or $(\Lambda^*, \beta)=\frac{1}{2}(\beta, \beta)$.*

PROOF. By (1°) of Lemma 4.1, we have

$$E_\rho\eta(Z_{-\beta}, Z_\beta)=\frac{1}{(\beta, \beta)}E_\rho\eta(\beta, \beta).$$

A direct calculation shows that

$$\rho^*(Z_\beta)\ v_{\Lambda^*-\beta}=\frac{2\sqrt{-1}}{(\beta, \beta)}(\beta, H)(\Lambda^*, \beta)\ v_{\Lambda^*}.$$

On the other hand, the left hand side of the above equality is equal to $\sqrt{-1}(\beta, H)\ v_{\Lambda^*}$ (see Lemma 4.1). Hence we have the lemma.

Q. E. D.

By Lemma 4.6, we know that $s(\Lambda^*)=0$, 1 or 2.

I. *The case where* $s(\Lambda^*)=2$. In this case we have $s(H)=2$ and $S(\Lambda^*)=S(H)$. Therefore the following three cases are possible:

(a) $\Lambda^*=m_1\lambda_1+m_2\lambda_2(m_1\geqq m_2>0)$; $H=h_1\lambda_1+h_2\lambda_2$ $(h_1\neq 0,\ h_2\neq 0)$.

(b) $\Lambda^*=m_1\lambda_1+m_r\lambda_r(m_1>0>m_r)$; $H=h_1\lambda_1+h_r\lambda_r$ $(h_1\neq 0,\ h_r\neq 0)$.

(c) $\Lambda^*=m_{r-1}\lambda_{r-1}+m_r\lambda_r(0>m_{r-1}\geqq m_r)$; $H=h_{r-1}\lambda_{r-1}+h_r\lambda_r$ $(h_{r-1}\neq 0,\ h_r\neq 0)$.

Case (*a*). We first assume that $r\geqq 3$. We set $\beta_1=\lambda_1-\lambda_r(\in\Delta)$ and $\beta_2=\lambda_2-\lambda_r(\in\Delta)$. Then since $(\beta_1, H)=ch_1\neq 0$, $(\beta_2, H)=ch_2\neq 0$, we obtain by Lemma 4.8 that $m_1=2(\Lambda^*,\beta_1)/(\beta_1,\beta_1)=1$ and $m_2=2(\Lambda^*,\beta_2)/(\beta_2,\beta_2)=1$. Hence $\Lambda^*=\lambda_1+\lambda_2$. Now we set $\beta=\lambda_1-\lambda_2$ $(\in\Delta)$. Then since $(\Lambda^*,\beta)=0$, we obtain by Lemma 4.8 that $(\beta, H)=c(h_1-h_2)=0$, i. e., $h_1=h_2$. On the other hand by (1°) of Lemma 4.2, we have

$$0=E_\rho\eta(\lambda_1,\lambda_2)=2\sqrt{-1}\,c^2h_1v_{\Lambda^*}.$$

This is a contradiction.

Next we examine the case where $r\leqq 2$.

$G=SO(5)$ *or* $Sp(2)$: We set $\beta_1=\lambda_1+\lambda_2$ $(\in\Delta)$ and $\beta_2=\lambda_1-\lambda_2$ $(\in\Delta)$. First we assume that $(\beta_1, H)=0$. Then we have $(\beta_1, H)=c(h_1+h_2)=0$, i. e., $h_2=-h_1$. Thus we have $(\beta_2, H)=c(h_1-h_2)=2ch_1\neq 0$. Therefore by Lemma 4.8, we get $2(\Lambda^*,\beta_2)/(\beta_2,\beta_2)=m_1-m_2=1$. On the other hand by (1°) of Lemma 4.2, we obtain

$$0=E_\rho\eta(\lambda_1,\lambda_2)=\sqrt{-1}\,c^2(m_1-m_2)h_1v_{\Lambda^*}$$
$$=\sqrt{-1}\,c^2h_1v_{\Lambda^*}.$$

This is a contradiction. Next we assume that $(\beta_1, H)\neq 0$. Then by Lemma 4.8, we obtain $2(\Lambda^*,\beta_1)/(\beta_1,\beta_1)=m_1+m_2=1$. Thus $2(\Lambda^*,\beta_2)/(\beta_2,\beta_2)=m_1-m_2\neq 1$, because $m_2\neq 0$. Hence by Lemma 4.8, we have $(\beta_2, H)=c(h_1-h_2)=0$, i. e., $h_1=h_2$. Therefore by (1°) or Lemma 4.2, we obtain

$$0=E_\rho\eta(\lambda_1,\lambda_2)=\sqrt{-1}\,c^2(m_1+m_2)h_1v_{\Lambda^*}$$
$$=\sqrt{-1}\,c^2h_1v_{\Lambda^*}.$$

This is a contradiction.

$G=Sp(1)$: This is not the case.

Case (*b*). This case can happen only if $G=U(n)$ $(n\geqq 3)$. We set $\beta_1=\lambda_1-\lambda_2$, $\beta_2=\lambda_2-\lambda_n$. Then we have $(\beta_1\ H)=ch_1\neq 0$, $(\beta_2, H)=-ch_n\neq 0$. Hence by Lemma 4.8, we have $2(\Lambda^*,\beta_1)/(\beta_1,\beta_1)=m_1=1$ and $2(\Lambda^*,\beta_2)/(\beta_2,\beta_2)=-m_n=1$. Hence $\Lambda^*=\lambda_1-\lambda_n$. We then have by (1°) of Lemma 4.2 that

$$0=E_\rho\eta(\lambda_1, \lambda_n)=\sqrt{-1}(h_n-h_1)v_{\Lambda^*}.$$

Hence we have $h_1=h_n$ and hence $H=h_1(\lambda_1+\lambda_n)$. Thus H is uniquely determined up to a constant multiple and hence η is uniquely determined up to a constant multiple. Therefore we have:

(4) $G=U(n)$ $(n\geqq 3)$: $\Lambda^*=\lambda_1-\lambda_n$; $\mathrm{M}=\lambda_1-\lambda_n$; $H=h_1(\lambda_1+\lambda_n)$.

Case (c). This case can happen only if $G=U(n)$ $(n\geqq 3)$. By a similar discussion as in *Case* (a), we also arrive at a contradiction.

II. *The case where* $s(\Lambda^*)=1$. In this case the following two cases are possible:

(d) $\Lambda^*=m\lambda_1(m>0)$.

(e) $\Lambda^*=-m\lambda_r(m>0)$.

Case (d). We first assume that $r\geqq 2$. Then for each $i(1<i\leqq r)$, we obtain by (1°) of Lemma 4. 2 that

$$0=E_\rho\eta(\lambda_1, \lambda_i)=\sqrt{-1}\,cm(H, \lambda_i)v_{\Lambda^*}.$$

Hence $(H, \lambda_i)=0$, i. e., $i\notin S(H)$. Therefore we have $H=h_1\lambda_1$. Thus H is uniquely determined up to a constant multiple, hence η is uniquely determined up to a constant multiple.

Let us set $\beta=\lambda_1-\lambda_2$ $(\in\Delta)$. Then since $(\beta, H)=ch_1\neq 0$, it follows that $2(\Lambda^*, \beta)/(\beta, \beta)=m=1$.

We next consider the case where $G=Sp(1)$. We note that each element of $\mathfrak{n}_e^c$ is given by a constant multiple of the inner product (,). Let us set $\beta=2\lambda_1$ $(\in\Delta)$. Then we have $(\beta, Z_{-\beta})=(Z_{-\beta}, Z_{-\beta})=0$. Therefore we have

$$0=E_\rho\eta(\beta, Z_{-\beta})=\sqrt{-1}(\Lambda^*-\beta,\ \beta)v_{\Lambda^*-\beta}+(\beta, H)\rho^*(Z_{-\beta})v_{\Lambda^*};$$

$$0=E_\rho\eta(Z_{-\beta}, Z_{-\beta})=2\,\rho^*(Z_{-\beta})v_{\Lambda^*-\beta}.$$

By the above equalities we obtain that $\rho^*(Z_{-\beta})\rho^*(Z_{-\beta})v_{\Lambda^*}=0$. This implies that $\Lambda^*-2\beta$ is not a weight of V_ρ^*. Hence we have $2(\Lambda^*, \beta)/(\beta, \beta)=m=1$. As in the case where $r\geqq 2$, η is uniquely determined up to a constant multiple. Thus we have:

(5) G (=*all*) ; $\Lambda^*=\lambda_1$; $\mathrm{M}=\lambda_1$; $H=h_1\lambda_1$ $(h_1\neq 0)$

Case (e). This case can happen only if $G=U(n)$ $(n\geqq 3)$. In the same manner as in *Case* (d), we obtain the following:

(6) $G=U(n)$ $(n\geqq 3)$: $\Lambda^*=-\lambda_n$; $\mathrm{M}=-\lambda_n$; $H=h_n\lambda_n(h_n\neq 0$.

III. *The case where* $s(\Lambda^*)=0$, *i. e.,* $\Lambda=\Lambda^*=0$.
Since ρ is a trivial representation of G, we know that $V_\rho^* \otimes \mathfrak{g}^{c*} \cong \mathfrak{g}^{c*} \cong \mathfrak{g}^c$ and that W_σ is the sum of irreducible $\rho_1(G)$-submodules with $\mathrm{M}=0$ as its highest weight. Hence this case can happen only if $\mathfrak{g}^c$ has a non-trivial center and H is an element of the center. Therefore we know that $G=U(n)(n\geqq 3)$ and hence $H=h(\sum\lambda_i)$. Thus we have

(7) $G=U(n)(n\geqq 3)$: $\Lambda^*=0$; $\mathrm{M}=0$; $H=h(\sum\lambda_i)(h\neq 0)$.

4. 4. The set $\mathscr{S}$. We now exhibit the results obtained in the previous paragraphs. We note that the set $\mathscr{S}_{A^+}$ is obtained by the results (1)~(3) in § 4. 2. The set $\mathscr{S}_{A^-}$ is automatically determined by the set $\mathscr{S}_{A^+}$ (see § 4. 1). The set $\mathscr{S}_B$ is obtained by the results (4)~(7) in § 4. 3. Therefore the set $\mathscr{S}$ is given as follows.

Table I : $G=SO(n)(n\geqq 5)$

Λ	M	$d(\Lambda)$	$d(\mathrm{M})$
$\lambda_1+\lambda_2$	$2\lambda_1$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}(n-1)(n+2)$
$2\lambda_1$	$\lambda_1+\lambda_2$	$\frac{1}{2}(n-1)(n+2)$	$\frac{1}{2}n(n-1)$
0	$\lambda_1+\lambda_2$	1	$\frac{1}{2}n(n-1)$
$\lambda_1+\lambda_2$	0	$\frac{1}{2}n(n-1)$	1
λ_1	λ_1	n	n

Table II : $G=U(n)(n\geqq 3)$

Λ	M	$d(\Lambda)$	$d(\mathrm{M})$
$2\lambda_1$	$-(\lambda_{n-1}+\lambda_n)$	$\frac{1}{2}n(n+1)$	$\frac{1}{2}n(n-1)$
$-(\lambda_{n-1}+\lambda_n)$	$2\lambda_1$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}n(n+1)$
$\lambda_1+\lambda_2$	$-2\lambda_n$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}n(n+1)$
$-2\lambda_n$	$\lambda_1+\lambda_2$	$\frac{1}{2}n(n+1)$	$\frac{1}{2}n(n-1)$
0	$\lambda_1-\lambda_n$	1	n^2-1
$\lambda_1-\lambda_n$	0	n^2-1	1
$\lambda_1-\lambda_n$	$\lambda_1-\lambda_n$	n^2-1	n^2-1
λ_1	$-\lambda_n$	n	n
$-\lambda_n$	λ_1	n	n
0	0	1	1

Table III : $G=Sp(n)$ $(n\geqq 1)$

Λ	M	$d(\Lambda)$	$d(\mathrm{M})$
$\lambda_1+\lambda_2$	$2\lambda_1$	$2n^2-n-1$	$2n^2+n$
$2\lambda_1$	$\lambda_1+\lambda_2$	$2n^2+n$	$2n^2-n-1$
0	$2\lambda_1$	1	$2n^2+n$
$2\lambda_1$	0	$2n^2+n$	1
λ_1	λ_1	$2n$	$2n$

In the above tables, we mean by $d(\mathrm{N})$ $(\mathrm{N}\in D(G))$ the dimensionality of V_τ, where $\tau: G\to \mathrm{GL}(V_\tau)$ is an irreducible representation of G with N as its highest weight. The integer $d(\mathrm{N})$ is calculated by the following Weyl's formula:

$$d(\mathrm{N})=\prod_{\alpha\in\Delta^+}\frac{(\mathrm{N}+\delta,\alpha)}{(\delta,\alpha)},$$

where Δ^+ means the set of positive roots and $\delta=\frac{1}{2}(\sum_{\alpha\in\Delta^+}\alpha)\in \mathfrak{t}$.

Let $\{\Lambda,\mathrm{M}\}\in\mathscr{S}$. We have $c(\Lambda,\mathrm{M})=1$ if $\{\Lambda,\mathrm{M}\}\in\mathscr{S}_A$ (see § 1). Moreover if $\{\Lambda,\mathrm{M}\}\in\mathscr{S}_B$, we also have $c(\Lambda,\mathrm{M})=1$. In fact, as we have remarked in § 4. 3, the extreme vector η in W_σ is uniquely determined up to a constant multiple. Hence we know that W_σ is $\rho_1(G)$-irreducible, i. e., $c(\Lambda,\mathrm{M})=1$.

Now let us consider the space *Ker L*. Since f is elliptic we know that $\dim\ Ker\ L<\infty$ (see [6], [9]). Hence we have $(Ker\ L)^c\subset \mathfrak{v}(T^{*c})$, because *Ker L* is $L(G)$-invariant (see Proposition 3. 1.). Therefore by Proposition 3. 7 we have

$$\dim\ Ker\ L=\sum_{\{\Lambda,\mathrm{M}\}}c(\Lambda,\mathrm{M})\cdot d(\Lambda)\cdot d(\mathrm{M})=\sum_{\{\Lambda,\mathrm{M}\}\in\mathscr{S}}d(\Lambda)\cdot d(\mathrm{M}).$$

By a direct calculation, we obtain that

$$\sum_{\{\Lambda,\mathrm{M}\}\in\mathscr{S}}d(\Lambda)\cdot d(\mathrm{M})=\frac{1}{2}N(N+1),$$

where we set $N=\dim_{\boldsymbol{R}} M(n,\boldsymbol{F})=n^2\cdot\dim_{\boldsymbol{R}}\boldsymbol{F}$.

Thus we have

THEOREM 4. 9. *Assume that G is either $SO(n)$ $(n\geqq 5)$, $U(n)$ $(n\geqq 3)$ or $Sp(n)$ $(n\geqq 1)$. Then the inclusion map $f: G\to M(n,\boldsymbol{F})$ is globally infinitesimally rigid.*

By the above theorem together with Theorem 1. 5, we know that the inclusion map f is globally rigid.

REMARK 1. Strictly speaking, if we denote by $\mathscr{S}'$ the set of pairs $\{\Lambda, \mathrm{M}\}$ in Table I, II or III, then we have $\mathscr{S} \subset \mathscr{S}'$, because $(1^\circ) \sim (6^\circ)$ in Lemma 4.3 are necessary conditions for η to be contained in $W(\rho)$. However our proof may be justified by the fact dim $Ker\ L \geqq \frac{1}{2}N(N+1)$ (see [6], [9]). Hence we have $\mathscr{S} = \mathscr{S}'$.

REMARK 2. Let $\tilde{G} = Spin(n)$ $(n \geqq 5)$ and let $\pi: \tilde{G} \to G = SO(n)$ be the natural covering homomorphism. Then we can show that the composite $\tilde{f} = f \cdot \pi$, which is an elliptic isometric immersion of $\tilde{G}$ into $M(n, \boldsymbol{R})$, is globally rigid. First we note that there is a difference between the sets $D(G)$ and $D(\tilde{G})$; $D(\tilde{G})$ contains elements Λ of the form $\Lambda = \sum_{i=1}^{r} m_i \lambda_i$ $(m_i \in (\frac{1}{2})\boldsymbol{Z})$, where we set $r = [\frac{n}{2}]$. However we remark that the assumption $m_i \in \boldsymbol{Z}$ is not used in the determination of the pairs $\{\Lambda, \mathrm{M}\}$ such that $c(\Lambda, \mathrm{M}) \neq 0$. Therefore the discussions developed in the case where $G = SO(n)$ $(n \geqq 5)$ are also applicable to this case. Then we have dim $Ker\ \tilde{L} = \frac{1}{2}n(n+1)$ $(N = \dim M(n, \boldsymbol{R}))$. Therefore we know that $\tilde{f}$ is rigid in the sense of Tanaka.

4.5. Other results. In this paragraph, we consider the case where the inclusion map $\boldsymbol{f}$: $G \to M(n, \boldsymbol{F})$ is non-dogenerate and is not elliptic, i. e., G is either $SO(n)$ $(n=3$ or $4)$ or $U(n)$ $(n=1$ or $2)$.

We first assume that G is either $SO(3)$ or $U(1)$. Then we know that the bundle $\mathfrak{n}$ of second fundamental forms coincides with the whole bundle $S^2 T^*$. Therefore the equation $L\phi = 0$ is trivial, i. e., every 1-form ϕ on G is a solution of this equation.

Now let us consider the case where $G = SO(4)$ or $U(2)$. We note that in either case the number of linearly independent relations in Lemma 4.2 is equal to dim $S^2\mathfrak{g}^*/\mathfrak{n}_e$. Thus the conditions $(1^\circ) \sim (5^\circ)$ in Lemma 4.2 are necessary and sufficient for η $(\in V_\rho^* \otimes \mathfrak{g}^{c*})$ to be contained in $W(\rho)$. Therefore in the same manner developed in the previous paragraphs, we can determine the integers $c(\Lambda, \mathrm{M})$ for all $\Lambda, \mathrm{M} \in D(G)$. We summarize our results in the following theorem.

THEOREM 4.10. *Assume that G is either $SO(4)$ or $U(2)$. Let Λ, $\mathrm{M} \in D(G)$. Then the integer $c(\Lambda, \mathrm{M})$ is given as follows:*

$$c(\Lambda, \mathrm{M}) = \begin{cases} 1 & \text{if the pair } \{\Lambda, \mathrm{M}\} \text{ is contained in the following Table IV or V.} \\ 0 & \text{otherwise.} \end{cases}$$

Table IV : $G=SO(4)$ $(m\geqq 0,\ m\in\boldsymbol{Z})$

Λ	M
$m(\lambda_1+\lambda_2)$	$(m+1)\lambda_1+(m-1)\lambda_2$
$m(\lambda_1-\lambda_2)$	$(m+1)\lambda_1-(m-1)\lambda_2$
$(m+1)\lambda_1+(m-1)\lambda_2$	$m(\lambda_1+\lambda_2)$
$(m+1)\lambda_1-(m-1)\lambda_2$	$m(\lambda_1-\lambda_2)$
λ_1	λ_1

Table V : $G=U(2)$ $(m\in\boldsymbol{Z})$

Λ	M
$-m(\lambda_1+\lambda_2)$	$(m+1)\lambda_1+(m-1)\lambda_2$
$(m+1)\lambda_1+(m-1)\lambda_2$	$-m(\lambda_1+\lambda_2)$
$m(\lambda_1-\lambda_2)$ $(m\geqq 0)$	$m(\lambda_1-\lambda_2)$
$-(m-1)\lambda_1-m\lambda_2$	$m\lambda_1+(m-1)\lambda_2$

By the above theorem, the space $(Ker\ L)^c\cap\mathfrak{o}(T^{*c})$ is completely determined. As a consequence we know that $Ker\ L$ is of infinite dimension.

References

[1] N. BOURBAKI, Groupes et Algèbres de Lie, Chapîtres 4, 5 et 6, Hermann (1968).

[2] C. CHEVALLEY, Theory of Lie Groups, Princeton Univ. Press (1946).

[3] S. HELGASON, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press (1978).

[4] N. IWAHORI, On real irreducible representations of Lie algebras, Nagoya Math. J. 14 (1959), 59-83.

[5] E. KANEDA, The spectra of 1-forms on simply connected compact irreducible Riemannian symmetric spaces, J. Math. Kyoto Univ. 23 (1983), 369-395.

[6] E. KANEDA and N. TANAKA, Rigidity for isometric imbeddings, J. Math. Kyoto Univ. 18 (1978), 1-70.

[7] S. KOBAYASHI, Isometric imbeddings of compact symmetric spaces, Tohoku Math. J. 20 (1958), 21-25.

[8] M. TAKEUCHI, Gendai no Kyu-kansu, Iwanami (1975) (in Japanese).

[9] N. TANAKA, Rigidity for elliptic isometric imbeddings, Nagoya Math. J. 51 (1973), 137-160.

[10] H. WEYL, The Classical Groups, Princeton Univ. Press (1946).

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