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Compact submanifolds of codimension p of a Sasakian space form.

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Introduction.

In [5] M. Morohashi has shown that an n-dimensional Euclidean sphere S^{n} admits a conformal Killing tensor field of degree p for any positive integer p such that $p<n$. Then H. Kôjyô [3] has constructed a conformal Killing tensor field of degree p inductively, on a Riemannian manifold of constant curvature which admits a conformal Killing vector field. In connection with these conformal Killing tensors of degree p, they and others [1, 6, 7, 9, 13, 15] have studied submanifolds of codimension p of a sphere or a Riemannian manifold of constant curvature and have proved that these submaifolds are totally umbilical under certain conditions.

In this paper, at first we point out that there naturally exist a conformal Killing tensor field of even degree and a Killing tensor field of odd degree on a Sasakian manifold (cf. [14]). Making use of these tensors, we prove that Theorem 5.1 which gives a sufficient condition for a compact submanifold of codimension p in a Sasakian space form to be totally umbilical.

§1. Sasakian space forms.

Let M be a $(2n+1)$-dimensional manifold endowed with an almost contact metric structure (Φ, ξ, η, G), where G is a Riemannian metric, η a 1-form, ξ a vector field and Φ a tensor field of type (1.1) on M which satisfy

$$(1.1)\qquad \Phi_{\lambda}{}^{\kappa}\xi^{\lambda}=0,\ \Phi_{\lambda}{}^{\kappa}\eta_{\kappa}=0,\ \xi^{\lambda}\eta_{\lambda}=1.$$
$$\Phi_{\lambda}{}^{\kappa}\Phi_{\kappa}{}^{\nu}=-\delta_{\lambda}^{\nu}+\eta_{\lambda}\xi^{\nu},\ G_{\lambda\kappa}\Phi_{\mu}{}^{\lambda}\Phi_{\nu}{}^{\kappa}=G_{\mu\nu}-\eta_{\mu}\eta_{\nu}.$$

If, in an almost contact metric manifold M, the structure tensors (Φ, ξ, η, G) satisfy

$$(1.2)\qquad \nabla_{\mu}\Phi_{\lambda\kappa}=\eta_{\lambda}G_{\mu\kappa}-\eta_{\kappa}G_{\mu\lambda},\ \nabla_{\lambda}\xi^{\kappa}=\Phi_{\lambda}{}^{\kappa},$$

where ∇ denotes the covariant derivative with respect to the Riemannian metric $G_{\lambda\kappa}$, the structure is called a Sasakian structure and the manifold M is called a Sasakian manifold (cf. [10]). Moreover, if a Sasakian manifold

M has the curvature tensor of the form

(1.3) $$R_{\nu\mu\lambda\kappa}=(k+1)(G_{\mu\lambda}G_{\nu\kappa}-G_{\nu\lambda}G_{\mu\kappa})+k(\Phi_{\mu\lambda}\Phi_{\nu\kappa}-\Phi_{\nu\lambda}\Phi_{\mu\kappa} -2\Phi_{\nu\mu}\Phi_{\lambda\kappa}+\eta_\nu\eta_\lambda G_{\mu\kappa}+\eta_\mu\eta_\kappa G_{\nu\lambda}-\eta_\mu\eta_\lambda G_{\nu\kappa}-\eta_\nu\eta_\kappa G_{\mu\lambda}),$$

where k is a constant, M is called a Sasakian space form. It is easy to see that if in a Sasakian space form M the constant $k=0$, M is a space of constant curvature 1.

§ 2. Killing tensor fields and conformal Killing tensor fields on a Sasakian manifold.

In this section, we construct a conformal Killing tensor field of degree p $(=2q)$ and a Killing tensor field of degree $p(=2q-1)$ on a Sasakian manifold $M^{2n+1}(2n+1>p)$.

First of all, we recall the definition of confomal Killing tensor fields of degree p(cf. [2]).

Let N^n $(n>3)$ be an n-dimensional Riemannian manifold whose metric tensor field is given by $g_{\lambda\kappa}$. We call a skew symmetric tensor field $u_{\lambda_1\lambda_2\ldots\lambda_p}$ a conformal Killing tensor field of degree p if there exists a skew-symmetric tensor field $\rho_{\lambda_1\lambda_2\ldots\lambda_{p-1}}$ such that

$$\nabla_\lambda u_{\lambda_1\lambda_2\ldots\lambda_p}+\nabla_{\lambda_1}u_{\lambda\lambda_2\ldots\lambda_p}=2\rho_{\lambda_2\ldots\lambda_p}g_{\lambda\lambda_1} -\sum_i(-1)^i(\rho_{\lambda_1\ldots\hat{\lambda}_i\ldots\lambda_p}g_{\lambda\lambda_i}+\rho_{\lambda\lambda_1\ldots\hat{\lambda}_i\ldots\lambda_p}g_{\lambda_1\lambda_i})$$

where $\hat{\lambda}_i$ means that λ_i is ommited. This $\rho_{\lambda_1\lambda_2\ldots\lambda_{p-1}}$ is called the associated tensor field of $u_{\lambda_1\lambda_2\ldots\lambda_p}$. Especially, if the associated tensor field $\rho_{\lambda_1\lambda_2\ldots\lambda_{p-1}}$ is identically zero, the conformal Killing tensor field $u_{\lambda_1\lambda_2\ldots\lambda_p}$ is called a Killing tensor field of degree p.

It is known that on a Sasakian manifold the structure tensor field $\Phi_{\lambda\kappa}$ is a conformal Killing tensor field of degree 2 with the associated tensor field η_λ and the 1-form η_λ is a Killing tensor field of degree 1 (cf. [12]).

Next, we adopt the symbol [ ] defined by Schouten (cf. [11]). For avoiding complicated coefficients in the following computations, we slightly change the meaning of brackets defined by Schouten, that is, we newly define brackets [ ] as $p!/2^{[\frac{p}{2}]}\left[\frac{p}{2}\right]!$ multiply of the old one, where [ ] means the Gauss symbol.

Now, we introduce the following two tensor fields on a Sasakian manifold.

(2.1) $$U_{\lambda_1\lambda_2\ldots\lambda_{2q}}=\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4}\ldots\Phi_{\lambda_{2q-1}\lambda_{2q}]}$$

(2.2) $$V_{\lambda_1\lambda_2\ldots\lambda_{2q-1}}=\eta_{[\lambda_1}\Phi_{\lambda_2\lambda_3}\ldots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}$$

About the brackets [ ], the following formulas are useful for the latter computations (cf. [11]).

$$(2.3)\qquad U_{\lambda_1\ldots\lambda_{2q}}=(-1)^{a-1}\Phi_{\lambda_a[\lambda_1}\Phi_{\lambda_2\lambda_3\ldots\hat{\lambda}_a}\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]}$$
$$=\sum_b(-1)^{a-1+b-2}\Phi_{\lambda_a\lambda_b}\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4\ldots\hat{\lambda}_a\ldots\hat{\lambda}_b}\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]}$$

$$(2.4)\qquad V_{\lambda\ldots\lambda_{2q-1}}=\sum_a(-1)^{a-1}\eta_{\lambda_a}\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4\ldots\hat{\lambda}_a}\cdots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}$$

S. Yamaguchi proved the following two Lemmas [14]. However, as we use the method in § 5, we give here another proof.

LEMMA 2. 1. *On a Sasakian manifold* M^{2n+1}, *the* $(2q\text{-}1)$*-form* $(q<n+1)$

$$V_{\lambda_1\ldots\lambda_{2q-1}}=\eta_{[\lambda_1}\Phi_{\lambda_2\lambda_3}\cdots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}$$

is a Killing tensor field.

PROOF. First we note that $\eta_{[\lambda_1}\Phi_{\lambda_2\lambda_3}\cdots\eta_{\lambda_i}\cdots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}=0$, because $[\lambda_1\ldots\lambda_{2q-1}]$ are skew-symmetric for all indices. Differentiating $V_{\lambda_1\ldots\lambda_{2q-1}}$ covariantly, we get

$$\begin{aligned}\nabla_\lambda V_{\lambda_1\ldots\lambda_{2q-1}}&=(\nabla_\lambda\eta_{[\lambda_1})\Phi_{\lambda_2\lambda_3}\cdots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}\\&\quad+\sum_i\eta_{[\lambda_1}\Phi_{\lambda_2\lambda_3}\cdots(\nabla_\lambda\Phi_{\lambda_i\lambda_{i+1}})\cdots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}\\&=\Phi_{\lambda[\lambda_1}\Phi_{\lambda_2\lambda_3}\cdots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}\\&\quad+\sum_i\eta_{[\lambda_1}\Phi_{\lambda_2\lambda_3}\cdots\eta_{\lambda_i}G_{|\lambda|\lambda_{i+1}}\cdots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}\\&\quad-\sum_i\eta_{[\lambda_1}\Phi_{\lambda_2\lambda_3}\cdots\eta_{\lambda_{i+1}}G_{|\lambda|\lambda_i}\cdots\Phi_{\lambda_{2q-2}\lambda_{2q-1}]}\\&=U_{\lambda\lambda_1\lambda_2\ldots\lambda_{2q-1}}.\end{aligned}$$

Similarly, we get

$$\nabla_{\lambda_1}V_{\lambda\lambda_2\ldots\lambda_{2q-1}}=U_{\lambda_1\lambda\lambda_2\ldots\lambda_{2q-1}}.$$

Thus, it follows

$$\nabla_\lambda V_{\lambda_1\lambda_2\ldots\lambda_{2q-1}}+\nabla_{\lambda_1}V_{\lambda\lambda_2\ldots\lambda_{2q-1}}=0.$$

This means that $V_{\lambda_1\ldots\lambda_{2q-1}}$ is a Killing tensor field of degree $2q$-1.

LEMMA 2. 2. *On a Sasakian manifold, the* $2q$*-form*

$$U_{\lambda_1\lambda_2\ldots\lambda_{2q}}=\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4}\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]}$$

is a conformal Killing tensor field.

PROOF.

$$
\begin{aligned}
\nabla_\lambda U_{\lambda_1\ldots\lambda_{2q}} &= \nabla_\lambda(\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4}\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]}) \\
&= \nabla_\lambda(\sum_b(-1)^{a-1+b-2}\Phi_{\lambda_a\lambda_b}\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4}\ldots\hat{\lambda}_a\ldots\hat{\lambda}_b\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]}) \\
&= \sum_{a<b}(-1)^{a-1+b-2}(\nabla_\lambda\Phi_{\lambda_a\lambda_b})\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4}\ldots\hat{\lambda}_a\ldots\hat{\lambda}_b\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]} \\
&= \sum_{a<b}(-1)^{a-1+b-2}(\eta_{\lambda_a}G_{\lambda\lambda_b}-\eta_{\lambda_b}G_{\lambda\lambda_a}) \\
&\qquad\cdot\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4}\ldots\hat{\lambda}_a\ldots\hat{\lambda}_b\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]} \\
&= \sum_{a,\ b}(-1)^{a-1+b-2}\eta_{\lambda_a}\Phi_{[\lambda_1\lambda_2}\Phi_{\lambda_3\lambda_4}\ldots\hat{\lambda}_a\ldots\hat{\lambda}_b\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]}G_{\lambda\lambda_b} \\
&= \sum_b(-1)^b\eta_{[\lambda}\Phi_{\lambda_2\lambda_3}\ldots\hat{\lambda}_b\cdots\Phi_{\lambda_{2q-1}\lambda_{2q}]}G_{\lambda\lambda_b} \\
&= \sum_b(-1)^b V_{\lambda_1\lambda_2\ldots\hat{\lambda}_b\ldots\lambda_{2q}}G_{\lambda\lambda_b}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\nabla_\lambda U_{\lambda_1\ldots\lambda_{2q}}+\nabla_{\lambda_1}U_{\lambda\lambda_2\ldots\lambda_{2q}} \\
&=-2V_{\lambda_2\lambda_3\ldots\lambda_{2q}}G_{\lambda\lambda_1} \\
&+\sum_b(-1)^b(V_{\lambda_1\ldots\hat{\lambda}_b\ldots\lambda_{2q}}G_{\lambda\lambda_b}+V_{\lambda\lambda_2\ldots\hat{\lambda}_b\ldots\lambda_{2q}}G_{\lambda_1\lambda_b}).
\end{aligned}
$$

This means that $U_{\lambda_1\ldots\lambda_{2q}}$ is a conformal Killing tensor field.

§ 3. Submanifolds of codimension p of a Riemannian manifold.

Let $\tilde{M}^{m+p}$ be a Riemannian manifold of dimension $m+p$ with local coordinates $\{y^\lambda\}$ and $G_{\lambda\kappa}$ be the Riemannian metric tensor of $\tilde{M}^{m+p}$. Let M^m be a differential submanifold of codimension p of $\tilde{M}^{m+p}$ and $\{x^i\}$ a local coordinates system of M^m. Then the immersion $\iota: M^m \to \tilde{M}^{m+p}$ is locally expressed by $y^\kappa = y^\kappa(x^1, \ldots, x^m)$, $\kappa=1, 2, \ldots, m+p$.

If we put $B_i^\kappa=\partial_i y^\kappa (\partial_i=\partial/\partial x^i)$, then m vectors B_i^κ span the tangent space of M^m at each point of M^m and the induced metric tensor g_{ji} of M^m is given by $g_{ji}=G_{\lambda\kappa}B_j^\lambda B_i^\kappa$. Assuming that manifolds M^m and $\tilde{M}^{m+p}$ are both orientable, we assume that $B_i^\kappa (i=1, 2, \ldots, m)$ give the positive orientation in M^m and we choose the mutually orthogonal unit normal vectors $N_a^\kappa (a=1, \ldots, p)$ to M^m in such a way that B_i^κ, N_a^κ give the positive orientation in $\tilde{M}^{m+p}$. We denote by (B_κ^i, N_κ^a) the basis dual to (B_i^κ, N_a^κ).

The van der Waerden-Bortolotti covariant derivatives $\nabla_j B_i^\kappa$ and $\nabla_j N_a^\kappa$ of B_i^κ and N_a^κ are respectively given by

$$
\begin{aligned}
\nabla_j B_i^\kappa &= \partial_j B_i^\kappa - \left\{ {h \atop j\ i} \right\} B_h^\kappa + \left\{ {\kappa \atop \mu\ \nu} \right\} B_j^\mu B_i^\nu, \\
\nabla_j N_a^\kappa &= \partial_j N_a^\kappa + \left\{ {\kappa \atop \mu\ \nu} \right\} B_j^\mu N_a^\nu.
\end{aligned}
$$

Let H_{aji} be the second fundamental tensor of M^m with respect to the normal N_a^κ and L_{abi} the third fundamental tensor of M^m. Then we have the following Gauss and Weingarten equations;

(3.1) $\nabla_j B_i^\kappa = \sum_a H_{aji} N_a^\kappa,$

(3.2) $\nabla_j N_a^\kappa = -H_{aj}{}^h B_h^\kappa + \sum_b L_{abj} N_b^\kappa.$

where $H_{aj}{}^h = g^{ih} H_{aji}$.

The mean curvature vector field H^κ of M^m in $\tilde{M}^{m+p}$ is defiend by

$$H^\kappa = \frac{1}{m} \sum_a H_{ak}{}^k N_a^\kappa.$$

which is independent of the choice of mutually orthogonal unit normal vectors N_a^κ. The length of the mean curvature vector field H^λ, i. e., $\frac{1}{m}(\sum_a (H_{ak}{}^k)^2)^{\frac{1}{2}}$ is called a mean curvature.

For a normal vector N^λ, if the normal part of $\nabla_j N^\lambda$ vanishes identically along M^m, then we say that N^λ is parallel with respect to the normal bundle.

When there exist mutually orthogonal unit nomal vector fields N_a^λ $(a=1, \dots, p)$ such that $L_{abj}=0$, we say that the connection of the normal bundle is trivial.

When the second fundamental tensor is of the form

$$H_{aji} = H_a g_{ji}$$

at each point of the submaifold M^m, we call the submanifold a totally umbilical submanifold.

Moreover if the functions H_a $(a=1, \dots, p)$ vanish identically, we call the submanifold a totally geodesic submanifold. The following facts are well known (for example. see [1]):

LEMMA 3.1. *A necessary and sufficient condition for M^m to be totally umbilical is that the following relations are satisfied*:

$$H_{aji} H_a{}^{ji} = \frac{1}{m} (H_{ak}{}^k)^2, \quad a=1, \dots p.$$

LEMMA 3.2. *In order that the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle, it is necessary and sufficient that*

$$\nabla_j H_{ak}{}^k = -\sum_b H_{bk}{}^k L_{baj}.$$

LEMMA 3.3. *If the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle, then the mean curvature of M^m is constant.*

We now recall the equations of Gauss, Mainardi-Codazzi and Ricci-Kühne:

$$(3.3)\qquad \tilde{R}_{\nu\mu\lambda\kappa} B_k^\nu B_j^\mu B_i^\lambda B_h^\kappa = R_{kjih} - \sum_a (H_{akh} H_{aji} - H_{aki} H_{ajh})$$

$$(3.4)\qquad \tilde{R}_{\nu\mu\lambda\kappa} B_k^\nu B_j^\mu B_i^\lambda N_a^\kappa = \nabla_k H_{aji} - \nabla_j H_{aki} + \sum_b (H_{bji} L_{bak} - H_{bki} L_{baj})$$

$$(3.5)\qquad \tilde{R}_{\nu\mu\lambda\kappa} B_k^\nu B_j^\mu N_a^\lambda N_b^\kappa = H_{ak}{}^i H_{bji} - H_{aj}{}^i H_{bki} + \nabla_k L_{abj} - \nabla_j L_{abk} + \sum_c (L_{acj} L_{cbk} - L_{ack} L_{cbj})$$

§ 4. Submanifolds of codimension p of Sasakian space forms.

Now, let $\tilde{M}^{2n+1}$ be a Sasakian space form and M^m be a submanifold of condimension p of $\tilde{M}^{2n+1}$, where $2n+1=m+p$.

The transform $\Phi_\lambda{}^\kappa B_i^\lambda$ of B_i^λ by $\Phi_\lambda{}^\kappa$ can be expressed as a linear combination of B_i^κ and N_a^κ. So, we can put

$$\Phi_\lambda{}^\kappa B_i^\lambda = f_i{}^h B_h^\kappa + \sum_a h_{ia} N_a^\kappa$$

from which we have

$$(4.1)\qquad f_i{}^h = \Phi_\lambda{}^\kappa B_i^\lambda B_\kappa^h, \quad h_{ia} = \Phi_\lambda{}^\kappa B_i^\lambda N_{a\kappa}.$$

Since $\Phi_{\lambda\kappa}$ is skew-symmetric, so is $f_{ij} = g_{hj} f_i{}^h$, and if we put $h_{ai} = \Phi_\lambda{}^\kappa N_a^\lambda B_{i\kappa}$, we get $h_{ia} + h_{ai} = 0$. We can also put the transform $\Phi_\lambda{}^\kappa N_a^\lambda$ of N_a^λ by $\Phi_\lambda{}^\kappa$ as follows;

$$\Phi_\lambda{}^\kappa N_a^\lambda = -h^i{}_a B_i^\kappa + r_a{}^b N_b^\kappa,$$

from which we have $h^i{}_a = g^{hi} h_{ha}$ and $r_a{}^b = \Phi_\lambda{}^\kappa N_a^\lambda B_\kappa^b$. We notice that since $\Phi_{\lambda\kappa}$ is skew-symmetric, p^2 functions $r_{ab} = \Phi_{\lambda\kappa} N_a^\lambda N_b^\kappa$ are skew-symmetric with respect to their indices a, b.

Next, we express the vector field ξ^λ on $\tilde{M}^{2n+1}$ as a linear combination of B_i^κ and N_a^κ as follows:

$$\xi^\kappa = u^h B_h^\kappa + \sum_a \omega_a N_a^\kappa$$

Then we have

$$(4.2) \qquad u^h = \xi^\kappa B_\kappa^h = g^{hi} B_i^\kappa \eta_\kappa, \quad \omega_a = N_a^\kappa \eta_\kappa.$$

Since $\tilde{M}^{2n+1}$ has the curvature tensor of the form (1.3), using the tensor field $f_i{}^h$, h_{ia}, r_{ab}, u^h and ω_a, we can rewrite the equations of Mainardi-Codazzi (3.4) and Ricci-Kühne (3.5) as follows:

$$(4.3) \qquad \begin{aligned} &\nabla_k H_{aji} - \nabla_j H_{aki} + \sum_b (H_{bji} L_{bak} - H_{bki} L_{baj}) \\ &= k(f_{ji} h_{ka} - f_{ki} h_{ja} - 2 f_{kj} h_{ia} + u_j \omega_a g_{ki} - u_k \omega_a g_{ji}) \end{aligned}$$

$$(4.4) \qquad \begin{aligned} &H_{ak}{}^i H_{bji} - H_{aj}{}^i H_{bki} + \nabla_k L_{abj} - \nabla_j L_{abk} + \sum_c (L_{acj} L_{cbk} - L_{ack} L_{cbj}) \\ &= k(h_{ja} h_{kb} - h_{ka} h_{jb} - 2 f_{kj} r_{ab}) \end{aligned}$$

Now, we recall the conformal Killing tensor field $U_{\lambda_1 \dots \lambda_p}$ where $p=2q$ and Killing tensor field $V_{\lambda_1 \dots \lambda_p}$ where $p=2q-1$ in § 2. In order to simplify the presentation, we shall denote both $U_{\lambda_1 \dots \lambda_p}$ and $V_{\lambda_1 \dots \lambda_p}$ by $F_{\lambda_1 \dots \lambda_p}$. We put

$$(4.5) \qquad r = F_{\lambda_1 \dots \lambda_p} N_1^{\lambda_1} \dots N_p^{\lambda_p},$$

$$(4.6) \qquad v_i = \sum_a H_{ai}{}^h F_{\lambda_1 \dots \lambda_a \dots \lambda_p} N_1^{\lambda_1} \dots B_h^{\lambda_a} \dots N_p^{\lambda_p},$$

$$(4.7) \qquad w_i = \sum_a H_{ak}{}^k F_{\lambda_1 \dots \lambda_a \dots \lambda_p} N_1^{\lambda_1} \dots B_i^{\lambda_a} \dots N_p^{\lambda_p}.$$

Then it is known that v_i and w_i are independent of the choice of mutually orthogonal unit normal vectors, so they are global vector fields (cf. [6]).

§ 5. Integral formulas.

To apply Green-Stokes' theorem, we compute the divergence of the vector fields v_i and w_i defined by (4.6) and (4.7) respectively. Making use of (3.1) and (3.2), we get:

$$\begin{aligned} \nabla_j v_i = &\sum_a \nabla_j H_{ai}{}^h F_{\lambda_1 \dots \lambda_a \dots \lambda_p} N_1^{\lambda_1} \dots B_h^{\lambda_a} \dots N_p^{\lambda_p} \\ &+ \sum_a H_{ai}{}^h B_j^\lambda \nabla_\lambda F_{\lambda_1 \dots \lambda_a \dots \lambda_p} \cdot N_1^{\lambda_1} \dots B_h^{\lambda_a} \dots N_p^{\lambda_p} \\ &+ \sum_a \mathrm{H}_{ai}{}^h \sum_{c,\ a \neq c} F_{\lambda_1 \dots \lambda_c \dots \lambda_a \dots \lambda_p} \\ &\cdot N_1^{\lambda_1} \dots (-H_{cj}{}^k B_k^{\lambda_c} + \sum_b L_{cbj} N_b^{\lambda_c}) \dots B_h^{\lambda_a} \dots N_p^{\lambda_p} \\ &+ \sum_a H_{ai}{}^h F_{\lambda_1 \dots \lambda_a \dots \lambda_p \dots \lambda_p} N_1^{\lambda_1} \dots \sum_b H_{bjh} N_b^{\lambda_a} \dots N_p^{\lambda_p} \end{aligned}$$

$$
\begin{aligned}
&=\sum_a(\nabla_j H_{ai}{}^h+\sum_b H_{bi}{}^h L_{baj})F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_h^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+\sum_a H_{ai}{}^h B_j^\lambda\nabla_\lambda F_{\lambda_1\dots\lambda_a\dots\lambda_6}N_1^{\lambda_1}\dots B_h^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad-\sum_{a,\,c} H_{ai}{}^h H_{cj}{}^k F_{\lambda_1\dots\lambda_c\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_h^{\lambda_a}\dots B_k^{\lambda_c}\dots N_p^{\lambda_p}\\
&\quad+r\sum_a H_{ai}{}^h H_{ajh}
\end{aligned}
$$

$$
\begin{aligned}
(5.1)\qquad \nabla_j v^j&=\sum_a(\nabla_j H_a{}^{jh}+\sum_b H_b{}^{jh}L_{baj})F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_h^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+\sum_a H_a{}^{jh}B_j^\lambda\nabla_\lambda F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_h^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad-\sum_{a,\,c} H_a{}^{jh}H_{cj}{}^k F_{\lambda_1\dots\lambda_c\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_h^{\lambda_a}\dots B_k^{\lambda_c}\dots N_p^{\lambda_p}\\
&\quad+r\sum_a H_{ajh}H_a{}^{jh}.
\end{aligned}
$$

$$
\begin{aligned}
\nabla_j w_i&=\sum_a\nabla_j H_{ak}{}^k F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_i^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+\sum_a H_{ak}{}^k B_j^\lambda\nabla_\lambda F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda}\dots B_i^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+\sum_a H_{ak}{}^k\sum_{c,\,a\neq c}F_{\lambda_1\dots\lambda_c\dots\lambda_a\dots\lambda_p}\cdot\\
&\quad\cdot N_1^{\lambda_1}\dots(-H_{cj}{}^h B_h^{\lambda_c}+\sum_b L_{cbj}N_b^{\lambda_c})\dots B_i^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+\sum H_{ak}{}^k F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots\sum_b H_{bji}N_b^{\lambda_a}\dots N_p^{\lambda_p}\\
&=\sum_a(\nabla_j H_{ak}{}^k+\sum_b H_{bk}{}^k L_{baj})F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_p^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+\sum_a H_{ak}{}^k B_j^\lambda\nabla_\lambda F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_i^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad-\sum_{a,\,c} H_{ak}{}^k H_{cj}{}^h F_{\lambda_1\dots\lambda_c\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_h^{\lambda_c}\dots B_i^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+r\sum_a H_{ak}{}^k H_{aji}
\end{aligned}
$$

$$
\begin{aligned}
(5.2)\qquad \nabla_j w^j&=\sum_a(\nabla_j H_{ak}{}^k+\sum_b H_{bk}{}^k L_{baj})F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B^{j\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+\sum_a H_{ak}{}^k B^{j\lambda}\nabla_\lambda F_{\lambda_1\dots\lambda_a\dots\lambda_p}N_1^{\lambda_1}\dots B_j^{\lambda_a}\dots N_p^{\lambda_p}\\
&\quad+r\sum_a (H_{ak}{}^k)^2
\end{aligned}
$$

On the other hand, since $F_{\lambda_1\dots\lambda_p}$ is a conformal Killing tensor field, the following equation is valid (cf. [6]);

$$
\begin{aligned}
&\nabla_\lambda F_{\lambda_1\dots\lambda_a\dots\lambda_p}+\nabla_{\lambda_a}F_{\lambda_1\dots\lambda\dots\lambda_p}\\
&\quad=-2(-1)^a\rho_{\lambda_1\dots\hat{\lambda}_a\dots\lambda_p}G_{\lambda\lambda_a}\\
&\qquad-\sum_{b,\,a\neq b}(-1)^b(\rho_{\lambda_1\dots\hat{\lambda}_b\dots\lambda_a\dots\lambda_p}G_{\lambda\lambda_b}+\rho_{\lambda_1\dots\hat{\lambda}_b\dots\lambda\dots\lambda_p}G_{\lambda_a\lambda_b})
\end{aligned}
$$

where $\rho_{\lambda_1\ldots\lambda_{p-1}}$ is the associated tensor field of $F_{\lambda_1\ldots\lambda_p}$. So, we have:

$$(5.3)\quad \sum_a H_a{}^{jh}B^\lambda_j B^{\lambda_a}_h \nabla_\lambda F_{\lambda_1\ldots\lambda_a\ldots\lambda_p} N^{\lambda_1}_1\ldots\hat{N}^{\lambda_a}_a\ldots N^{\lambda_p}_p$$
$$=\frac{1}{2}\sum_a H_a{}^{jh}B^\lambda_j B^{\lambda_a}_h(\nabla_\lambda F_{\lambda_1\ldots\lambda_a\ldots\lambda_p}+\nabla_{\lambda_a}F_{\lambda_1\ldots\lambda\ldots\lambda_p}).\ N^{\lambda_1}_1\ldots\hat{N}_a{}^{\lambda_a}\ldots N^{\lambda_p}_p$$
$$=-\sum_a(-1)^a H_{ak}{}^k\rho_{\lambda_1\ldots\hat{\lambda}_a\ldots\lambda_p}N^{\lambda_1}_1\ldots\hat{N}_a{}^{\lambda_a}\ldots N^{\lambda_p}_p$$

$$(5.4)\quad \sum_a H_{ak}{}^k B^{j\lambda}B^{\lambda_a}_j\nabla_\lambda F_{\lambda_1\ldots\lambda_a\ldots\lambda_p}N^{\lambda_1}_1\ldots\hat{N}_a{}^{\lambda_a}\ldots N^{\lambda_p}_p$$
$$=\frac{1}{2}\sum_a H_{ak}{}^k B^{h\lambda}B^{\lambda_a}_h(\nabla_\lambda F_{\lambda_1\ldots\lambda_a\ldots\lambda_p}+\nabla_{\lambda_a}F_{\lambda_1\ldots\lambda\ldots\lambda_p}).\ N^{\lambda}_1\ldots\hat{N}_a{}^{\lambda_a}\ldots N^{\lambda_p}_p$$
$$=-m\sum_a(-1)^a H_{ak}{}^k\rho_{\lambda_1\ldots\hat{\lambda}_a\ldots\lambda_p}N^{\lambda_1}_1\ldots\hat{N}_a{}^{\lambda_a}\ldots N^{\lambda_p}_p$$

Especially, in the case that $F_{\lambda_1\ldots\lambda_p}$ is a Killing tensor field, we have

$$(5.3)'\quad \sum_a H_a{}^{jh}B^\lambda_j B^{\lambda_a}_h\nabla_\lambda F_{\lambda_1\ldots\lambda_a\ldots\lambda_p}N^{\lambda_1}_1\ldots\hat{N}_a{}^{\lambda_a}\ldots N^{\lambda_p}_p=0.$$
$$(5.4)'\quad \sum_a H_{ak}{}^k B^{j\lambda}B^{\lambda_a}_j\nabla_\lambda F_{\lambda_1\ldots\lambda_a\ldots\lambda_p}N^{\lambda_1}_1\ldots\hat{N}_a{}^{\lambda_a}\ldots N^{\lambda_p}_p=0.$$

Now suppose that the mean curvature vector field H^λ of M^m is parallel with respect to the connection of the normal bundle. Then, from Lemma 3. 2 and (4. 3) we get

$$(5.5)\quad \nabla_j H_{ak}{}^k+\sum_b H_{bk}{}^k L_{baj}=0,$$
$$(5.6)\quad \nabla_j H_{ai}{}^j+\sum_b H_{bji}L_{ba}{}^j=k(3f_{ij}h^j{}_a+(m-1)u_i\omega_a).$$

Moreover we assume that the connection of the normal bundle is trivial. Then, from (4. 4) we get

$$H_{aj}{}^h H_c{}^{jk}-H_a{}^{jk}H_{cj}{}^h=k(h^k{}_a h^h{}_c-h^h{}_a h^k{}_c-2f^{hk}r_{ac}).$$

So, we have

$$(5.7)\quad \sum_{a,\,c}H_a{}^{jh}H_{cj}{}^k F_{\lambda_1\ldots\lambda_c\ldots\lambda_a\ldots\lambda_p}N^{\lambda_1}_1\ldots B^{\lambda_a}_h\ldots B^{\lambda_c}_k\ldots N^{\lambda_p}_p$$
$$=\frac{1}{2}\sum_{a,\,c}(H_a{}^{jh}H_{cj}{}^k-H_a{}^{jk}H_{cj}{}^h)F_{\lambda_1\ldots\lambda_c\ldots\lambda_a\ldots\lambda_p}.\ N^{\lambda_1}_1\ldots B^{\lambda_a}_h\ldots B^{\lambda_c}_k\ldots N^{\lambda_p}_p$$
$$=\frac{1}{2}k\sum_{a,\,c}(h^k{}_a h^h{}_c-h^h{}_a h^k{}_c-2f^{hk}r_{ac})F_{\lambda_1\ldots\lambda_c\ldots\lambda_a\ldots\lambda_p}.$$
$$N^{\lambda_1}_1\ldots B^{\lambda_a}_h\ldots B^{\lambda_c}_k\ldots N^{\lambda_p}_p$$
$$=k\sum_{a,\,c}(h^k{}_a h^h{}_c-h^{hk}r_{ac})F_{\lambda_1\ldots\lambda_p}N^{\lambda_1}_1\ldots B^{\lambda_a}_h\ldots B^{\lambda_c}_k\ldots N^{\lambda_p}_p.$$

At first, in the case that $F_{\lambda_1\ldots\lambda_p}$ is a conformal Killing tensor field, substituting (5.3), (5.6) and (5.7) into (5.1) and also (5.4) and (5.5) into (5.2), we get:

$$(5.8)\quad \nabla_j v^j = k\sum_a (3f^i{}_j h^j{}_a + (m-1)u^i\omega_a)F_{\lambda_1\ldots\lambda_a\ldots\lambda_p}\cdot N_1^{\lambda_1}\ldots B_i^{\lambda_a}\ldots N_p^{\lambda_p}$$
$$-\sum_a(-1)^a H_{ak}{}^k \rho_{\lambda_1\ldots\lambda_a\ldots\lambda_p} N_1^{\lambda_1}\ldots \hat{N}_a^{\lambda_a}\ldots N_p^{\lambda_p}$$
$$-k\sum_{a,c}(h^k{}_a h^h{}_c - f^{hk}r_{ac})F_{\lambda_1\ldots\lambda_c\ldots\lambda_a\ldots\lambda_p} N_1^{\lambda_1}\ldots B_h^{\lambda_a}\ldots B_k^{\lambda_c}\ldots N_p^{\lambda_p}$$
$$+r\sum_a H_{aji}H_a{}^{ji}$$

$$(5.9)\quad \nabla_j w^j = -m\sum_a(-1)^a H_{ak}{}^k \rho_{\lambda_1\ldots\hat{\lambda}_a\ldots\lambda_p} N_1^{\lambda_1}\ldots\hat{N}_a^{\lambda_a}\ldots N_p^{\lambda_p}$$
$$+r\sum_a(H_{ak}{}^k)^2$$

From (5.8) and (5.9), we obtain:

$$(5.10)\quad \nabla_j v^j - \nabla_j w^j$$
$$= r\sum_a\{H_{aji}H_a{}^{ji} - \frac{1}{m}(H_{ak}{}^k)^2\}$$
$$+k\sum_a(3f^i{}_j h^j{}_a + (m-1)u^i\omega_a)F_{\lambda_1\ldots\lambda_a\ldots\lambda_p} N_1^{\lambda_1}\ldots B_i^{\lambda_a}\ldots N_p^{\lambda_p}$$
$$-k\sum_{a,c}(h^k{}_a h^k{}_c - f^{kh}r_{ac})F_{\lambda_1\ldots\lambda_c\ldots\lambda_a\ldots\lambda_p} N_1^{\lambda_1}\ldots B_h^{\lambda_a}\ldots B_k^{\lambda_c}\ldots N_p^{\lambda_p}$$

Next, in the case that $F_{\lambda_1\ldots\lambda_p}$ is a Killing tensor field, we get:

$$(5.8)'\quad \nabla_j v^j = k\sum_a(3f^i{}_j h^j{}_a + (m-1)u^i\omega_a)F_{\lambda_1\ldots\lambda_a\ldots\lambda_p} N_1^{\lambda_1}\ldots B_i^{\lambda_a}\ldots N_p^{\lambda_p}$$
$$-k\sum_{a,c}(h^k{}_a h^h{}_c - f^{hk}r_{ac})F_{\lambda_1\ldots\lambda_c\ldots\lambda_a\ldots\lambda_p} N_1^{\lambda_1}\ldots B_h^{\lambda_a}\ldots B_k^{\lambda_c}\ldots N_p^{\lambda_p}$$
$$+r\sum_a H_{aji}H_a{}^{ji}$$

$$(5.9)'\quad \nabla_j w^j = r\sum_a(H_{ak}{}^k)^2$$

Now, we substitute $U_{\lambda_1\ldots\lambda_{2q}}$ into $F_{\lambda_1\ldots\lambda_p}$ in the equation (5.10). Making use of (2.3), (2.4), (1.1), (4.1) and (4.2), after long complicated computations, (5.10) becomes

$$(5.11)\quad \nabla_j v^j - \nabla_j w^j$$
$$= r\left[\sum_a\{H_{aji}H_a{}^{ji} - \frac{1}{m}(H_{ak}{}^k)^2\} + k(m-1)(p - \sum_a\omega_a^2)\right].$$

Next, substituting $V_{\lambda_1\ldots\lambda_{2q-1}}$ into $F_{\lambda_1\ldots\lambda_p}$ in the equation (5.8)′, by similar computations, (5.8)′ reduces to

$$(5.12) \qquad \nabla_j v^j = r\left[\sum_a H_{aji}H_a{}^{ji} + k\{(m+2)(1-\sum_a \omega_a^2)+(p-1)m\}\right].$$

Applying Green-Stokes' theorem to (5.11), (5.9)′ and (5.12), we get the following two sets of integral formulas;

$$(5.13) \qquad \int r\left[\sum_a\{H_{aji}H_a{}^{ji} - \frac{1}{m}(H_{ak}{}^k)^2\} + k(m-1)(p-\sum_a \omega_a^2)\right] dM = 0,$$

$$(5.14) \qquad \begin{cases} \int r\sum_a (H_{ak}{}^k)^2 dM = 0, \\ \int r\left[\sum_a H_{aji}H_a{}^{ji} + k\{(m+2)(1-\sum \omega_a^2)+(p-1)m\}\right] dM = 0. \end{cases}$$

THEOREM 5.1. *Let $\tilde{M}^{m+p}$ $(m \neq 1)$ be an $(m+p)$-dimensional Sasakian space form and M^m be a compact orientable submanifold of codimension p of $\tilde{M}^{m+p}$. We suppose that k is non-negative, the mean curvature vector field is parallel with respect to the connection of the normal bundle, the connection of the normal bundle is trivial and r is almost everywhere non-zero valued function whose sign dose not change, then we have:*

1) *The ambient manifold $\tilde{M}^{m+p}$ is necessarilly a space of constant curvature* 1.

2) *If the codimension p is even, M^m is a totally umbillical submanifold and if the codimension p is odd, M^m is a totally geodesic submanifold.*

Theorem 5.1 follows from the integral formulas (5.13), (5.14) and the fact that $u_i u^i = 1 - \sum_a \omega_a^2 \geqq 0$.

COROLLARY 5.1. *Let $\tilde{M}^{2n+1}$ be a $(2n+1)$-dimensional Sasakian space form and M^{2n} be a compact orientable hypersurface. We suppose that the mean curvature of M^{2n} is constant and $\xi_\lambda N^\lambda$ has fixed sign on M^{2n}, then we have*

1) *The ambient manifold $\tilde{M}^{2n+1}$ is necessarilly a space of constant curvature* 1.

2) *M^{2n} is totally geodesic.*

This result is obtained directly, computing the divergences of two vector fields v_i and w_i which are defined as follows:

$$V_i = H_i{}^j \xi^\lambda B_{\lambda j}$$
$$W_i = H_k{}^k \xi^\lambda B_{i\lambda}$$

REMARK. The case of codimension 2 has been proved by the author [4].

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