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Wiener functionals and probability limit theorems, II: Term-wise multiplication and its applications

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§ 1. Unconditionally convergent multiplication

In the preceding paper [**13**], the author studied central limit theorems for a class of Wiener functionals, that is, measurable functions of Gaussian white noise. The present paper consists of the first part (§ 1-§ 2), an English presentation of § 6-§ 7 of [**11**] and the second part (§ 3-§ 4), central limit theorems (CLT's) as an application of the first part. Theorem 3 generalizes Theorem 1 in [**13**]. In the first part we consider a multiplication procedure in the classes, N_{2m}, $N'_{2m}(1\leq m<\infty)$ of Wiener functionals.

We are concerned with $L^{p}(1\leq p<\infty)$, the space of real random variables X, furnished with $\|X\|_{p}=(E|X|^{p})^{1/p}$ as norm, subordinate to a real Gaussian stationary process

$$(1.1)\qquad \xi(t)=\int\exp i\lambda td\beta(\lambda),\ -\infty<t<\infty,$$

with $E\xi(t)=0$, complex spectral random measure $d\beta$, and spectral measure $d\sigma=E|d\beta(\lambda)|^{2}$, which is absolutely continuous with respect to Lebesgue measure, $d\sigma(\lambda)=f(\lambda)d\lambda$. Define $\mathscr{L}_{k,2}(1\leq k<\infty)$ to be the set of complex symmetric Borel functions h on $\boldsymbol{R}^{k}$ satisfying (i) $\overline{h(\lambda)}=h(-\lambda)$ (ii) $h\in L^{2}(d^{k}\sigma)$. An arbitrary $X\in L^{2}$ is represented as

$$(1.2)\qquad X=c_{0}+\sum_{k\geq 1}X_{k},\ X_{k}\equiv I(c_{k})=\int c_{k}(\lambda)d^{k}\beta,\ \lambda\in R^{k},$$

where $c_{k}\in\mathscr{L}_{k,2}$(c. f. [**13**]), X_{k} is a k-fold multiple Itô integral (or homogeneous polynomial of degree k). Sometimes, the notation $X=(c_{0}, c_{1}, \cdots)$ or $X=(c_{k}, 0\leq k<\infty)$ is used to abreviate the expression (1.2). $X^{*}=(|c_{0}|, |c_{1}|, \cdots)$ is at the same time an element of L^{2}. If the right-hand side of (1.2) is a finite sum, X is said to be finite; $d=\max(k:\|c_{k}\|_{2}\neq 0)$ is the degree of X, where $\|c_{k}\|_{2}^{2}=\int|c_{k}(\lambda)|^{2}d^{k}\sigma$. We succeed notational conventions of [**13**]: $X_{(n)}$ denotes the partial sum of (1.2) up to X_{n}; whole spaces, such as $(-\infty, \infty)$ in (1.1) or $\boldsymbol{R}^{k}$ in (1.2) as integration domains are suppressed; c, c_{1}, c_{2}, $\cdots$ will denote constants, of which the same symbol may stand for

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different values.

Writing $\lambda^{(d)}$ or $\dim(\lambda)=d$ $(d>0)$ we mean that the relevant vectors are d-dimensional. However, in many cases, dimensionalities are not explicit, when there is no danger of confusion. If $\lambda^{(m)}=(\lambda_1,\cdots,\lambda_m)$, $\mu^{(n)}=(\mu_1,\cdots,\mu_n)$ are vectors, the notations $c_{m+n}(\lambda_1,\cdots,\lambda_m,\mu_1,\cdots,\mu_n)$ and $c_{m+n}(\lambda^{(m)},\mu^{(n)})$ are synonymously used.

Suppose that we are given finite elements $X^{(j)}\in L^2(1\leq j\leq m)$, $X^{(j)}=(c_0^{(j)}, c_1^{(j)},\cdots)$. Let

$$(1.3)\qquad \prod_{j=1}^{m} X^{(j)}=\sum_{k=0}^{\infty}\int C_k(\lambda)d^k\beta$$

be the Itô-Wiener expansion of $X^{(1)}\cdots X^{(m)}$. C_k, the kernel of kth homogeneous polynomial, is obtained by the multiplication rule (p. 388, [**12**]). To get an idea, consider the case $m=4$. Then

$$(1.4)\qquad \begin{aligned} &C_k(\lambda)=\sum_{\bar{\mathbf{u}}=k} C(\lambda,\mathbf{u})\\ &\lambda=(\lambda^{(u_1)},\cdots,\lambda^{(u_4)}),\ \mathbf{u}=(u_1,\cdots,u_4),\ \bar{\mathbf{u}}=u_1+\cdots+u_4, \end{aligned}$$

where

$$C(\lambda,\mathbf{u})=\sum_{P} C(P;\lambda,\mathbf{u}),\ P=\|p_{ij}\|$$

$$(1.5)\qquad \begin{aligned} C(P;\lambda,\mathbf{u})=&\prod_{1\leq i<j\leq 4} p_{ij}!\binom{u_1+p_1}{p_1}\frac{p_1!}{p_{12}!\,p_{13}!\,p_{14}!}\binom{u_2+p_2}{p_2}\\ &\times\frac{p_2!}{p_{12}!\,p_{23}!\,p_{24}!}\binom{u_3+p_3}{p_3}\frac{p_3!}{p_{13}!\,p_{23}!\,p_{34}!}\binom{u_4+p_4}{p_4}\frac{p_4!}{p_{14}!\,p_{24}!\,p_{34}!}\\ &\times\int_{\mathbf{R}^k} c_{v_1}^{(1)}(\lambda_{12},\lambda_{13},\lambda_{14},\lambda^{(u_1)})c_{v_2}^{(2)}(-\lambda_{12},\lambda_{23},\lambda_{24},\lambda^{(u_2)})\\ &\times c_{v_3}^{(3)}(-\lambda_{13},-\lambda_{23},\lambda_{34},\lambda^{(u_3)})c_{v_4}^{(4)}(-\lambda_{14},-\lambda_{24},-\lambda_{34},\lambda^{(u_4)})d^k\sigma, \end{aligned}$$

where

$$(\mathrm{P})\qquad \begin{aligned} &\dim(\lambda_{ij})=p_{ij},\ v_i=u_i+p_i,\ p_i=\sum_{j=1}^{4}p_{ij},\ p_{ii}=0,\\ &p_{ij}=p_{ji}(1\leq i,j\leq 4),\ k=\bar{p}/2,\ \bar{p}=p_1+\cdots+p_4,\\ &\mathbf{R}^0=\{0\},\ d^0\sigma=\delta_0\ (\text{unit mass at }\{0\}), \end{aligned}$$

the summation of (1.5) is taken over all matrices subject to the above conditions. Those λ_{ij} for which $d(\lambda_{ij})=0$ are absent in the right-hand side of (1.5).

To have a systematic expression of a kernel function in (1.3), introduce a measure $d\tau$ on the space

$$E=\bigcup_{n=0}^{\infty} E_n,\ E_n=\{n\}\times \boldsymbol{R}^n\ (n\geq 0),\ \boldsymbol{R}^0=\{0\}.$$

The σ-algebra on E is composed of $\{n\}\times\mathscr{B}(\boldsymbol{R}^n)$ on E_n, where $\mathscr{B}(\boldsymbol{R}^n)$ are the Borel families on $\boldsymbol{R}^n$. Define τ to be such a measure that

$$\tau(\{n\}\times A)=n!\int_A d^n\sigma(n\geq 1),\ A\in\mathscr{B}(\boldsymbol{R}^n),\ \tau(E_0)=1.$$

Let $P=\|p_{ij}\|$ be a symmetric $m\times m$-matrix whose elements p_{ij} are non-negative integers, with $p_{ii}=0(1\leq i, j\leq n)$. Define an $m\times m$-matrix $\Xi=\Xi(\mathbf{x})=\|x_{ij}\|$ whose (i, j)-elements are from E, $x_{ij}=(p_{ij}, \lambda_{ij})$, $\dim(\lambda_{ij})=p_{ij}$, $\lambda_{ji}=-\lambda_{ij}$, and $\mathbf{x}=(x_{ij}, 1\leq i<j\leq m)$ is an $m(m-1)/2$-dimensional composite variable; $x_{ji}(1\leq i<j\leq m)$ is a function of x_{ij}. Let $\{e_1=(1,0,\cdots,0),\cdots,e_m=(0,\cdots,0,1)\}$ be the standard basis of $\boldsymbol{R}^m$. For $l\in\boldsymbol{N}_0=(0,1,\cdots)$, an ordered partition π of l is a map $\pi: l\to(l_1,\cdots,l_m)\in\boldsymbol{N}_0^m$, with $l_1+\cdots+l_m=l$. The corresponding partition of $\lambda=(\lambda_1,\cdots,\lambda_l)\in\boldsymbol{R}^l$ is the map: $\lambda\to(\lambda^{(l_1)},\cdots,\lambda^{(l_m)})\in\boldsymbol{R}^{l_1}\times\cdots\times\boldsymbol{R}^{l_m}$, where $\lambda^{(l_1)}=(\lambda_1,\cdots,\lambda_{l_1})$, $\lambda^{(l_2)}=(\lambda_{l_1+1},\cdots,\lambda_{l_1+l_2})$, etc. Introduce functions C_l and $B_l^{(j)}(1\leq j\leq m)$ defined respectively on $E^m\times\boldsymbol{R}^l$ and $E^s\times\boldsymbol{R}^l(s=m(m-1)/2)$ by

$$C_l^{(j)}(y_1,\cdots y_m,\mu)=\binom{l+\bar{q}}{\bar{q}}\frac{\bar{q}!}{q_1!\cdots q_m!}c_{l+\bar{q}}^{(j)}(\lambda_1,\ldots,\lambda_m,\mu),\ 1\leq j\leq m,$$

$$\begin{aligned}(1.6)\qquad & y_k=(q_k,\lambda_k)\in E\ (1\leq k\leq m),\ \mu\in\boldsymbol{R}^l,\\ & B_l^{(j)}(\mathbf{x},\mu)=C_l^{(j)}(e_j\Xi(\mathbf{x}),\mu),\ \mathbf{x}\in E^s,\ \mu\in\boldsymbol{R}^l.\end{aligned}$$

λ_j is absent in the right-hand expression of (1.6) if $q_j=0$. C_l of $X^{(1)}\cdots X^{(m)}$ is then written as

$$(1.7)\qquad C_l(\lambda)=\sum_{\pi}\int_{E^s}\prod_{j=1}^{m}B_{l_j}^{(j)}(\mathbf{x},\lambda^{(l_j)})\prod_{1\leq i<j\leq m}d\tau(x_{ij}),$$

where π runs over the set of ordered partitions of l.

Definition 1. Let $X=(c_0, c_1,\cdots)\in L^2$. Define $|||X|||_{2m}$, $|||X|||'_{2m}(m\geq 1)$ and associated subclasses of L^2:

$$(1.8)\qquad |||X|||_{2m}=\sum_{k\geq 0}(2m-1)^{k/2}\sqrt{k!}\|c_k\|_2,$$

$$|||X|||'_{2m}=(\sup_{0\leq n<\infty}E(X^*_{(n)})^{2m})^{1/2m},$$

(1.9) $\quad N_{2m}=\{X\in L^2 : |||X|||_{2m}<\infty\},$
$\quad N'_{2m}=\{X\in L^2 : |||X|||'_{2m}<\infty\}.$

X and X^* belong to the same subclass with equal relevant norms. Obviously $N_2\subset N'_2=L^2$, the latter two with coincident norm.

An infinite series in a Banach space is said to be unconditionally convergent when it converges to the same limit, regardless of the order of summation.

THEOREM 1. *Given $X^{(j)}\in N'_{2m}$ with $X^{(j)}=(c_0^{(j)}, c_1^{(j)}, \cdots)(1\leq j\leq m)$, multiply the Itô-Wiener expansions of $X^{(j)}$ term by term to get a formal series of homogeneous polynomials. Then the series is unconditionally L^2-convergent to $X^{(1)}\cdots X^{(m)}$. The sum of thus obtained homogeneous polynomials of degree $l(0\leq l<\infty)$ is equal to the l th homogeneous polynomial of $X^{(1)}\cdots X^{(m)}$.*

Denote by $S(X^{(1)},\cdots, X^{(p)})$, $1\leq p\leq m$, the cumulant of $X^{(1)},\cdots, X^{(p)}$, i. e.

$$S(X^{(1)}, \cdots , X^{(p)})$$
$$=i^{-p}\left(\frac{\partial^p}{\partial\alpha_1\cdots\partial\alpha_p}\right)\log E\{\exp i[\alpha_1X^{(1)}+\cdots+\alpha_pX^{(p)}]\}\,|_{\alpha_1=\cdots=\alpha_p=0}.$$

Then we have

$$(1.10)\qquad E\Big(\prod_{j=1}^{p} X^{(j)}\Big)=\sum_{k(1),\cdots,k(p)\geq 0} E\Big(\prod_{j=1}^{p} I_{k(j)}^{(j)}\Big),$$

$$(1.11)\qquad S(X^{(1)}, \cdots , X^{(p)})=\sum_{k(1),\cdots,k(p)\geq 0} S(I_{k(1)}^{(1)}, \cdots , I_{k(p)}^{(p)}),\quad I_k^{(j)}=I(c_k^{(j)}),$$
$$2\leq p\leq m,$$

where the right-hand members are absolutely convergent.

For the present and later use we prepare propositions 1- I -VI.

1- I . *Let $(X, \mathscr{B}, m)$ be the product measure space of the measure spaces $(X_i, \mathscr{B}_i, m_i)$, $1\leq i\leq n$, and f_i $(1\leq i\leq k)$ be $\mathscr{B}$-measurable complex -valued functions of $x_1, \cdots , x_n$, where x_i is the generic point of X_i. Suppose further that each x_i is involved in certain distinct two but no more factors of the product $f_1\cdots f_k$.*

Then

$$(1.12)\qquad \Big|\int f_1\cdots f_k dm\Big|\leq \|f_1\|_2\cdots\|f_k\|_2,$$
$$\|f_i\|_2^2=\int |f_i|^2,\ 1\leq i\leq k,$$

where the last expression is the integral with respect to the arguments involved in f_i and relevant product measure.

PROOF. A repeated use of Schwarz's inequality with respect to $m_1, \cdots ,$

m_n or mathematical induction on n connected with Schwarz's inequality suffices to derive the conclusion ([**11**]).

1-II. *Let $X^{(j)} \in L^2$ $(1 \leq j \leq l)$ be finite. Then*

$$(1.13)\qquad |S(X^{(1)}, \cdots, X^{(l)})| \leq S(X^{(1)*}, \cdots, X^{(l)*}) \leq E(\prod_{j=1}^{l} X^{(j)*}),$$

$$(1.14)\qquad |E(\prod_{j=1}^{l} X^{(j)})| \leq E(\prod_{j=1}^{l} X^{(j)*}).$$

PROOF. Write $I_k^{(j)}$ for the kth homogeneous polynomial of $X^{(j)}$. (1.13), (1.14) follow from the multi-linearity of cumulants, moments, and integral representations of $S(I_{k(1)}^{(1)}, \cdots, I_{k(l)}^{(l)})$, $E(I_{k(1)}^{(1)}, \cdots, I_{k(l)}^{(l)})$ (c.f. (2.1), (2.2), [**13**]).

1-III. (i) *Each of* $|||\cdot|||_{2m}$, $|||\cdot|||'_{2m}$ $(1 \leq m < \infty)$ *is non-decreasing; each of* N_{2m}, N'_{2m} $(1 \leq m < \infty)$ *is non-increasing; if* $X, Y \in N'_{2m}$, $|||X^* - Y^*|||'_{2m} \leq |||X - Y|||'_{2m}$;

(ii) $\|\cdot\|_{2m} \leq |||\cdot|||'_{2m} \leq |||\cdot|||_{2m}$; $L^{2m} \supset N'_{2m} \supset N_{2m}$.

PROOF. Except the last one, conclusions of (i) directly follow from the relevant definitions. For the proof of the last conclusion it suffices to notice that $E((X^* - Y^*)_{(n)})^{2m} = E(X_{(n)}^* - Y_{(n)}^*)^{2m} \leq E\{X_{(n)} - Y_{(n)})^*\}^{2m}$, of which the last inequality follows by using $||c_k^{(1)}(\lambda)| - |c_k^{(2)}(\lambda)|| \leq |c_k^{(1)}(\lambda) - c_k^{(2)}(\lambda)|$ $(0 \leq k \leq n)$ in (1.6)-(1.7), where $X = (c_k^{(1)}, 0 \leq k < \infty)$, $Y = (c_k^{(2)}, 0 \leq k < \infty)$.

To prove (ii), make use of Nelson's inequality (p. 113, [**9**]) to have

$$\|X_k\|_{2m} \leq (2m-1)^{k/2} \|X_k\|_2, \quad m = 1, 2, \dots,$$

where X_k is the kth homogeneous polynomial in the expansion of X. Then, if X is finite

$$\begin{aligned} EX^{2m} \leq E(X^*)^{2m} &= \sum_{k(1), \cdots, k(2m)} E(X_{k(1)}^* \cdots X_{k(2m)}^*) \\ &\leq \sum_{k(1), \cdots, k(2m)} \|X_{k(1)}^*\|_{2m} \cdots \|X_{k(2m)}^*\|_{2m} \\ &= (\sum_{k \geq 1} (2m-1)^{k/2} \|X_k\|_2)^{2m}. \end{aligned}$$

In general, given $X \in L^2$, put $X = X_{(n)}$ in the last inequalities, and let $n \to \infty$ along a subsequence of $1, 2, \cdots$ Then one easily gets the conclusions of (ii).

1-IV. *N_{2m} $(1 \leq m < \infty)$ is a Banach space.*

PROOF. Obviously $|||\cdot|||_{2m}$ is a norm. Suppose $X^{(n)} = \{c_k^{(n)}, 0 \leq k < \infty\}$ is a Cauchy sequence in N_{2m}, and put $\alpha_k=(2m-1)^{k/2}$. Since $\|X^{(n)}-X^{(p)}\|_2\leq \|\!|X^{(n)}-X^{(p)}|\!\|_{2m}$, $\{X^{(n)}\}$ being a Cauchy sequence in L^2, we have $\lim_{n\to\infty} X^{(n)}=X$ in L^2 for some $X=(c_0, c_1, \cdots)\in L^2$. $|\!|\!|X^{(n)}|\!|\!|_{2m}$ being bounded for $n\to\infty$,

$$\sum_{k\geq 0}\alpha_k\sqrt{k!}\|c_k^{(n)}\|_2\leq c,$$

with a constant $c>0$ independent of n. On making $n\to\infty$,

$$\sum_{k\geq 0}\alpha_k\sqrt{k!}\|c_k\|_2\leq c,$$

i. e. $X\in N_{2m}$. On the other hand, given $\varepsilon>0$, one can find $n_0=n_0(\varepsilon)$ such that

$$(1.15)\qquad \sum_{k\geq 0}\alpha_k\sqrt{k!}\sqrt{\int|c_k^{(n)}-c_k^{(p)}|^2 d^k\sigma}\leq\varepsilon,\ p,\ n\geq n_0.$$

If we let $p\to\infty$, (1.15) leads to

$$|\!|\!|X^{(n)}-X|\!|\!|_{2m}\leq\varepsilon \text{ for } n\geq n_0,$$

which concludes the completeness of N_{2m}.

1-V. (i) *If* $X\in L^2$, $p>q\geq 1$, *then*

$$(1.16)\qquad E(X_{(p)}^*)^{2m}-E(X_{(q)}^*)^{2m}\geq E\{X_{(p)}-X_{(q)})^*\}^{2m}.$$

(ii) *If* $X\in N'_{2m}$, *then* $|\!|\!|X|\!|\!|'_{2m}=\|X^*\|_{2m}$, $|\!|\!|X-X_{(n)}|\!|\!|'_{2m}\to 0$ *as* $n\to\infty$.

(iii) $(N'_{2m}, |\!|\!|\cdot|\!|\!|'_{2m})$ *is a Banach space.*

PROOF. (i) The obvious equality $X_{(p)}^*=(X_{(p)}-X_{(q)})^*+X_{(q)}^*$ leads to

$$(X_{(p)}^*)^{2m}=((X_{(p)}-X_{(q)})^*)^{2m}+\sum_{k=1}^{2m-1}\binom{2m}{k}(X_{(p)}-X_{(q)})^*)^{2m-k}(X_{(q)}^*)^k$$
$$+(X_{(q)}^*)^{2m}.$$

The second term of the last expression being non-negative, one obtains (1.16).

(ii) Suppose $X\in N'_{2m}$, and apply (1.16) to have

$$(1.17)\qquad 0\leq\lim_{p>q,\, q\to\infty} E((X_{(p)}-X_{(q)})^*)^{2m}$$
$$\leq\lim_{p>q,\, q\to\infty}(E(X_{(p)}^*)^{2m}-E(X_{(q)}^*)^{2m})=0,$$

whence by (1.14)

$$\lim_{p>q,\, q\to\infty} E(X_{(p)}-X_{(q)})^{2m}=0.$$

On the other hand, since $X_{(n)}\to X$, a. e. as $n\to\infty$ along a subsequence of 1, 2, ⋯, one obtains

$$(1.18)\qquad \lim_{p\to\infty} E(X_{(p)}-X)^{2m}=0.$$

This implies also $X^*_{(n)}\to X^*$ in L^{2m}, as $n\to\infty$. Substituting this into $|||X|||'_{2m}=(\lim_{n\to\infty} E(X^*_{(n)})^{2m})^{1/2m}$, we have

$$(1.19)\qquad |||X|||'_{2m}=\|X^*\|_{2m}.$$

Given $\varepsilon>0$, (1. 17) implies that there exists $q_0=q_0(\varepsilon)$ such that

$$\varepsilon\geq\|X^*_{(n)}-X^*_{(q)}\|_{2m} \text{ for } n,\ q\geq q_0.$$

Since $X^*_{(q)}\to X^*$ $(q\to\infty)$ in L^{2m}, the last relation leads to

$$\varepsilon\geq\|X^*_{(n)}-X^*\|_{2m}=\|(X_{(n)}-X)^*\|_{2m} \text{ for } n\geq q_0.$$

Then using (1. 19),

$$\lim_{n\to\infty}|||X_{(n)}-X|||'_{2m}=0.$$

Finally we will prove the completeness of N'_{2m}.

Let $\{Y^{(j)}=(c_0^{(j)}, c_1^{(j)}, \cdots),\ 1\leq j<\infty\}\subset L^2$ be a sequence of finite elements with bounded degrees, such that $Y^{(j)}\to Y^{(\infty)}=(c_0^{(\infty)}, c_1^{(\infty)}, \cdots)$ $(j\to\infty)$ in L^2. Using (1. 6), (1. 7) $\|Y^{(j)*}-Y^{(\infty)*}\|_{2m}\leq\|(Y^{(j)}-Y^{(\infty)})^*\|_{2m}$ and $J=E\{Y^{(j)}-Y^{(\infty)})^*\}^{2m}$ is a sum of bounded number of integrals by product measures $dm=d^p\sigma$ whose kernels are of the same character as those in Proposition 1- I . As a simple application of (1. 12) one concludes that J is bounded by a polynomial in $\|c_k^{(j)}-c_k^{(\infty)}\|_2$ $(k\geq1)$ with coefficients independent of j, so that

$$(1.20)\qquad Y^{(j)*}\to Y^{(\infty)*} \text{ in } L^{2m}, \text{ as } j\to\infty.$$

Let $\{X^{(j)},\ j\geq1\}$ be a Cauchy sequence of N'_{2m}, then by the first inequality of (ii), Proposition 1-III, there exists $X\in L^{2m}$, such that $X^{(j)}\to X$ in L^{2m} $(j\to\infty)$ and moreover as a real Cauchy sequence $|||X^{(j)}|||'_{2m}$ so that by (1. 19) $\|X^{(j)*}\|_{2m}$ being bounded,

$$(1.21)\qquad E(X^{(j)*}_{(n)})^{2m}\leq E(X^{(j)*})^{2m}\leq c,$$

with $c>0$, independent of n, j. Since $X^{(j)}_{(n)}\to X_{(n)}$ in L^2, as $j\to\infty$, by (1. 20),

$X_{(n)}^{(j)*} \to X_{(n)}^{*}$ in L^{2m}, whence from (1.21)

$$E(X_{(n)}^{*})^{2m} \leq c \text{ for all } n \geq 1,$$

i.e. $X \in N'_{2m}$.

Given $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon)$ such that

$$(1.22) \qquad \|(X_{(n)}^{(k)} - X_{(n)}^{(j)})^{*}\|_{2m} \leq |||X^{(k)} - X^{(j)}|||'_{2m} \leq \varepsilon$$

for every $n \geq 1$, whenever $k, j \geq k_0$. On the other hand, since $X_{(n)}^{(k)} - X_{(n)}^{(j)} \to X_{(n)} - X_{(n)}^{(j)}$ in L^2 $(k \to \infty)$, $(X_{(n)}^{(k)} - X_{(n)}^{(j)})^{*} \to (X_{(n)} - X_{(n)}^{(j)})^{*}$ in L^2, on making $k \to \infty$ in (1.22),

$$\sup_{n \geq 1} \|(X_{(n)} - X_{(n)}^{(j)})^{*}\|_{2m} = \sup_{n \geq 1} \|(X - X^{(j)})_{(n)}^{*}\|_{2m} \leq \varepsilon \text{ for } j \geq k_0,$$

or

$$|||X - X^{(j)}|||'_{2m} \leq \varepsilon \text{ for } j \geq k_0.$$

This means that N'_{2m} is complete.

1-IV. *Let $f_{k,l}$, $0 \leq k < \infty$, be a sequence of functions satisfying*

(i) $f_{k,l} \in \mathscr{L}_{k,2} (1 \leq l < \infty)$ $(\mathscr{L}_{0,2} \equiv \boldsymbol{R})$,

(ii) $g_k(\lambda) \equiv \sum_{l=1}^{\infty} |f_{k,l}(\lambda)| \in \mathscr{L}_{k,2}$,

$$\sum_{k=0}^{\infty} k!\, \|g_k\|_2^2 < \infty \quad (\|g_0\|_2 = |g_0|).$$

Then the double series

$$(1.23) \qquad \sum_{k=0,\, l=1}^{\infty} \int f_{k,l}(\lambda)\, d^k\beta \quad \left(\int f_{0,l}(\lambda)\, d^0\beta \text{ represent real numbers}\right) \text{ is}$$

unconditionally convergent to an $X \in L^2$, with

$$\int \left(\sum_{l=1}^{\infty} f_{k,l}\right) d^k\beta \quad (0 \leq k < \infty)$$

as its kth homogeneous polynomial.

Proof. As a standard way of summation for (1.23) consider

$$(1.24) \qquad \sum_{k=0}^{\infty} \left(\sum_{l=1}^{\infty} \int f_{k,l}(\lambda)\right) d^k\beta$$

whose interior and exterior series are easily checked to be convergent in L^2. Then, through a comparison of an arbitrary summation of (1.23) with this, it is easy to see that the former is L^2-convergent to (1.24), with the designated expression as its kth homogeneous polynomial.

PROOF OF THEOREM 1. Having the second inclusion of (ii), 1-III, we are sufficient to prove the theorem under the assumption that all $X^{(j)} \in N'_{2m}$.

Let $\mathscr{P}$ denote the set of matrices P subject to the condition (P), set $\Pi X^{(j)} = X^{(1)} \cdots X^{(m)}$, and write $\mathfrak{K}(\Pi X^{(j)}; P; \lambda, \mathbf{u})$ for $C(P; \lambda, \mathbf{u})$ of (1.5), and moreover $\mathfrak{K}(\Pi X^{(j)}; \lambda, \mathbf{u})$, $C_k(\Pi X^{(j)}; \lambda)$ respectively for $C(\lambda, \mathbf{u})$, $C_k(\lambda)$ in (1.3) when $X^{(j)}$, $1 \leq j \leq m$, are finite. Then refering to (1.3)-(1.5) we have

$$(1.25) \qquad \prod_{j=1}^{m} X^{(j)}_{(n)} = \sum_{k=0}^{\infty} \int C_k(\Pi X^{(j)}_{(n)}; \lambda) d^k \beta$$

$$(1.26) \qquad C_k(\Pi X^{(j)}_{(n)}; \lambda) = \sum_{\bar{\mathbf{u}}=k} \sum_{P \in \mathscr{P}} \mathfrak{K}(\Pi X^{(j)}_{(n)}; \lambda, \mathbf{u}).$$

Put

$$\mathfrak{S}_n(k) = \{\mathfrak{K}(\Pi X^{(j)}_{(n)}; P; \lambda, \mathbf{u}) \mid \bar{\mathbf{u}} = k, P \in \mathscr{P}\},$$
$$\mathfrak{S}(k) = \{\mathfrak{K}(\Pi X^{(j)}; P; \lambda, \mathbf{u}) \mid \bar{\mathbf{u}} = k, P \in \mathscr{P}\}.$$

$\mathfrak{S}(k)$ is the set of kernels arising from multiplying term-wise the expansions of $X^{(j)}$, $1 \leq j \leq m$. Obviously $\mathfrak{S}_n(k) \uparrow \mathfrak{S}(k)$, as $n \to \infty$, and

$$|\mathfrak{K}(\Pi X^{(j)}_{(n)}; P; \lambda, \mathbf{u})| \leq \mathfrak{K}(\Pi X^{(j)*}_{(n)}; P; \lambda, \mathbf{u}),$$
$$|\mathfrak{K}(\Pi X^{(j)}; P; \lambda, \mathbf{u})| \leq \mathfrak{K}(\Pi X^{(j)*}; P; \lambda, \mathbf{u}).$$

Let $\{f_{k,l}, 1 \leq l < \infty\}$ be a linearly ordered enumeration of $\mathfrak{S}(k)$. For every $n \geq 1$

$$(1.27) \qquad \sum_{k \geq 0} k! \int (\sum_{\bar{\mathbf{u}}=k} \sum_{P \in \mathscr{P}} \mathfrak{K}(\Pi X^{(j)*}_{(n)}; P, \mathbf{u}, \lambda)^2 d^k \sigma$$
$$= \sum_{k \geq 0} k! \int |C_k(\Pi X^{(j)*}_{(n)}; \lambda)|^2 d^k \sigma$$
$$= E(X^{(1)*}_{(n)} \cdots X^{(m)*}_{(n)})^2 \leq \prod_{j=1}^{m} |||X^{(j)}|||'_{2m}.$$

Making $n \to \infty$ in (1.27), we see that

$$\sum_{\bar{\mathbf{u}}=k} \sum_{P \in \mathscr{P}} \mathfrak{K}(\Pi X^{(j)*}; P; \mathbf{u}, \lambda)$$

is $d^k\sigma$-a. e. and $\mathscr{L}_{k,2}$ convergent to an $h_k(\lambda) \in \mathscr{L}_{k,2}$, so does

$$\sum_{\bar{\mathbf{u}}=k} \sum_{P \in \mathscr{P}} |\mathfrak{K}(\Pi X^{(j)}; P; \mathbf{u}, \lambda)|$$

to a $g_k(\lambda) \in \mathscr{L}_{k,2}$, and

$$\sum_{k\geq 0} k!\|g_k\|_2^2<\infty.$$

Therefore, appealing to 1-VI, we have the equality

$$(1.28)\qquad \sum_{k\geq 0}\int C_k(\lambda)d^k\beta=\sum_{k\geq 0}\sum_{\bar{\mathbf{u}}=k}\sum_{P\in\mathscr{P}}\int \mathfrak{R}(\Pi X^{(j)};P;\mathbf{u},\lambda)d^k\beta,$$

where

$$(1.29)\qquad C_k(\lambda)=\sum_{\bar{\mathbf{u}}=k}\sum_{P\in\mathscr{P}}\mathfrak{R}(\Pi X^{(j)};P;\mathbf{u},\lambda),$$

and the right-hand side of (1. 28) is unconditionally convergent. On the other hand, the right-hand members of (1. 25) being, as $n\to\infty$, exhausting the partial sums of the right-hand side of (1. 28), after making $n\to\infty$ in (1. 25), we conclude that (1. 28) is equal to $X^{(1)}\cdots X^{(m)}$.

By the multi-linearity of moments and cumulants

$$(1.30)\qquad E(\prod_{j=1}^{P}X_{(n)}^{(j)})=\sum_{D(n)}E(\prod_{j=1}^{P}I_{k(j)}^{(j)})$$

$$(1.31)\qquad S(X_{(n)}^{(1)},\cdots,X_{(n)}^{(p)})=\sum_{D(n)}S(I_{k(1)}^{(1)},\cdots,I_{k(p)}^{(p)}),$$
$$D(n)=\{k(1),\cdots,k(p)\}:0\leq k(1),\cdots,k(p)\leq n\}.$$

On the other hand, using (1. 13), (1. 14), we have

$$\sum_{D(n)}|E(\prod_{j=1}^{p}I_{k(j)}^{(j)})|,\ \sum_{D(n)}|S(I_{k(1)}^{(1)},\cdots,I_{k(p)}^{(p)})|$$
$$\leq\sum_{D(n)}E(\prod_{j=1}^{p}I_{k(j)}^{(j)*})=E(X_{(n)}^{(1)*}\cdots X_{(n)}^{(p)*})\leq\prod_{j=1}^{p}|||X^{(j)}|||_{2m}'.$$

Making $n\to\infty$ we get the desired conclusion in the second half of Theorem 1.

§ 2. Cumulant spectral densities

Let $X=\{X(t),\ -\infty<t<\infty\}$ be a real stationary process subordinate to (1. 1) with $EX(t)=0$. Then it is represented in the form

$$(2.1)\qquad X(t)=\sum_{k\geq 1}X_k(t),\ X_k(t)=I(c_k(\cdot)e_k(\cdot,t))=\int c_k(\lambda)e_k(\lambda,t)d^k\beta,$$
$$\lambda=(\lambda_1,\cdots,\lambda_k)\in\boldsymbol{R}^k,\ e_k(\lambda,t)=\exp i\bar{\lambda}t,\ \bar{\lambda}=\lambda_1+\cdots+\lambda_k.$$

As will be easily seen from the Fourier integral representations for the summands in (2. 1) it presents a natural way of obtaining concrete expressions of moment and cumulant spectral densities.

Let $\mathscr{F}=\{c_{p_i},1\leq i\leq n\}$, $c_{p_i}\in\mathscr{L}_{p_i,2}$, and $I_i=I(c_{p_i})$. After (1. 5) with $u_1=\cdots=u_4=0$

(2.2.1) $S(I_1, \cdots, I_n) = \sum_{\Gamma \in \mathscr{C}(p_1, \cdots, p_n)} \gamma(\Gamma)\mathscr{K}(\Gamma ; \mathscr{F}),$

where the summation is taken over $\mathscr{C}(p_1, \cdots, p_n)$, the set of connected graphs Γ based on $\mathscr{F}$,

(2.2.2) $\gamma(\Gamma) = \{\prod_{k=1}^{n} \frac{p_k\,!}{p_{k1}\,! \cdots p_{kn}\,!}\} \prod_{1 \leq i < j \leq n} p_{ij}\,! = \frac{\prod_{k=1}^{n} p_k\,!}{\prod_{1 \leq i < j \leq n} p_{ij}\,!},$

$$K(\Gamma ; \mathscr{F}) = \int \prod_{k=1}^{n} c_{p_k}(\lambda_{k1}, \cdots, \lambda_{kn}) d^k\sigma,$$

where

$$k = \bar{p}/2,\ \bar{p} = p_1 + \cdots + p_n,\ p_i = \sum_{j=1}^{n} p_{ij},\ p_{ii} = 0,$$
$$p_{ij} = p_{ji},\ \lambda_{ji} = -\lambda_{ij}\ (1 \leq i < j \leq n)\,;$$

$\|p_{ij}\|$ corresponds to Γ.

From now on a connected graph and the corresponding matrix will be denoted by the same symbol.

DEFINITTION 2. Let $X(t)$, $-\infty < t < \infty$, be a real strictly stationary process such that $EX(t) = 0$, $E|X(t)|^m < \infty$. If the function of $(t_1, \cdots, t_{m-1})$, $S(X(t_1), \cdots, X(t_{m-1}), X(0))$ $(m \geq 2)$ admitts the Fourier integral representation

$$\begin{aligned}(2.3)\qquad & S(X(t_1), \cdots, X(t_{m-1}), X(0)) \\ &= \int f_m(x) \exp(i \sum_{k=1}^{m-1} t_k x_k) dx,\ x = (x_1, \cdots, x_{m-1}),\end{aligned}$$

with $f_m \in L(\boldsymbol{R}^{m-1})$, f_m is called the mth cumulant spectral density (CSD) of X.

f_2 is the usual spectral density. Under the same assumption as above on the moments, the mth moment spectral density (MSD) is defined as the L^1 -kernel of the Fourier integral representation of $E(X(t_1) \cdots X(t_{m-1}) X(0))$ if it exists. By means of known algebraic relations between moments and cumulants [7], MSD's are represented in terms of CSD's and vice versa. From the symmetry of moments and cumulants follows that of MND's and CSD'S.

THEOREM 2. *Suppose $X(0) \in N'_{2m}$ for $X(t)$ of* (2.1). *Then it posseses the nth CSD's f_n $(2 \leq n \leq 2m)$ of the form*

(2.4) $f_n(x) = \sum_{\Gamma \in \mathscr{C}} \gamma(\Gamma) f_n(\Gamma, x),\ \mathscr{C} = \bigcup_{1 \leq p_1, \cdots, p_n < \infty} \mathscr{C}(p_1, \cdots, p_n),$

where $\mathscr{C}(p_1, \cdots, p_n)$ and $\gamma(\Gamma)$ $(\Gamma = \|p_{ij}\|)$ are defined in (2.2.1), (2.2.2),

$f_n(\Gamma, x) \in L^1(\boldsymbol{R}^{n-1})$, and it is explicitly written in terms of $\mathscr{T}$ and Γ. The series of (2.4) converges absolutely a. e. (relative to Lebesgue measure) and also in $L^1(\boldsymbol{R}^{n-1})$ to $f_n(x)$.

PROOF. For notational simplification deal with the case $m=2$. Associated with $X(t)$, consider an auxiliary process

$$Y(t)=\sum_{k\geq 1} I(|c_k| e_k(\lambda, t)).$$

By Theorem 1 we have two absolutely convergent series

$$(2.5.1)\quad S(X(t_1), X(t_2), X(t_3), X(0))=\sum_{1\leq p_1, \cdots, p_4<\infty} S(\mathscr{I}),$$

$$(2.5.2)\quad S(Y(t_1), Y(t_2), Y(t_3), Y(0))=\sum_{1\leq p_1, \cdots, p_4<\infty} S(\mathscr{J}),$$

where $S(\mathscr{I})$, $S(\mathscr{J})$ are respectively cumulants of the set $\mathscr{I}=\{I_1, \cdots, I_4\}$, $\mathscr{J}=\{J_1, \cdots, J_4\}$, $I_i=I(c_{p_i}(\cdot)e_{p_i}(\cdot, t_i))$, $J_i=I(|c_{p_i}(\cdot)|e_{p_i}(\cdot, t_i))$, $1\leq i\leq 4$, with $t_4=0$. The set of connected graphs $\mathscr{C}(p_1, \cdots, p_4)$ based on $\mathscr{T}'$ coincides with that of those based on $\mathscr{G}'$, where $\mathscr{T}'=\{c_{p_i}(\cdot)e_{p_i}(\cdot, t_i), 1\leq i\leq 4\}$, $\mathscr{G}'=\{|c_{p_i}(\cdot)|e_{p_i}(\cdot, t_i), 1\leq i\leq 4\}$, with $t_4=0$. Then $S(\mathscr{I})$, $S(\mathscr{J})$ are written in the forms

$$(2.6.1)\quad S(\mathscr{I})=\sum_{\Gamma\in\mathscr{C}(p_1, \cdots, p_4)} \gamma(\Gamma)U(\Gamma, t),$$

$$(2.6.2)\quad S(\mathscr{J})=\sum_{\Gamma\in\mathscr{C}(p_1, \cdots, p_4)} \gamma(\Gamma)V(\Gamma, t),$$

where

$$(2.7)\quad \begin{aligned} &U(\Gamma, t)=K(\Gamma; \mathscr{T}'),\ V(\Gamma, t)=K(\Gamma; \mathscr{G}'),\\ &\gamma(\Gamma)=\prod_1^4 p_i!/\prod_{1\leq i<j\leq 4} p_{ij}!,\ \Gamma=\|p_{ij}\|,\ t=(t_1, t_2, t_3). \end{aligned}$$

One knows that for every $\Gamma\in\mathscr{C}(p_1, \cdots, p_4)$, $U(\Gamma, t)$, $V(\Gamma, t)$ are represented as the Fourier transforms of some $f(\Gamma, x)$, $g(\Gamma, x)\in L^1(\boldsymbol{R}^3)$.

$$(2.8)\quad \begin{aligned} &U(\Gamma, x)=\int \exp it\cdot x\, f(\Gamma, x)dx\\ &V(\Gamma, x)=\int \exp it\cdot x\, g(\Gamma, x)dx,\ x=(x_1, x_2, x_3). \end{aligned}$$

For example, consider such a Γ that $p_{ij}>0$ for all $1\leq i<j\leq 4$, for which one may write $\lambda_{12}=(a, a')$, $\lambda_{13}=(b, b')$, $\lambda_{14}=(c, c')$, $\lambda_{23}=(d, d')$, $\lambda_{24}=(e, e')$, $\lambda_{34}=(f, f')$, with $\dim(a)=\cdots=\dim(f)=1$, $\dim(a'), \cdots, \dim(f')\geq 0$. Then

$$(2.9)\qquad U(\Gamma, t)=\int c_{p_1}(a, b, c, a', b', c')\ c_{p_2}(-a, d, e, -a', d', e')$$
$$\times c_{p_3}(-b, -d, f, -b', -d', f')\ c_{p_4}(-c, -e, -f, -c', -e', -f')$$
$$\times \exp\{it_1(a+l_3)+it_2(-a+d+l_2)+it_3(-d+f+l_1)\}\, WdV,$$

where W is a non-negative function of $a, \cdots, f'$, dV its reference Lebesgue measure, l_1, l_2, l_3 linear functions of the other vectors. There are several different ways of writing the exponential factor. Make a linear transformation from a, d, f to x_1, x_2, x_3:

$$(2.10)\qquad x_1=a+l_1,\ x_2=-a+d+f+l_2,\ x_3=-d+f+l_3.$$

Its inverse enables one to express $c_{p_1}, c_{p_2}, c_{p_3}$, W in the ritht-hand member of (2.9) as a function of $x_1, x_2, x_3, b, c, e, a', \cdots, f'$, say $W'(x, b, \cdots, f')$, $x=(x_1, x_2, x_3)$, and then write $U(\Gamma, t)$ as

$$(2.11)\qquad U(\Gamma, t)=\int \exp\ it\cdot x\ f(\Gamma, x)\,dx,$$

where $f(\Gamma, x)$ is W' integrated out by the variables other than x. That $f\in L^1(\boldsymbol{R}^3)$ is implied by Fubini's theorem used on the passage leading to (2.11), or by the estimate

$$\int |f(\Gamma, x)|\,dx\le\int \prod_{k=1}^{4} |c_{p_k}|\, WdV\le \prod_{k=1}^{4} \|c_{p_k}\|_2,$$

which is obtained as an easy consequence of 1-I. Since $U(\Gamma, t)$, $V(\Gamma, t)$ are constructed on the same graph, $g(\Gamma, x)$ is obtained by writing $|c_{p_i}|$ in place of c_{p_i} contained in $f(\Gamma, x)$. This implies

$$(2.12)\qquad |f(\Gamma, x)|\le g(\Gamma, x).$$

From (2.6.1), (2.6.2) we have

$$S(X(t_1), \cdots, X(0))=\sum_{\Gamma\in\mathscr{C}} \gamma(\Gamma)\int f(\Gamma, x)\exp\ it\cdot x\ dx,$$
$$S(Y(t_1), \cdots, Y(0))=\sum_{\Gamma\in\mathscr{C}} \gamma(\Gamma)\int g(\Gamma, x)\exp\ it\cdot x\ dx,$$
$$\mathscr{C}=\bigcup_{1\le p_1, \cdots, p_4<\infty} \mathscr{C}(p_1, \cdots, p_4).$$

Then, since

$$\sum_{\Gamma\in\mathscr{C}} \gamma(\Gamma)\int g(\Gamma, x)\,dx=S(Y(0), \cdots, Y(0))<\infty,$$

by (2.12)

$$f_4(x)=\sum_{\Gamma\in\mathscr{C}}\gamma(\Gamma)f(\Gamma,x)$$

is an L^1-function, the right-hand side being convergent in the requested manner to f_4, the CSD of X.

§ 3. The local behaviors of the spectral density and related Abelian and Tauberian theorems

Throughout this section $X=\{X(t),\ -\infty<t<\infty\}$ is a square integrable real stationary process with zero mean. Let $\varphi(\lambda)/2$ be the spectral density of X. As we have seen in [**13**], the growth of

$$V(T)=V\left(\int_0^T X(t)\,dt\right)=\int_0^\infty D_T^2(\lambda)\varphi(\lambda)\,d\lambda, \tag{3.1}$$
$$D_T(\lambda)=\frac{\sin T\lambda/2}{\lambda/2},$$

as $T\to\infty$, is closely related with the local behavior of $\varphi(\lambda)$ at 0.

A positive Borel function f on $(0,a]$, $0<a\le\infty$ ($[b,\infty)$, $0\le b<\infty$), is slowly varying (SV) at $0(\infty)$ if it is locally bounded and there exists $\lim_{x\to 0}f(cx)/f(x)=1$ ($\lim_{x\to\infty}f(cx)/f(x)=1$) for any $c>0$. Slowly varying functions at ∞ correspond in 1-1 way to those at 0 through the map $y=x^{-1}$ from $[0,\infty)$ onto $(0,\infty]$.

As will be made clear in § 4, the slow variation of $V(T)$ at ∞ is an essential character for the functional central limit theorem (FCLT). That SV property is, as propositions in this section will clarify, intimately connected with allied properties of $\varphi(\lambda)$ at 0.

When α is a real constant, a positive function g on $(0,a]$ ($[b,\infty)$) of the form $g(x)=x^\alpha f(x)$, f SV at $0(\infty)$, is said to be regularly varying (RV) at $0(\infty)$ with exponent α. The only case $\alpha=1$ arrising in the present paper, by RV we mean exclusively this type of variation.

$h_0(x)$ is SV at 0 iff it is represented in the form (Feller [**2**], Ibragimov-Linnik [**4**])

$$h_0(x)=c(x)s_0(x), \tag{3.2}$$

$$s_0(x)=\exp\left(\int_x^a\frac{\eta(u)}{u}du\right),\quad 0<x\le a, \tag{3.3}$$

where $c(x)$, $\eta(x)$ are bounded having

$$\lim_{x\to+0}c(x)>0,\quad \lim_{x\to+0}\eta(x)=0. \tag{3.4}$$

$s_0(x)$ itself is SV at 0. $s_0(x)$ will be called canonically SV. If in the above the first condition is relaxed to $\underline{\lim}_{x\to+0}\, c(x)>0$, $h_0(x)$ is said to be SV in the wide sense.

$h_\infty(x)$ is SV at ∞ iff $h_\infty(x)=h_0(1/x)$, with some h_0, SV as 0, or equivalently

$$(3.5)\qquad h_\infty(x)=d(x)s_\infty(x)$$

$$(3.6)\qquad s_\infty(x)=\exp\Big(\int_b^x \frac{\varepsilon(u)}{u}du\Big),\ b\le x<\infty,$$

where d, ε are bounded having

$$\underline{\lim}_{x\to\infty} d(x)>0,\ \lim_{x\to\infty}\varepsilon(x)=0.$$

From now on by h_0, s_0, η, c, h_∞, s_∞, ε, d respectively we denote the functions standing in (3. 2)-(3. 6) in reference to slow variation.

When evaluating expressions involving SV functions, frequent uses are made of the fact that for an arbitrary $\varepsilon>0$

$$(3.7)\qquad c_1x^\varepsilon<h_0(x)<c_2\frac{1}{x^\varepsilon}$$

if $x>0$ is small enough.

We prepare several propositions for the use in § 4.

3- I . *Let $\varphi(\lambda)/2$ be the spectral density of X. If $V(T)$ is RV at ∞, or*

$$(3.8)\qquad V(T)\sim Th_\infty(T),\ T\to\infty,$$

($\sim$ *means that the ratio of the both members tends to* 1) *then* $H(x)=\int_0^x\varphi(\lambda)d\lambda$ *is RV at* 0, *more precisely*

$$(3.9)\qquad H(x)\sim\frac{1}{\pi}xh_0(x),\ h_0(x)=h_\infty(1/x),\ x\to+0.$$

PROOF.

$$(3.10)\qquad \int_0^\infty V(T)e^{-sT}dT=\int_0^\infty\varphi(\lambda)d\lambda\int_0^\infty D_T^2(\lambda)e^{-sT}dT$$
$$=\frac{1}{s}\int_0^\infty\frac{2\varphi(\lambda)}{s^2+\lambda^2}d\lambda.$$

On the other hand, by L'hospital's rule

$$\lim_{T\to\infty}\frac{1}{T^2s_\infty(T)}\int_0^T ud(u)s_\infty(u)du$$

$$= \lim_{T\to\infty} \frac{Td(T)s_\infty(T)}{2Ts_\infty(T)+T^2s_\infty(T)(-\varepsilon(T)/(T)} = \frac{1}{2}d(\infty).$$

This means that

$$\int_0^T V(u)\,du \sim t^2 h_\infty(t)/2, \ t\to\infty,$$

whence by the Abelian theorem (Theorem 2, p. 421, Feller [**2**]),

$$\int_0^\infty V(T)e^{-sT}dT \sim h_0(s)/s^2 \ (s\to+0).$$

Substitute this into (3. 10) to have

$$\int_0^\infty \frac{2\varphi(\lambda)}{s^2+\lambda^2}\,d\lambda \sim h_0(s)/s \ \ (s\to+0),$$

or

$$(3.11) \qquad \int_0^\infty \frac{\varphi(\sqrt{x})}{\sigma+x}\,\frac{dx}{\sqrt{x}} = \int_0^\infty e^{-\sigma t}u(t)\,dt \sim \tilde{h}_\infty(1/\sigma)/\sqrt{\sigma}, \ \sigma\to+0,$$

where we set

$$u(t) = \int_0^\infty e^{-xt}\frac{\varphi(\sqrt{x})}{\sqrt{x}}dx,$$
$$\tilde{h}_\infty(1/\sigma) = \tilde{h}_0(\sigma) \equiv h_0(\sqrt{\sigma}).$$

Apply Theorem 4, p. 423, [**2**] to obtain

$$u(t) = \int_0^\infty e^{-xt}\frac{\varphi(\sqrt{x})}{\sqrt{x}}dx \sim \frac{t^{-1/2}}{\Gamma(1/2)}\tilde{h}_\infty(t), \ t\to\infty.$$

Then by the Tauberian theorems (Theorem 2, Theorem 3, pp. 421-423, [**2**]), this implies that

$$\int_0^x \frac{\varphi(\sqrt{y})}{\sqrt{y}}dy \sim \frac{\sqrt{x}\,\tilde{h}_\infty(1/x)}{\Gamma(1/2+1)\Gamma(1/2)} = \frac{2}{\pi}\sqrt{x}\,h_0(\sqrt{x}), \ x\to+0,$$

or

$$\int_0^x \varphi(\lambda)\,d\lambda \sim \frac{x}{\pi}h_0(x), \ x\to+0.$$

The following propositions are motivated by asking question if the converse to 3- I is true. Although a satisfactory answer to this has not been obtained, we have singled out local behaviors of $\varphi(\lambda)$ which are sufficient for CLT and FCLT.

3-II. *Let $h_0(x)$ be SV near zero, $h_0(x)=c(x)s_0(x)$, $c=c(+0)>0$, then on a right-hand neighborhood of zero, one can find $\bar{c}(x)$, $\tilde{c}(x)$ such that*

$$\int_0^x h_0(y)\,dy = x\bar{c}(x)s_0(x), \tag{3.12}$$

where

$$\bar{c}(x)=\frac{1}{x}\int_0^x \tilde{c}(y)\,dy,\quad \tilde{c}(+0)=c, \tag{3.13}$$

$$\lim_{x\to+0} x\bar{c}'(x)=0.$$

PROOF. The L'hospital rule gives

$$\lim_{x\to+0}\frac{\int_0^x h_0(y)\,dy}{cxs_0(x)}=\lim_{x\to+0}\frac{h_0(x)}{cs_0(x)+cs_0(x)(-\eta(x))} \tag{3.14}$$
$$=\lim_{x\to+0}\frac{c(x)s_0(x)}{cs_0(x)(1-\eta(x))}=1.$$

So that there exists $\bar{c}(x)$ which satisfies (3.12) with $\bar{c}(+0)=c$. Differentiation of (3.12) leads to

$$h_0(x)=\bar{c}(x)s_0(x)+x\bar{c}'(x)s_0(x)-\bar{c}(x)s_0(x)\eta(x).$$

Substituting this into (3.14)

$$\lim_{x\to+0}\frac{\bar{c}(x)+x\bar{c}'(x)-\bar{c}(x)\eta(x)}{c(1-\eta(x))}=1,$$

whence

$$\lim_{x\to+0} x\bar{c}'(x)=0.$$

On the other hand, using (3.12)

$$\left(\frac{\int_0^x h_0(y)\,dy}{s_0(x)}\right)'=\frac{c(x)s_0(x)+\eta(x)\bar{c}(x)s_0(x)}{s_0(x)}=c(x)+\eta(x)\bar{c}(x).$$

Therefore

$$\int_0^x h_0(y)\,dy=s_0(x)\int_0^x \tilde{c}(y)\,dy,$$
$$\tilde{c}(y)=c(y)+\eta(y)\bar{c}(y).$$

$\bar{c}$, $\tilde{c}$ satisfy the requested conditions.

3-III. *Let $\varphi(\lambda)\in L[0,\infty]$ and satisfy*
$$|\varphi(\lambda)|\leq a(\lambda)s_0(\lambda)$$
on a right-hand neighborhood of zero, with a non-negative bounded a. Then

$$\int_0^x|\varphi(\lambda)|d\lambda\leq\frac{3}{2}\Big(\int_0^x\bar{a}(\lambda)d\lambda\Big)s_0(x),\ 0\leq x\leq\delta$$

$$\bar{a}(\lambda)=\sup_{0\leq u\leq\lambda}a(u)$$

for some $\delta>0$.

Proof. By integration by parts

$$\int_0^x|\varphi(\lambda)|d\lambda\leq\int_0^x\bar{a}(\lambda)s_0(\lambda)d\lambda=A(x)+B(x),$$
$$A(x)=\int_0^x\bar{a}(\lambda)d\lambda\cdot s_0(x),$$
$$B(x)=\int_0^x\Big(\int_0^\lambda\bar{a}(\mu)d\mu\Big)s_0(\lambda)\frac{\eta(\lambda)}{\lambda}d\lambda.$$

Take $\delta>0$ so small that $m(\delta)=\sup_{0<x\leq\delta}|\eta(x)|<1/3$. For $0<x\leq\delta$ we have

$$A'(x)\geq s_0(x)\bar{a}(x)(1-m(\delta))\geq\frac{2}{3}s_0(x)\bar{a}(x),$$
$$|B(x)|\leq C(x),$$
$$C(x)=m(\delta)\int_0^x\Big(\frac{1}{\lambda}\int_0^\lambda\bar{a}(\mu)d\mu\Big)s_0(\lambda)d\lambda,$$

and

$$C'(x)=m(\delta)\frac{1}{x}\int_0^x\bar{a}(\lambda)d\lambda\cdot s_0(x)\leq\frac{1}{3}\bar{a}(x)s_0(x)\leq\frac{1}{2}A'(x).$$

This implies

$$C(x)\leq\frac{1}{2}A(x).$$

Then

$$\int_0^x|\varphi(\lambda)|d\lambda\leq A(x)+|B(x)|\leq A(x)+\frac{1}{2}A(x)=\frac{3}{2}A(x).$$

This completes the proof.

3-IV. *Let $\varphi(\lambda)\in L[0,\infty)$, and on a right-hand neighborhood of zero*

$$H(x)=\int_0^x |\varphi(\lambda)|\,d\lambda \leq xc(x)s_0(x),$$

with a non-negative bounded Borel $c(x)$, canonically SV $s_0(x)$ near zero.
Then

$$(3.15)\qquad \overline{\lim_{T\to\infty}}\ \left|\int_0^\infty D_T^2(\lambda)\varphi(\lambda)\,d\lambda\right|/Ts_0(1/T)\leq 9\ \overline{\lim_{x\to+0}}\ c(x).$$

PROOF. Write

$$\int_0^\infty D_T^2(\lambda)\varphi(\lambda)\,d\lambda=I_1+I_2,$$
$$I_1=\int_0^{1/T} D_T^2(\lambda)\varphi(\lambda)\,d\lambda,\ I_2=\int_{1/T}^\infty D_T^2(\lambda)\varphi(\lambda)\,d\lambda.$$

We have

$$(3.16)\qquad I_1\leq T^2\int_0^{1/T}|\varphi(\lambda)|\,d\lambda\leq m(1/T)\,Ts_0(1/T),$$
$$m(x)=\sup_{0<u\leq x} c(u).$$

Take a small $\delta>0$. Then by integration by parts

$$(3.17)\qquad \frac{1}{4}|I_2|\leq\int_{1/T}^\infty |\varphi(\lambda)|/\lambda^2 d\lambda\leq 2\int_{1/T}^\infty \frac{H(x)}{x^3}dx$$
$$=2\left(\int_\delta^\infty+\int_{1/T}^\delta\right)H(x)/x^3dx\leq\frac{1}{\delta^2}\|\varphi\|_L+2\int_{1/T}^\delta H(x)/x^3dx.$$

Extend $s_0(x)$, originally defined on a right-hand neighborhood of zero, to $(0,\infty)$ having

$$s_0(x)=\exp\left(\int_x^\infty\frac{\eta(u)}{u}du\right),\ 0<x<\infty,$$

with a bounded η satisfying $\eta(+0)=0$. Thus

$$(3.18)\qquad \int_{1/T}^\delta H(x)/x^3dx\leq m(\delta)\int_{1/T}^\delta s_0(x)/x^2dx.$$

Integration by parts gives

$$\int_{1/T}^\infty s_0(x)/x^2dx=A(T)-B(T),$$
$$A(T)=Ts_0(1/T),$$
$$B(T)=\int_{1/T}^\infty\frac{\eta(x)}{x^2}dx\ \exp\left(\int_x^\infty\frac{\eta(u)}{u}du\right).$$

First, (3. 7) implies that $A\to\infty$ and

$$\int_{1/T}^{\infty} \frac{1}{x^2} \exp\left(\int_x^{\infty} \frac{\eta(u)}{u} du\right) dx \geq c_1 \int_{1/T}^{\infty} \frac{1}{x^{3/2}} dx \to \infty, \text{ as } T \to \infty.$$

Second, since $\eta(+0)=0$, when $T\to\infty$

$$|B| \leq o(C(T)),$$
$$C(T)=\int_{1/T}^{\infty} \frac{1}{x^2} \exp\left(\int_x^{\infty} \frac{\eta(u)}{u} du\right) dx.$$

Moreover by L'hospital's rule

$$\lim_{T\to\infty} \frac{C}{A} = \lim_{T\to\infty} \frac{T^2 s_0(1/T)(-1/T^2)}{s_0(1/T)+s_0(1/T)\eta(1/T)} = -1.$$

Therefore

$$(3.19) \qquad \overline{\lim_{T\to\infty}} \int_{1/T}^{\delta} \frac{H(x)}{x^3} dx / T s_0(1/T) \leq m(\delta)$$

Putting (3. 16)-(3. 19) together we get (3. 15).

3-V. *Let $\varphi(\lambda)/2$ be the spectral density of X.*

(i) *Suppose that $\int_0^x \varphi(\lambda)d\lambda$ is RV near zero,*

$$(3.20) \qquad \int_0^x \varphi(\lambda)d\lambda = x h_0(x),\ h_0(x)=c(x)s_0(x),\ c(+0)=c>0,\ 0<x\leq\delta$$

for some $\delta>0$, and moreover

$$(3.21) \qquad yc'(y) \in L_{loc}[0,\infty),\ \int_0^x |yc'(y)|dy = o(x),\ x\to+0.$$

Then on $(0,\delta]$ φ is decomposed into two (not always positive) parts

$$(3.22) \qquad \varphi(\lambda)=\varphi_1(\lambda)+\varphi_2(\lambda),\ \varphi_1,\ \varphi_2 \in L[0,\delta],$$

such that on $[0,\delta]$

$$\int_0^x |\varphi_1(\lambda)|d\lambda \leq x c_1(x) s_0(x),$$

where $c_1(x)$ is bounded, with $c_1(+0)=0$, and

$$(3.23) \qquad \varphi_2(x) = c(x s_0(x))'.$$

We have

$$(3.24) \qquad \int_0^{\infty} D_T^2(\lambda)\varphi(\lambda)d\lambda \sim \pi T h_0(1/T),\ T\to\infty.$$

(ii) *If* $\varphi(\lambda)$ *itself is SV near zero, the assumptions in* (i) *are satisfied. So that* (3.24) *is true.*

PROOF. (ii) is obvious from 3-II. From (3.20)

$$\varphi(x)=\varphi_1(x)+\varphi_2(x),$$
$$\varphi_1(x)=(c(x)-c)s_0(x)+x(c(x)-c)s_0'(x)+xc'(x)s_0(x),$$
$$\varphi_2(x)=c(xs_0(x))'.$$

Then

$$|\varphi_1(x)|\leq a_0(x)s_0(x),$$
$$a_0(x)=|c(x)-c|+|(c(x)-c)\eta(x)|+|xc'(x)|,$$

and by (3.21)

$$\int_0^x a_0(y)\,dy=o(x),\quad x\to+0.$$

By integration by parts

$$\int_0^x|\varphi_1(\lambda)|\,d\lambda\leq J_1+J_2, \tag{3.25}$$

where

$$J_1=xa_1(x)s_0(x),\quad a_1(x)=\frac{1}{x}\int_0^x a_0(\lambda)\,d\lambda\to 0,\quad x\to+0, \tag{3.26}$$
$$J_2=\int_0^x\Big(\frac{1}{\lambda}\int_0^\lambda a_0(\mu)\,d\mu\Big)s_0(\lambda)\eta(\lambda)\,d\lambda,$$

whence by 3-III

$$|J_2|=\Big|\int_0^x\Big(\frac{1}{\lambda}\int_0^\lambda a_0(\mu)\,d\mu\Big)s_0(\lambda)\eta(\lambda)\,d\lambda\Big| \tag{3.27}$$
$$\leq\frac{3}{2}xa_2(x)a_3(x)s_0(x),$$

where

$$a_2(x)=\sup_{0<\lambda\leq x}\frac{1}{\lambda}\int_0^\lambda a_0(\mu)\,d\mu,\quad a_3(x)=\sup_{0<\lambda\leq x}|\eta(\lambda)|.$$

Finally from putting (3.25)-(3.27) together we have

$$\int_0^x|\varphi_1(\lambda)|\,d\lambda\leq xc_1(x)s_0(x),$$

with

$$c_1(x)=a_1(x)+\frac{3}{2}a_2(x)a_3(x),$$

which clearly satisfies the required conditions.

To proceed to the proof of (3.24), extend $\varphi_i(i=1,2)$ in such a way that $\varphi_1(\lambda)=\varphi(\lambda)$, $\varphi_2(\lambda)=0$ on $[\delta,\infty)$, and notice that $\delta>0$ in the following can be chosen arbitrarily small. Write

$$(3.28)\qquad \int_0^\infty D_T^2(\lambda)\varphi(\lambda)d\lambda=\int_0^\infty D_T^2(\lambda)\varphi_1(\lambda)d\lambda+\int_0^\infty D_T^2(\lambda)\varphi_2(\lambda)d\lambda$$

and observe that 3-IV means

$$(3.29)\qquad \int_0^\infty D_T^2(\lambda)\varphi_1(\lambda)d\lambda=o(Ts_0(1/T)).$$

On the other hand

$$(3.30)\qquad \int_0^\infty D_T^2(\lambda)\varphi_2(\lambda)d\lambda=I_1+I_2,$$

$$I_1=c\int_0^\delta D_T^2(\lambda)s_0(\lambda)d\lambda,\quad I_2=-c\int_0^\delta D_T^2(\lambda)s_0(\lambda)\eta(\lambda)d\lambda.$$

Since $\eta(+0)=0$, by 3-III, 3-IV

$$(3.31)\qquad I_2=o(Ts_0(1/T)).$$

On the other hand

$$\int_0^\delta D_T^2(\lambda)s_0(\lambda)d\lambda=T\int_0^{\delta T}D^2(\lambda)s_0(\lambda/T)d\lambda$$
$$=Ts_0(1/T)\int_0^{\delta T}D^2(\lambda)\frac{s_0(\lambda/T)}{s_0(1/T)}d\lambda,\quad D(\lambda)\equiv D_1(\lambda).$$

Write

$$(3.32)\qquad \int_0^{\delta T}D^2(\lambda)\frac{s_0(\lambda/T)}{s_0(1/T)}d\lambda=\left(\int_0^1+\int_0^{\delta T}\right)D^2(\lambda)\frac{s_0(\lambda/T)}{s_0(1/T)}d\lambda.$$

First, on $0<\lambda\leq 1$, when $T\to\infty$

$$(3.33)\qquad D^2(\lambda)\frac{s_0(\lambda/T)}{s_0(1/T)}\leq c_0\exp\left(\int_{\lambda/T}^{1/T}\frac{\eta(u)}{u}du\right)\leq c_0\exp\left(\varepsilon\int_{\lambda/T}^{1/T}\frac{du}{u}\right)$$
$$=c_0\frac{1}{\lambda^\varepsilon},$$

with an arbitrarily small constant $\varepsilon>0$. Second, on $1\leq\lambda\leq\delta T$, $\delta>0$ having been sufficiently small, (3.34) $D^2(\lambda)\frac{s_0(\lambda/T)}{s_0(1/T)}\le\frac{4}{\lambda^2}\exp(-\int_{1/T}^{\lambda/T}\frac{\eta(u)}{u}du)\le\frac{4}{\lambda^2}\exp(\varepsilon\int_{1/T}^{\lambda/T}\frac{du}{u})$

$$=\frac{4}{\lambda^{2-\varepsilon}},$$

where $\varepsilon>0$ can be made as small as we please with δ. Obviously

$$D^2(\lambda)\frac{s_0(\lambda/T)}{s_0(1/T)}\to D^2(\lambda),\ T\to\infty,\ \text{for every } \lambda>0.$$

So that with the right-hand members on (3.33), (3.34) as majorants, the Lebesgue convergence theorem applied to (3.32) concludes that

$$\lim_{T\to\infty}\int_0^{\delta T}D^2(\lambda)\frac{s_0(\lambda/T)}{s_0(1/T)}d\lambda=\int_0^\infty D^2(\lambda)d\lambda=\pi,$$

whence

$$\int_0^\delta D_T^2(\lambda)s_0(\lambda)d\lambda\sim\pi Ts_0(1/T).$$

This together with (3.28)-(3.31) proves (3.24).

3-VI. *Let $\varphi(\lambda)/2$ be the spectral density of X. Suppose that $H(x)=\int_0^x\varphi(\lambda)d\lambda$ is RV at zero, specifically*

(3.35) $H(x)=xh_0(x),\ h_0(x)=c(x)s_0(x)$ *on* $(0,\delta)$,

for some $\delta>0$, and

(3.36) $c(x)-c(+0)=O(x^q),\ x\to+0,$

where q is a positive constant.

Then

$$V(T)\sim\pi Th_0(1/T).$$

PROOF. Rewrite the second equality in (3.35) into

$$h_0(x)=k(1+\gamma(x))s_0(x),$$

where

$$k=c(+0),\ \gamma(x)=(c(x)-c(+0))/c(+0).$$

(3.36) is then equivalent to

$$\gamma(x)=O(x^{1-p}),\ -\infty<p<1,$$

but in proving the assertion, we may and will assume that $0<p<1$.

Extend $s_0(x)$ to $(0, \infty)$ by

$$s_0(x)=\exp\left(\int_x^\infty \eta(u)/u\,du\right),\quad 0<x<\infty,$$

where η is bounded, identically zero near ∞. Define γ on $[\delta, \infty)$ so that $H(x)=xk(1+\gamma(x))s_0(x)$ throughout $(0, \infty)$. Then γ is bounded on $(0, \infty)$. Decompose φ on $(0, \delta)$ into three parts φ_j, $1\leq j\leq 3$,

$$\begin{aligned}&\varphi_1(x)=kx\gamma'(x)s_0(x),\quad \varphi_2(x)=\gamma(x)s_0(x)+x\gamma(x)s_0'(x)\\&\varphi_3(x)=k(xs_0(x))',\end{aligned}$$

and then extend them in such a way that $\varphi_1(x)=\varphi(x)$, $\varphi_2(x)=\varphi_3(x)=0$ on $[\delta, \infty)$, so that $\varphi(x)=\varphi_1(x)+\varphi_2(x)+\varphi_3(x)$ over $(0, \infty)$.

Since φ_2 satisfies the condition $\bar{a}(+0)=0$ in 3-III in adddition to the conditions imposed on φ there, by IV we conclude that

$$\int_0^\infty D_T^2(\lambda)\varphi_2(\lambda)\,d\lambda=o(Ts_0(1/T)),\quad T\to\infty.$$

By the final step in the proof of V

$$\int_0^\infty D_T^2(\lambda)\varphi_3(\lambda)\,d\lambda\sim k\pi Ts_0(1/T),\quad T\to\infty.$$

So that for the proof of the present propositon we are sufficed to show

$$V_0(T)\equiv\int_0^\infty D_T^2(\lambda)\varphi_1(\lambda)\,d\lambda=o(Ts_0(1/T)). \tag{3.37}$$

Write

$$\begin{aligned}&V_0(T)=I_1+I_2,\\&I_1=\int_0^a D_T^2(\lambda)\varphi_1(\lambda)\,d\lambda,\quad I_2=\int_a^\infty D_T^2(\lambda)\varphi_1(\lambda)\,d\lambda.\end{aligned}$$

To evaluate I_2 take $a=T^{-1/2+\varepsilon_0}$, where ε_0 $(0<\varepsilon_0<1/2)$ will be specified later. Then

$$\begin{aligned}D_2&=\int_a^\infty D_T^2(\lambda)\varphi_1(\lambda)\,d\lambda\leq\frac{4}{a^2}\|\varphi_1\|_L=O(T^{1-\varepsilon_0})\\&=o(Ts_0(1/T)),\quad T\to\infty.\end{aligned} \tag{3.38}$$

Turning to I_1, if we set $d(\lambda)=\lambda\gamma'(\lambda)$

$$\begin{aligned}\frac{I_1}{4k}&=\frac{1}{4}\int_0^a D_T^2(\lambda)d(\lambda)s_0(\lambda)\,d\lambda\\&=\frac{1}{4}Ts_0(1/T)\int_0^{aT}D^2(\lambda)d(\lambda/T)\frac{s_0(\lambda/T)}{s_0(1/T)}d\lambda\end{aligned} \tag{3.39}$$

$$= Ts_0(1/T) \int_{1/aT}^{\infty} \sin^2(1/2\lambda)\, d(1/\lambda T)\frac{s_0(1/\lambda T)}{s_0(1/T)}d\lambda.$$

First, to evaluate the last expression observe that

$$(3.40)\quad \int_0^x d(1/\lambda T)d\lambda = \frac{1}{T}\int_0^{xT} d(1/\lambda)d\lambda = \frac{1}{T}\int_{1/xT}^{\infty} \frac{d(\lambda)}{\lambda^2}d\lambda$$
$$= \frac{1}{T}\int_{1/xT}^{\infty} \frac{\gamma'(\lambda)}{\lambda}d\lambda$$
$$= \frac{1}{T}\{xT\gamma(1/xT) + \int_{1/xT}^{\infty} \gamma(\lambda)/\lambda^2 d\lambda\},$$

where on passing to the last expression we have used integration by parts and the boundedness of γ.

Second, if $xT>1$, $|xT\gamma(1/xT)| \le c_1 xT(1/xT)^{1-p} = c_1(xT)^p$,

$$\left|\int_{1/xT}^{\infty} \frac{\gamma(\lambda)}{\lambda^2}d\lambda\right| \le \left|\int_1^{\infty}\frac{\gamma(\lambda)}{\lambda^2}d\lambda\right| + \left|c_1\int_{1/xT}^{1} \frac{\lambda^{1-p}}{\lambda^2}d\lambda\right|$$
$$\le \|\gamma\|_\infty + \frac{c_1}{p}(xT)^p \le (\|\gamma\|_\infty + c_1/p)(xT)^p.$$

Substitute this into (3. 40) to have

$$(3.41)\quad \int_0^x d(1/\lambda T)d\lambda \le c_2\frac{1}{T}(xT)^p \text{ if } xT>1.$$

Third, by integration by parts

$$(3.42)\quad \int_{1/aT}^{\infty} \sin^2(1/2\lambda)\, d(1/\lambda T)\frac{s_0(1/\lambda T)}{s_0(1/T)}d\lambda$$
$$= \sin^2(1/2\lambda)\frac{s_0(1/T)}{s_0(1/T)}\int_0^{\lambda} d(1/\mu T)d\mu\Big]_{\lambda=1/aT}^{\infty}$$
$$+\frac{1}{2}\int_{1/aT}^{\infty} \sin(1/\lambda)\frac{1}{\lambda^2}\frac{s_0(1/\lambda T)}{s_0(1/T)}d\lambda \int_0^{\lambda} d(1/\mu T)d\mu$$
$$-\int_{1/aT}^{\infty} \sin^2(1/2\lambda)\left(\frac{s_0(1/\lambda T)}{s_0(1/T)}\right)' d\lambda \int_0^{\lambda} d(1/\mu T)d\mu.$$

Since by (3. 41) $\int_0^{\lambda} d(1/\mu T)d\mu \le c_2(\lambda T)^p/T$, and by (3. 7) $s_0(1/\lambda T) \le c_3\lambda$, as $\lambda\to\infty$, the first term of (3. 42) is equal to

$$(3.43)\quad -\sin^2(aT/2)\frac{s_0(a)}{s_0(1/T)}\int_0^{1/aT} d(1/\mu T)d\mu.$$

By (3. 7) and (3. 41), one can find ε_1, $0<\varepsilon_1<1/2$ such that the expression in (3. 43) is in absolute value less than $c_4 T^{-1+p/2+\varepsilon_1}$, as $T\to\infty$, whence

(3.44) the first term in (3. 42)$\rightarrow 0$, $T\rightarrow\infty$.

The second term of (3. 42) is by (3. 41) less than

$$c_5\left(\int_{1/aT}^{1}+\int_{1}^{\infty}\right)\frac{1}{\lambda^2}\frac{s_0(1/\lambda T)}{s_0(1/T)}\frac{1}{T}(\lambda T)^p d\lambda=c_5(J_1+J_2),$$

$$J_1=\frac{1}{T^{1-p}}\int_{1/aT}^{1}\frac{1}{\lambda^{2-p}}\frac{s_0(1/\lambda T)}{s_0(1/T)}d\lambda,$$

$$J_2=\frac{1}{T^{1-p}}\int_{1}^{\infty}\frac{1}{\lambda^{2-p}}\frac{s_0(1/\lambda T)}{s_0(1/T)}d\lambda.$$

To evaluate these make use of

(3.45) $$\frac{s_0(1/\lambda T)}{s_0(1/T)}\leq c_6\frac{1}{\lambda^{\varepsilon_3}}\quad\text{for }\frac{1}{aT}<\lambda<1,$$
$$\leq c_6\lambda^{\varepsilon_3}\qquad\text{for }\lambda>1,$$

with $\varepsilon_3>0$, which can be chosen arbitrarily small, as $T\rightarrow\infty$.

First, remembering $aT=T^{1/2+\varepsilon_0}$, by (3. 45)

$$J_1=O\left(\frac{1}{T^{1-p}}\right)\int_{1/aT}^{1}\frac{1}{\lambda^{2-p+\varepsilon_3}}d\lambda=O(T^{\xi}),$$

where $\xi=-(1-p)/2+\varepsilon_3/2+\varepsilon_0(1-p)+\varepsilon_0\varepsilon_3$. Take ε_0, $\varepsilon_3>0$ so small that $\xi<0$, then $J_1\rightarrow 0$, as $T\rightarrow\infty$.

Second, take $\varepsilon_3>0$ so small that $2-p-\varepsilon_3>1$, then by (3. 45)

$$J_2=O\left(\frac{1}{T^{1-p}}\right)\int_{1}^{\infty}\frac{1}{\lambda^{2-p-\varepsilon_3}}d\lambda=O(1/T^{1-p})=o(1),\quad T\rightarrow\infty.$$

Thus

(3.46) the second term of (3. 42)$\rightarrow 0$, as $T\rightarrow\infty$.

The third term of (3. 42) is in absolute value less than

$$\int_{1/aT}^{1}\sin^2(1/2\lambda)\frac{s_0(1/\lambda T)}{s_0(1/T)}\lambda T\,|\eta(1/\lambda T)|\frac{1}{\lambda^2 T}d\lambda\int_{0}^{\lambda}d(1/\mu T)d\mu$$
$$\leq c_7(K_1+K_2),$$
$$K_1=\int_{1/aT}^{1}\sin^2(1/2\lambda)\frac{s_0(1/\lambda T)}{s_0(1/T)}\frac{1}{(\lambda T)^{1-p}}|\eta(1/\lambda T)|\,d\lambda,$$
$$K_2=\int_{1}^{\infty}\sin^2(1/2\lambda)\frac{s_0(1/\lambda T)}{s_0(1/T)}\frac{1}{(\lambda T)^{1-p}}|\eta(1/\lambda T)|\,d\lambda,$$

where we have made use of (3. 41).

First, since $\lambda T\rightarrow\infty$ for $1/aT<\lambda<\infty$, by (3. 45)

$$K_1 \leq \frac{1}{T^{1-p}} \int_{1/aT}^{1} \frac{1}{\lambda^{1-p+\varepsilon_3}} |\eta(1/\lambda T)| d\lambda$$
$$= o(\frac{1}{T^{1-p}}) \int_{1/aT}^{1} \frac{d\lambda}{\lambda^{1-p+\varepsilon_3}} \leq o(1) T^{-1+\varepsilon_3} \int_{1/a}^{T} \frac{d\lambda}{\lambda^{\varepsilon_3}} = o(1), \quad T \to \infty,$$

Second, since $\sin^2(1/2\lambda) \leq 1/4\lambda^2$, by (3.45),

$$K_2 \leq o(1/T^{1-p}) \int_1^{\infty} \frac{d\lambda}{\lambda^{3-p-\varepsilon_3}} = o(1/T^{1-p}) = o(1).$$

Thus,

(3.47) the third term of (3.42)$\to 0$, as $T \to \infty$.

Collecting (3.44), (3.46), (3.47), we know that the left-hand side of (3.42) tends to 0, as $T \to \infty$. Substituting this into (3.39) $I_1 = o(Ts_0(1/T))$, $T \to \infty$, which together with (3.38) proves (3.37). This completes the proof of VI.

3-VII. *Let $\varphi(\lambda)/2$ be the spectral density of X. Let $H(x) = \int_0^x \varphi(\lambda) d\lambda$ be RV at zero in the wide sense, $H(x) = xh_0(x)$, $h_0(x) = c(x)s_0(x)$.*

Then

(3.48) $$(\frac{2}{\pi})^2 R_\infty(T) \leq V(T) \leq c_0 R_\infty(T), \quad T \to \infty,$$

where $R_\infty(T) = Th_0(1/T)$ is RV in the wide sense at ∞, and c_0 is a positive constant depending on h.

PROOF. Extend η involved in (3.3) to $(0, \infty)$ by setting $\eta(\lambda) = 0$ $(a < \lambda < \infty)$ and, then $s_0(x)$ to $(0, \infty)$ by

$$s_0(x) = \exp(\int_x^{\infty} \eta(u)/u \, du).$$

Finally define $c(x)$ to satisfy

$$H(x) = xc(x)s_0(x).$$

Of course, the above $c(x)$ is an extension of the original one defined near zero and is bounded on $(0, \infty)$.

Write $V(T) = I_1 + I_2$,

$$I_1 = \int_0^{1/T} D_T^2(\lambda)\varphi(\lambda) d\lambda, \quad I_2 = \int_{1/T}^{\infty} D_T^2(\lambda)\varphi(\lambda) d\lambda.$$

Then first,

$$(3.49)\qquad I_1 \leq T^2 \int_0^{1/T} \varphi(\lambda)\, d\lambda = R_\infty(T),$$

$$(3.50)\qquad V(T) \geq I_1 \geq \left(\frac{2T}{\pi}\right)^2 \int_0^{1/T} \varphi(\lambda)\, d\lambda = \left(\frac{2}{\pi}\right)^2 R_\infty(T).$$

Second, by integration by parts

$$(3.51)\qquad I_2 \leq 4 \int_{1/T}^{\infty} \frac{\varphi(\lambda)}{\lambda^2} d\lambda \leq 8A(T),$$

where

$$A(T) = \int_{1/T}^{\infty} \frac{H(x)}{x^3} dx = \int_{1/T}^{\infty} \frac{h_0(x)}{x^2} dx.$$

$c(x)$ being bounded, integration by parts gives

$$A(T) = B(T) + C(T),$$

$$B(T) = \int_{1/T}^{\infty} \frac{c(y)}{y^2} dy \cdot s_0(1/T),$$

$$C(T) = C_0 + C_1(T),\quad C_0 = \int_\delta^\infty \left(\int_x^\infty \frac{c(y)}{y^2} dy\right) s_0(x) \frac{\eta(x)}{x} dx,\quad \delta > 0,$$

$$C_1(T) = \int_{1/T}^{\delta} \left(\int_x^\infty \frac{c(y)}{y^2} dy\right) s_0(x) \frac{\eta(x)}{x} dx.$$

By (3.7), $B(T) \to \infty$, $T \to \infty$. On the other hand

$$|C_1(T)| \leq \sup_{0 < x \leq \delta} |\eta(x)|\, C_2(T),$$

$$C_2(T) = \int_{1/T}^{\delta} \left(x \int_{x^2}^{\infty} \frac{c(y)}{y^2} dy\right) \frac{s_0(x)}{x^2} dx,$$

and if $\delta > 0$ is sufficiently small

$$C_2(T) \geq \frac{1}{2} \lim_{x \to +0} c(x) \int_{1/T}^{\delta} \frac{s_0(x)}{x^2} dx \to \infty, \text{ as } T \to \infty.$$

So that L'hospital's rule gives rise to

$$\overline{\lim_{T\to\infty}} \frac{C(T)}{B(T)} \leq \sup_{0 < x \leq \delta} |\eta(x)|\, \overline{\lim_{T\to\infty}} \frac{C_2(T)}{B(T)}$$

$$= \sup_{0 < x \leq \delta} |\eta(x)| \lim_{T \to \infty} \frac{T^{-1} \int_{1/T}^{\infty} c(y) y^{-2} dy}{c(1/T) + T^{-1} \int_{1/T}^{\infty} c(y) y^{-2} dy \cdot \eta(1/T)}$$

$$= \sup_{0 < x \leq \delta} |\eta(x)|.$$

The last expression being arbitrarily small with δ,

$$(3.52)\qquad \lim_{T\to\infty}\frac{C(T)}{B(T)}=0.$$

Rewrite $B(T)$,

$$B(T)=T\Big(\frac{1}{T}\int_0^T c(1/u)\,du\Big)s_0(1/T).$$

Then by (3.51), (3.52)

$$(3.53)\qquad \overline{\lim_{T\to\infty}}\; I_2/8R_\infty(T)\le \overline{\lim_{T\to\infty}}\; A(T)/R_\infty(T)\le \overline{\lim_{T\to\infty}}\; B(T)/R_\infty(T)$$
$$=\overline{\lim_{T\to\infty}}\;\Big(T^{-1}\int_0^T c(1/u)\,du\Big)/c(T)\le \overline{\lim_{x\to+0}}\; c(x)/\underline{\lim_{x\to+0}}\; c(x).$$

(3.49), (3.50), (3.53) prove (3.48).

§4. Central limit theorems

We are going to show how the FCLT is naturally formulated for X of (1.2). On a preliminary step, as a by-product, we attain a refinement of the CLT in [**13**].

Let $T\ge 1$, $\theta_k(\lambda)=c_k(\lambda)/|c_k(\lambda)|$ or $=0$ according as $c_k(\lambda)\ne 0$ or $=0$, and define

$$(c_k\wedge\eta)(\lambda)\equiv(|c_k(\lambda)|\wedge\eta)\theta_k(\lambda)\quad(\eta>0),$$

$$(4.1)\qquad c_k^\varepsilon=c_k\wedge T^\varepsilon,\ \Delta c_k^\varepsilon=c_k-c_k^\varepsilon,$$
$$\delta[|c_k|^2,\alpha^2](\lambda)=|c_k(\lambda)|^2-|c_k(\lambda)|^2\wedge\alpha^2\quad(\alpha>0),$$
$$(4.2)\qquad \Delta X_k^\varepsilon(t)=X_k(t)-X_k^\varepsilon(t),$$
$$X_k^\varepsilon(t)=\int c_k^\varepsilon(\lambda)e_k(\lambda,t)\,d^k\beta,\ 0<\varepsilon\le\infty,\ 1\le k<\infty,$$

where $c_k^\infty(\lambda)=c_k(\lambda)$, $X_k^\infty(t)=X_k(t)$. Furthermore, write

$$(4.3)\qquad V(T)=V\Big(\int_0^T X(t)\,dt\Big),\ v_k(T)=V\Big(\int_0^T X_k(t)\,dt\Big),$$
$$V_n(T)=V\Big(\int_0^T S_n(t)\,dt\Big),\ S_n(t)=\sum_{k=1}^n X_k(t),$$
$$S_n^\varepsilon(t)=\sum_{k=1}^n X_k^\varepsilon(t),$$
$$(4.4)\qquad \Delta V_n(T)=V\Big(\int_0^T R_n(t)\,dt\Big),\ R_n(t)=X(t)-S_n(t),\ 1\le k,\ n<\infty,$$

where V denotes variance. If we denote by $\varphi_k(\lambda)/2$ the spectral density of $X_k(t)$, then

$$\varphi_k(\lambda)/2=\varphi(|c_k|^2;\ \lambda)=k!\int |c_k(\lambda-\bar{\lambda}',\ \lambda_1,\ \cdots,\ \lambda_{k-1})|^2 \times f(\lambda-\bar{\lambda}')f(\lambda_1)\cdots f(\lambda_{k-1})d\lambda_1\cdots d\lambda_{k-1},$$

$\lambda'=(\lambda_1,\ \cdots,\ \lambda_{k-1})$, $\bar{\lambda}'=\lambda_1+\cdots+\lambda_{k-1}$ (c.f. Section I, [**13**]).

THEOREM 3. *Suppose that the process* X *in* (1.2) *satisfies the conditions*:

(i) $f(\lambda)$ *is bounded,*

(ii) $H(x)=\int_0^x \varphi(\lambda)d\lambda$ *is* RV *in the wide sense at* 0, $H(x)=xh_0(x)$, $h_0(x)=c(x)s_0(x)$, *where* $\varphi(\lambda)/2$ *is the spectral density of* X,

(iii) $\lim_{n\to\infty}\overline{\lim_{T\to\infty}}\ V(T)^{-1}\Delta V_n(T)=0$,

(iv) *there exists an* ε_0, $0<\varepsilon_0<1/2$, *such that*
$\Phi(\delta[|c_k|^2,\ T^{2\varepsilon_0}];\ x)=o(H(x))$, $x=1/T$, *as* $T\to\infty$,

$1\le k<\infty$, *where* $\Phi(|c_k|^2;\ x)$ *is the functional of* $|c_k|^2$ *defined by*

$$\Phi(|c_k|^2;\ x)=\int_0^x \varphi(|c_k|^2;\ \lambda)d\lambda.$$

Then

(4.5) $\quad$ dist $\bar{X}(T)\to N(0,1)$ (*weakly*), *as* $T\to\infty$,

where

$$\bar{X}(T)=\frac{1}{\sqrt{V(T)}}\int_0^T X(t)dt,$$

dist *denotes probability distribution, and* $N(0,1)$ *the normal law with zero mean, variance* 1.

For the proof, along a similar line to Section II, [**13**], we prepare several propositions.

4-I. *Under the assumption* (ii) *of Theorem* 3, *for* $x=1/T$, $T\to\infty$,

(4.6) $\quad \Phi(|c_k-c_k\wedge T^{\varepsilon}|^2;\ x)=o(H(x))$

if and only if

(4.7) $\quad \Phi(\delta[|c_k|^2,\ T^{2\varepsilon}];\ x)=o(H(x))$,

where ε *is arbitrary positive contant.*

PROOF. Suppose that (4.6) is true, and notice that

$$\delta[|c_k|^2,\ T^{2\varepsilon}]=(|c_k|+|c_k|\wedge T^{\varepsilon})(|c_k|-|c_k|\wedge T^{\varepsilon}) \le 2|c_k||c_k-c_k\wedge T^{\varepsilon}|.$$

Then repeated use of Schwarz's inequality yields

$$\begin{aligned}(4.8)\qquad &\varphi(\delta[|c_k|^2,\ T^{2\varepsilon}];\ \lambda)\le 2\varphi^{1/2}(|c_k|^2;\ \lambda)\varphi^{1/2}(|c_k-c_k\wedge T^{\varepsilon}|^2;\ \lambda),\\ &\Phi(\delta[|c_k|^2,\ T^{2\varepsilon}];\ x)\le 2\varphi^{1/2}(|c_k|^2;\ x)\Phi^{1/2}(|c_k-c_k\wedge T^{\varepsilon}|^2;\ x).\end{aligned}$$

On the other hand the assumption (ii) implies

$$\begin{aligned}&\varphi(|c_k|^2;\ \lambda)\le\varphi(\lambda)/2,\\ &\Phi(|c_k|^2;\ x)\le H(x).\end{aligned}$$

Substituting the last inequality and (4.6) into the right-hand member of (4.8) we obtain (4.7).

Conversely assume (4.7). Then, since

$$\begin{aligned}|c_k-c_k\wedge T^{\varepsilon}|^2&=(|c_k|-|c_k|\wedge T^{\varepsilon})^2\\ &\le(|c_k|-|c_k|\wedge T^{\varepsilon})(|c_k|+|c_k|\wedge T^{\varepsilon})\\ &=\delta[|c_k|^2,\ T^{2\varepsilon}],\end{aligned}$$

we have

$$\begin{aligned}\Phi(|c_k-c_k\wedge T^{\varepsilon}|^2;\ x)&\le\Phi(\delta[|c_k|^2,\ T^{2\varepsilon}];\ x)\\ &=o(H(x)),\ \text{as } T\to\infty,\end{aligned}$$

i. e. (4.6).

In addition to random variables in (4.1)-(4.4), define

$$\begin{aligned}&\Delta S_n^{\varepsilon}(t)=S_n(t)-S_n^{\varepsilon}(t)\\ (4.9)\qquad &\bar{X}_k(T)=\frac{1}{\sqrt{V(T)}}\int_0^T X_k(t)\,dt,\ \bar{X}_k^{\varepsilon}(T)=\frac{1}{\sqrt{V(T)}}\int_0^T X_k^{\varepsilon}(t)\,dt,\\ &\Delta\bar{X}_k^{\varepsilon}(T)=\bar{X}_k(T)-\bar{X}_k^{\varepsilon}(T),\\ &\bar{S}_n(T)=\sum_{k=1}^{n}\bar{X}_k(T),\ \bar{S}_n^{\varepsilon}=\sum_{k=1}^{n}\bar{X}_k^{\varepsilon}(T),\\ (4.10)\qquad &\Delta\bar{S}_n^{\varepsilon}(T)=\bar{S}_n(T)-\bar{S}_n^{\varepsilon}(T),\\ &1\le k,\ n<\infty,\ 0<\varepsilon\le\infty,\end{aligned}$$

where

$$\bar{S}_n^{\infty}(T)=\bar{S}_n(T),\ S_n^{\infty}(t)=S_n(t),\ 1\le n<\infty.$$

4-II. *Assume that* (ii), (iv), *Theorem* 3 *are satisfied. Then for every* $n\ge 1$

$$(4.11)\qquad \lim_{T\to\infty} V(\Delta\bar{S}_n^{\varepsilon_0}(T))=0.$$

Proof. By virtue of (4.10), it is enough to show that

$$(4.12)\qquad \lim_{T\to\infty} V(\Delta\bar{X}_k^{\varepsilon_0}(T))=0 \qquad \text{for every } k\geq 1.$$

From (4.1), (4.2)

$$\frac{1}{2}V(\Delta\bar{X}_k^{\varepsilon_0}(T))=\frac{1}{V(T)}\int_0^\infty D_T^2(\lambda)\varphi(|\Delta c_k^{\varepsilon_0}|^2;\ \lambda)d\lambda=I_1+I_2,$$

$$I_1=\frac{1}{V(T)}\int_0^{r\tau} D_T^2(\lambda)\varphi(|\Delta c_k^{\varepsilon_0}|^2;\ \lambda)d\lambda,$$

$$I_2=\frac{1}{V(T)}\int_{r\tau}^\infty D_T^2(\lambda)\varphi(|\Delta c_k^{\varepsilon_0}|^2;\ \lambda)d\lambda,\ r>1,\ \tau=1/T.$$

Since $T^{\varepsilon_0}=(rT')^{\varepsilon_0}$, $T'=T/r$, by (ii), Theorem 3, and 4-I

$$\int_0^{r\tau}\varphi(|\Delta c_k^{\varepsilon_0}|^2;\ \lambda)d\lambda\leq\int_0^{1/T'}\varphi(|c_k-c_k\wedge(T')^{\varepsilon_0}|^2;\ \lambda)d\lambda$$
$$=o(H(1/T'))=o(H(r/T)),\ T\to\infty.$$

Therefore

$$(4.13)\qquad \int_0^{r\tau}D_T^2(\lambda)\varphi(|\Delta_k^{\varepsilon_0}|^2;\ \lambda)d\lambda=o(T^2H(r/T))=o(Th_0(r/T))$$
$$=o(V(T)),\ T\to\infty,$$

where we have used (3.48).

Turn to I_2.

$$(4.14)\qquad \overline{\lim_{T\to\infty}}\ I_2\leq\overline{\lim_{T\to\infty}}\frac{R_\infty(T)}{V(T)}\ \overline{\lim_{T\to\infty}}\frac{R_\infty(T')}{R_\infty(T)}$$
$$\times\overline{\lim_{T'\to\infty}}\frac{1}{R_\infty(T')}\int_{1/T'}^\infty D_T^2(\lambda)\varphi(|c_k^{\varepsilon_0}|^2;\ \lambda)d\lambda.$$

By 3-VII,

$$(4.15)\qquad \overline{\lim_{T\to\infty}}\frac{R_\infty(T)}{V(T)}\leq\frac{\pi^2}{4},$$

and by the wide-sense SV property of h_0

$$(4.16)\qquad \overline{\lim_{T\to\infty}}\frac{R_\infty(T')}{R_\infty(T)}\leq\frac{1}{r}\ \overline{\lim_{x\to+0}}\ c(x)/\underline{\lim_{x\to+0}}\ c(x).$$

Since $\varphi(|c_k^{\varepsilon_0}|^2;\ \lambda)\leq\varphi(\lambda)/2$, the last factor on the right-hand member of (4.14) is less than

$$(4.17)\qquad 2\ \overline{\lim_{T\to\infty}}\frac{1}{R_\infty(T)}\int_{1/T}^\infty\frac{\varphi(\lambda)}{\lambda^2}d\lambda\leq 4\ \overline{\lim_{x\to+0}}\ c(x)/\underline{\lim_{x\to+0}}\ c(x),$$

where we have used (3. 51) and (3. 53). Combination of (4. 14)-(4. 17) leads to

$$(4.18)\qquad \overline{\lim_{T\to\infty}}\ I_2 \leq \frac{\pi^2}{r}(\overline{\lim_{x\to+0}}\ c(x)/\underline{\lim}_{x\to+0}\ c(x))^2.$$

r being arbitrary, (4. 13), (4. 18) together prove (4. 12), which completes the proof of 4-II.

Let us write $S_{m(1)\cdots m(k)}$ $(\xi_1,\cdots,\xi_k)$ for the cumulant of k real random variables $\xi_1,\cdots,\xi_k$ of respective orders $m(1),\cdots,m(k)$; c. f. Section 2, [**13**], for these notations.

4-III. *Let $X=(\xi_1,\cdots,\xi_k)$ be an $\boldsymbol{R}^k$-valued random variable which has $S_{m(1)\cdots m(k)}$ $(\xi_1,\cdots,\xi_k)$ for every $m(1),\cdots,m(k)\geq 1$. If there is an integer $k_0\geq 0$, such that $S_{m(1)\cdots m(k)}$ $(\xi_1,\cdots,\xi_k)=0$ for $m(1)+\cdots+m(k)\geq k_0+1$, then X is Gaussian.*

In some sense or other this seems known.

PROOF. If $t_1,\cdots,t_k$ are real parameters, the nth cumulant of ξ $S_n(\xi)$, $\xi=t_1\xi_1+\cdots+t_k\xi_k$, is a sum of $S_{m(1)\cdots m(k)}$ $(\xi_1,\cdots,\xi_k)$ multiplied by homogeneous polynomials of $t_1,\cdots,t_k$. So that $S_n(\xi)=0$ for $n\geq k_0+1$. This means that no loss of generality we are sufficed to deal with the special case $k=1$. Let X be a real random variable which has the nth moment $\mu(n)$ for all $n\geq 1$, and assume that there exists an integer $k_0\geq 0$ such that $S_n(X)=0$ for all $n\geq k_0+1$. We will show that X is Gaussian.

Let $I=\{i\in\boldsymbol{N}: S_i(X)\neq 0\}$, take n, $p\in\boldsymbol{N}$, and define

$$\begin{aligned}
&A(p)=\{J=(a_1,\cdots,a_p): a_j\in I\ (1\leq j\leq p) \text{ are distinct}\},\\
&B(J)=\{\mathbf{x}=(x_1,\cdots,x_p): a_1x_1+\cdots+a_px_p=n,\ x_j\in\boldsymbol{N}\ (1\leq j\leq p)\},\\
&J=(a_1,\cdots,a_p)\in A(p),\\
&S(\mathbf{x})=\{S_{a_1}(X)\}^{x_1}\cdots\{S_{a_p}(X)\}^{x_p}\quad (\mathbf{x}\in B(J),\ J\in A(p)).
\end{aligned}$$

μ(n) is a sum of constant multiples of $S(\mathbf{x})$,

$$\begin{aligned}
&\mu(n)=\sum_{p=1}^{|I|}\sum_{J\in A(p)}\sum_{\mathbf{x}\in B(J)}\varepsilon(\mathbf{x})\frac{n!\,S(\mathbf{x})}{(a_1!)^{x_1}\cdots(a_p!)^{x_p}x_1!\cdots x_p!},\\
&|\varepsilon(\mathbf{x})|=1,
\end{aligned}$$

where $|I|$ is the cardinality of I. From the equality $a_1x_1+\cdots+a_px_p=n$, we have

$$p(n)\equiv\min_{1\leq p\leq|I|}\ \max_{\substack{\mathbf{x}\in B(J)\\ J\in A(p)}}(x_1,\cdots,x_p)\asymp\text{n, or } \rho_1 n\leq p(n)\leq\rho_2 n,$$

$$\max_{J\in A(p),\ 1\leq p\leq |I|} |B(J)|\leq c_1 n^m,$$

and

$$\max_{\substack{\mathbf{x}\in B(J),\ J\in A(p)\\ 1\leq p\leq |I|}} |S(\mathbf{x})|\leq \sigma^n,\ \sigma=\max_{i\in I} |S_i(X)|,$$

where $\rho_1,\ \rho_2,\ c_1>0,\ m\in\boldsymbol{N}$ are independent of n. If φ is the characteristic function of X, using the Taylor expansion of exp $it\ X$

$$\begin{aligned}
&|\varphi(t)-\sum_{k=0}^{n-1}\mu(k)\frac{(it)^k}{k!}|\leq |t|^n\frac{\mu(n)}{n!}\\
&\leq |I||t|^n \max_{1\leq p\leq |I|} |A(p)| \max_{J\in A(p)} |B(J)|\sigma^n\frac{1}{p(n)!}\\
&\leq c_2|t|^n\sigma^n n^m\frac{e^{p(n)}}{\{p(n)\}^{p(n)}\sqrt{p(n)}}\leq c_2\left(\frac{c_3}{n^{\rho_1}}\right)^n\rightarrow 0,\ n\rightarrow\infty,
\end{aligned}$$

that is, φ is entire. Through the definition of $S_n(X)$, we see that the derivatives at $t=0$ of the analytic function

$$H(t)=\log\varphi(t)-P(t),\ P(t)=\sum_{n=1}^{k_0}\frac{(it)^n}{n!}S_n(X),$$

vanish, hance $H(t)=0$, or $\varphi(t)=\exp\ P(t)$. Then as a corollary of Marcinkiewicz's theorem (p. 65, [**8**]), X must be Gaussian.

Proof of Theorem 3. Let $m(1),\cdots,\ m(k)$ be natural numbers, and $S(\bar{X}^{\varepsilon}_{m(1)}(T),\ \cdots,\ \bar{X}^{\varepsilon}_{m(k)}(T))$ be the cumulant of joint variables in the bracket $(0<\varepsilon\leq\infty)$. It vanishes if $m(1)+\cdots+m(k)$ is odd, while in the notations in Section 3, [**13**]

$$\begin{aligned}
(4.19)\quad & S(\bar{X}^{\varepsilon}_{m(1)}(T),\ \cdots,\ \bar{X}^{\varepsilon}_{m(k)}(T))\\
&=\sum_{Q}\left(\frac{1}{\sqrt{V(T)}}\right)k\int Q\{c^{\varepsilon}_{m(1)}(x_1)\cdots c^{\varepsilon}_{m(k)}(x_k)\mathscr{D}_T(\bar{x}_1)\cdots\mathscr{D}_T(\bar{x}_k)\}d^p\sigma,\\
&x_j\in\boldsymbol{R}^{m(j)}\ (1\leq j\leq k),\ 2p=m(1)+\cdots+m(k),
\end{aligned}$$

if $m(1)+\cdots+m(k)$ is even; $Q\{\ \bullet\ \}$ denotes the connected kernel corresponding to a connected graph Q based on $\{c^{\varepsilon}_{m(j)}(x_j)\mathscr{D}_T(\bar{x}_j),\ 1\leq j\leq k\}$, and Q runs over all connected graphs. By 1-I in Section 1 and the fact $\sum_{k=1}^{\infty}v_k(T)=V(T)$

$$|\left(\frac{1}{\sqrt{V(T)}}\right)^k\int Q\{c^{\varepsilon}_{m(1)}(x_1)\cdots c^{\varepsilon}_{m(k)}(x_k)\mathscr{D}_T(\bar{x}_1)\cdots\mathscr{D}_T(\bar{x}_k)\}d^p\sigma|$$

$$\leq \prod_{j=1}^{k}\{\frac{1}{V(T)}\int |c^{\varepsilon}_{m(j)}(x_j)D^2_T(\bar{x}_j)|d^{m(j)}\sigma\}^{1/2}$$
$$=\prod_{j=1}^{k}\{\frac{v_{m(j)}(T)}{V(T)}\}^{1/2}\leq 1,$$

whence

$$\sup_{\substack{0<\varepsilon\leq\infty\\ 1\leq T<\infty}} |S(\bar{X}^{\varepsilon}_{m(1)}(T),\ \cdots,\ \bar{X}^{\varepsilon}_{m(k)}(T))|<\infty.$$

The multilinearity of the cumulant and functional relations between cumulants and moments tells us

$$\sup_{\substack{0<\varepsilon\leq\infty\\ 1\leq T<\infty}} |E\{(\bar{S}^{\varepsilon}_1(T)^{p_1}\cdots(\bar{S}^{\varepsilon}_n(T))^{p_n}\}|<\infty \tag{4.20}$$

for an arbitrary multi-index $(p_1,\ \cdots,\ p_n)$ of integer entries, and arbitrary n. (4.20) implies that $\{(\bar{S}^{\varepsilon}_k(T))^p,\ 0<\varepsilon\leq\infty,\ 1\leq T<\infty\}$ is uniformly integrable for any $p>0$, $1\leq k<\infty$. Take natural numbers $m(1),\cdots,m(p+1)$, $p\geq 0$, such that $2r=m(1)+\cdots+m(p+1)$ and write after (4.19)

$$S(\bar{X}^{\varepsilon_0}_{m(1)}(T),\ \cdots,\ \bar{X}^{\varepsilon_0}_{m(p+1)}(T))=\sum_{Q} J(Q),$$

where

$$J(Q)=(\frac{1}{\sqrt{V(T)}})^{p+1}\int Q(c^{\varepsilon_0}_{m(1)}(x_1)\cdots c^{\varepsilon_0}_{m(p+1)}(x_{p+1})$$
$$\times\mathscr{D}_T(\bar{x}_1)\cdots\mathscr{D}_T(\bar{x}_{p+1}))d^r\sigma,$$
$$x_j\in\boldsymbol{R}^{m(j)}\ (1\leq j\leq p+1),$$

Q changes over the set of connected graphs. Evaluate $J(Q)$ by the same device as in the proof of Theorem 1, [**13**]. The coupling diagram of Q and the fact that $|c^{\varepsilon_0}_{m(j)}|\leq T^{\varepsilon_0}$provides that

$$|J(Q)|\leq(\frac{1}{\sqrt{V(T)}})^{p+1}T^{(p+1)\varepsilon_0}\int|D_T(l_1)\ldots D_T(l_{p+1})| \tag{4.21}$$
$$\times g_1(u_1)\cdots g_{p+l}(u_{p+l})du_1\cdots du_{p+l},$$

where $u_1,\ \cdots,\ u_{p+l}$ correspond to the $p+l$ edges (Q consists of $p+1$ vertices and $p+l$ edges), $g_j=f^{d_j*}$ (d_j-fold convolution of f), and d_j the multiplicity of the edge corresponding to u_j (c.f. Section 2, [**13**] for these terminologies). $l_1,\ \cdots,\ l_{p+1}$ are linearly dependent forms of $u_1,\ \cdots,\ u_{p+l}$, indeed

$$\sum_{j=1}^{p+1} l_j = 0.$$

However, as a consequence of the connectedness of Q, any p members of $l_1, \cdots, l_{p+1}$ are linearly independent. Choose l linear forms $\tilde{l}_1, \cdots, \tilde{l}_l$ of $u_1, \cdots, u_{p+l}$ in such a way that the $p+l$ functions $l_1, \cdots, l_p, \tilde{l}_1, \cdots, \tilde{l}_l$ are linearly independent. Make a linear transformation from $u_1, \cdots, u_{p+l}$ to $x=(x_1, \cdots, x_{p+l})$

$$x_i = l_i \ (1 \leq i \leq p), \ x_j = \tilde{l}_{j-p} \ (p+1 \leq j \leq p+l).$$

The inverse transformation is

$$u_i = u_i(x) = \sum_{j=1}^{p+l} a_{ij} x_j \ (1 \leq i \leq p+l).$$

The last l column vectors in $A = \|a_{ij}\|$ being linearly independent, from $(p+l) \times l$-matrix $\|a_{ij}\|$, $1 \leq i \leq p+l$, $p+1 \leq j \leq p+l$, one can pick a non-singular square submatrix of order l. With no loss of generality, we may and we will assume that $A' = \|a_{ip+j}\|$, $1 \leq i$, $j \leq l$ is non-singular. Let us write $u_i = v_i + w_i$ $(1 \leq i \leq p+l)$, where v_i, w_i $(1 \leq i \leq p+l)$ are respectively linear functions of $(x_1, \cdots, x_p)$ and $(x_{p+1}, \cdots, x_{p+l})$; write $v_i(x)$, $w_i(x)$ for v_i, w_i $(1 \leq i \leq p+l)$ if necessary. Then

$$\begin{aligned}
&\int |D_T(l_1) \cdots D_T(l_{p+1})| g_1(u_1) \cdots g_{p+l}(u_{p+l}) du_1 \cdots du_{p+l} \\
&= |\det A| \int |D_T(x_1) \cdots D_T(x_p) D_T(x_1 + \cdots + x_p)| \\
&\qquad \times g_1(u_1(x)) \cdots g_{p+l}(u_{p+l}(x)) dx_1 \cdots dx_{p+l} \\
&= |\det A| \prod_{j=l+1}^{l+p} \|g_j\|_\infty \int |D_T(x_1) \cdots D_T(x_p) D_T(x_1 + \cdots + x_p)| \\
&\qquad \times dx_1 \cdots dx_p \int g_1(v_1(x) + w_1) \cdots g_l(v_l(x) + w_l) \\
&\qquad \times |\det A'|^{-1} dw_1 \cdots dw_l \\
&= |\det A| \, |\det A'|^{-1} \prod_{j=l+1}^{l+p} \|g_j\|_\infty \prod_{k=1}^{l} \|g_k\|_{L^1} \|\Psi^{(p)}\|_{L^1} \, T,
\end{aligned}$$

where

$$\Psi^{(p)}(x_1, \cdots, x_p) = D(x_1) \cdots D(x_p) D(x_1 + \cdots + x_p),$$
$$D(x) = \frac{\sin x/2}{x/2}.$$

Therefore

(4.22) $\quad |J(Q)| \leq c_3 (\frac{1}{\sqrt{V(T)}})^{p+1} T^{(p+1)\varepsilon_0} \cdot T.$

By the slow variation of h_0, one can find a δ such that $0<\delta<1-2\varepsilon_0$, and

$$c_4 T^{-\delta} < h_0(1/T) < c_5 T^{\delta} (T \geq 1).$$

Then

$$|J(Q)| \leq c_6 \left(\frac{1}{T^{(1-\delta)/2-\varepsilon_0}} \right)^{p+1} T,$$

whence

(4.23) $\quad \lim_{T\to\infty} J(Q)=0 \qquad \text{for } p \geq k_0+1,\ k_0=[2/(1-\delta-2\varepsilon_0)].$

Define $\mathscr{D}_0$ to be the set of sequences on $[1, \infty)$ tending to ∞. Since $V(\bar{X}(T))=1$, $\{\text{dist } \bar{X}(T),\ T\geq 1\}$ is relatively compact. Let μ be a limit point of dist $\bar{X}(T)$, as $T\to\infty$, and $D_1 \in \mathscr{D}_0$ be such that dist $\bar{X}(T) \to \mu$ (weakly), as $T\to\infty$ on D_1. On the other hand as was noticed before $\mathscr{F}=\{\bar{S}_n^{\varepsilon}(T))^p,\ 0<\varepsilon\leq\infty,\ 1\leq T<\infty\}$ is uniformly integrable for any n, $p\geq 1$. Then $\{\text{dist } \{\bar{S}_1^{\varepsilon}(T), \cdots, \bar{S}_n^{\varepsilon}(T)\},\ T \geq 1\}$ being relatively compact for every n, by a diagonal procedure, one can find $D_2 \in \mathscr{D}_0$, $D_2 \subset D_1$, such that for every n $\{\bar{S}_1^{\varepsilon_0}(T), \cdots, \bar{S}_n^{\varepsilon_0}(T),\ \bar{X}(T)\}$ goes to a limit in distribution, as $T\to\infty$ on D_2, so does $\{\bar{S}_1(T), \cdots, \bar{S}_n(T),\ \bar{X}(T)\}$ to the same limit by 4-II. According to 4-III, and (4.23), as $T\to\infty$ on D_2, $F=\lim \text{dist } \{\bar{S}_1(T), \cdots, \bar{S}_n(T)\}$ must be Gaussian. In addition, the above-mentioned uniform integrability of $\mathscr{F}$ gives rise to the moment convergence

(4.24) $\quad E\{(\bar{S}_1(T))^{m(1)} \cdots (\bar{S}_n(T))^{m(n)}\} \to \int x_1^{m(1)} \cdots x_n^{m(n)} dF,\ T\to\infty \text{ on } D_2,$

$$x=(x_1, \cdots, x_n),$$

for all $m(1), \cdots, m(n) \geq 0$, $n\geq 1$. In particular F has zero mean.

Consider the discrete-time processes $\{S_l(T), 1\leq l\leq\infty\}$ indexed by $T\geq 1$, with $\bar{S}_\infty(T) \equiv \bar{X}(T)$. From the preceding results, all of its finite-dimensional marginal distributions are convergent, when $T\to\infty$ on D_2, and moreover the marginal limit distributions of $\{\bar{S}_l(T),\ 1\leq l<\infty\}$ are Gaussian with zero mean. So that by Kolmogorov's theorem, there exists a stochastic process $\{U_l,\ 1\leq l\leq\infty\}$, whose finite-dimensional marginal distributions are the limits of the corresponding ones of $\{\bar{S}_l(T),\ 1\leq l\leq\infty\}$, as $T\to\infty$ on D_2. The finite-time section $\{U_l,\ 1\leq l<\infty\}$ is Gaussian and the probability law of U_∞ is μ.

To determine μ introduce

$$g(x)=x^2/(1+x^2)(-\infty<x<\infty),$$

and observe that

$$(4.25)\quad \lim_{l\to\infty} Eg(|U_l-U_\infty|)=\lim_{l\to\infty}\lim_{\substack{T\to\infty\\ T\in D_2}} Eg(|\bar{S}_l(T)-\bar{S}_\infty(T)|)$$
$$\leq \lim_{l\to\infty}\overline{\lim_{T\to\infty}}\, E(\frac{1}{\sqrt{V(T)}}\int_0^T S_l(x)\,ds-\frac{1}{\sqrt{V(T)}}\int_0^T X(s)\,ds)^2$$
$$=\lim_{l\to\infty}\overline{\lim_{T\to\infty}}\,\frac{1}{V(T)}V(\int_0^T R_l(s)\,ds)=0$$

concluding that U_l converges in probability to U_∞, as $l\to\infty$. Then since the set of Gaussian distributions is closed under weak convergence, $(U_1, \cdots, U_l, U_\infty)$ $(1\leq l<\infty)$ as the limit in probability of $(U_1, \cdots, U_l, U_{l'})$, as $l'\to\infty$, is Gaussian with zero mean. The relation $U_l-U_\infty\to 0$ in probability leads to $E\,(U_l-U_\infty)^2\to 0$, as $l\to\infty$.

If we define

$$\|\Delta(l, T)\|^2=E(\bar{S}_l(T)-\bar{S}_\infty(T))^2,$$

$$K(l)=\overline{\lim_{\substack{T\to\infty\\ T\in D_2}}}\|\Delta(l, T)\|^2,$$

then by (iii)

$$\lim_{l\to\infty} K(l)=0.$$

On the other hand, $\|\bar{S}_\infty(T)\|=\|\bar{X}(T)\|=1$,

$$\|\bar{S}_\infty(T)\|-\|\Delta(l, T)\|\leq\|\bar{S}_l(T)\|\leq\|\bar{S}_\infty(T)\|+\|\Delta(l, T)\|,$$

and by (4. 24)

$$\|\bar{S}_l(T)\|\to\|U_l\|, \text{ as } T\to\infty \text{ on } D_2.$$

Therefore

$$1-K(l)\leq\|U_l\|\leq 1+K(l),$$

whence on making $l\to\infty$,

$$\lim_{l\to\infty}\|U_l\|=1.$$

So that

$$\|U_\infty\|=1,$$

namely $\mu=N(0,1)$. This means that the set of the limit points of $\bar{X}(T)$, as $T\to\infty$, consists of a single element $N(0,1)$, or $\bar{X}(T)$ converges in distribution to $N(0,1)$, as $T\to\infty$. This completes the proof of the theorem.

Before passing to the main theorem, we will make a few remarks on the standard normalization in the FCLT. Let $\{X(t),\ -\infty<t<\infty\}$ be a stationary process and $B(T)$, $1\le T<\infty$, be a normalizing function for our FCLT. This means that if we put

$$(4.26)\qquad \bar{X}(T,t)=\frac{1}{B(T)}\int_0^{Tt}X(s)\,ds,$$

$\bar{X}(T)=\{\bar{X}(T,t),\ 0\le t\le 1\}$ converges in distribution, $T\to\infty$, on the space $C[0,1]$ of continuous functions, to the standard Brownian motion $\boldsymbol{W}=\{W(t),\ 0\le t\le 1\}$. If in addition we assume that for every $t\{(\bar{X}(T,t))^2,\ 1\le T<\infty\}$ is uniformly integrable, then as $T\to\infty$

$$(4.27)\qquad E(\bar{X}(T,1))^2\to E(W(1))^2=1,$$

$$(4.28)\qquad V(\bar{X}(T,t)-\bar{X}(T,s))\to E(W(t)-W(s))^2=t-s,\ 0\le s<t\le 1.$$

(4.27) suggests us to take $B(T)=\sqrt{V(T)}$ in (4.26). Then under this normalization combined with the stationarity, (4.28) implies that as $T\to\infty$

$$\frac{1}{V(T)}V(T(t-s))\to t-s,$$

or $V(T)$ is RV at ∞.

THEOREM 4. *Suppose that* X *of* (2.1) *satisfies the conditions* (A), (B), (C). (A) *the condition* (i), (iii), (iv) *of Theorem* 3. (B) *one of the conditions* (B_1)-(B_3) :

(B_1) $V(T)=V(\int_0^T X(s)ds)$ *is* RV *at* ∞, *with* $V(T)=Th_\infty(T)$, $h_\infty(T)=c(1/T)s_0(1/T)$;

(B_2) *if we write* $\varphi(\lambda)/2$ *for the spectral density of* X, *then* $H(x)=\int_0^x\varphi(\lambda)d\lambda$ *is* RV *at* 0, $H(x)=xh_0(x)$, $h_0(x)=c(x)s_0(x)$ *on some interval* $(0,\delta)$ $(\delta>0)$, *and* $c(x)$ *fulfils*

$(B_2$-$1)$ $yc'(y)\in L(0,\delta)$, $\int_0^x|yc'(y)|dy=o(x)$, $x\to+0$, *or*

$(B_2\text{-}2)\quad c(x)-c(+0)=O(x^q),\ x\to+0$

for some constant $q>0$;

$(B_3)\quad \varphi(\lambda)$ *is RV at* 0.

$(C) X(0)\in N_4'$ *and there exists an a,* $0<a<1$, *such that*

$$\sum_{k\geq 1} 3^{k/2}\zeta_k(a)<\infty,$$

where

$$\zeta_k^2(a)=\sup_{0<x<a} x^{-1}\int_0^x \varphi_k(\lambda)/h_0(\lambda)d\lambda\quad(\zeta_k(a)\geq 0).$$

Then

$$\bar{\boldsymbol{X}}(T)=\{\bar{X}(T,t),\ t\in I\},$$
$$\bar{X}(T,t)=\frac{1}{\sqrt{V(T)}}\int_0^{Tt} X(s)ds,\ I=[0,1]$$

converges in distribution on $C(I)$, *the space of continuous functions on I, to standard Brownian motion* $\boldsymbol{W}=(W(t),\ 0\leq t\leq 1)$.

Since (B_2) or (B_3) implies (B_1) we are sufficed to prove the theorem under (A), (B_1), and (C). Using the notations in the preceding paragraphs, define further processes, depending on time $t\in I$, indexed by $T>1$. Let

$$(4.29)\quad \begin{aligned}&\bar{\boldsymbol{S}}_l(T)=\{S_l(T,t),\ t\in I\},\ (1\leq l\leq\infty),\\ &\bar{S}_\infty(T,t)\equiv\bar{X}(T,t),\ \bar{S}_\infty(T)\equiv\bar{\boldsymbol{X}}(T),\\ &\bar{S}_l(T,t)=\frac{1}{\sqrt{V(T)}}\int_0^{Tt} S_l(s)ds\ (1\leq l<\infty),\end{aligned}$$

and

$$(4.30)\quad \begin{aligned}&\bar{S}_l^\varepsilon(T)=\{\bar{S}_l^\varepsilon(T,t),\ t\in I\},\\ &\bar{S}_l^\varepsilon(T,t)=\frac{1}{\sqrt{V(T)}}\int_0^{Tt} S_l^\varepsilon(s)ds\ (1\leq l<\infty,\ 0<\varepsilon\leq\infty),\\ &\bar{S}_l^\infty(t)\equiv\bar{S}_l(t),\ S_l^\infty(t)=S_l(t)\ (1\leq l<\infty).\end{aligned}$$

Take an increasing sequence $t_1,\cdots,t_m$ on I and set

$$(4.31)\quad \begin{aligned}&\operatorname{marg}\bar{\boldsymbol{X}}(T)=(\bar{X}(T,t_1),\cdots,\bar{X}(T,t_m)),\\ &\operatorname{marg}\boldsymbol{W}=(W(t_1),\cdots,W(t_m)),\\ &\operatorname{marg}\boldsymbol{S}_l(T)=(\bar{S}_l(T,t_1),\cdots,\bar{S}_l(T,t_m)),\end{aligned}$$

similarly for $\bar{S}^{\varepsilon}(T)$.

Since the assumptions of Theorem 4 are stronger than those of Theorem 3, we can mkae use of the arguments and conclusions in the proof of the latter. As usual, the proof consists of two parts, i. e. the first part for marginal convergence and the second for compactness.

PROOF OF THEOREM 4. Step 1 (marginal convergence). By the continuity of $T\to V(X(T,t))$ $(t>0$ fixed) and RV property of $V(T)$ $\sup_{T>1} V(Tt)/V(T)<\infty$ for every $t\geq 0$. Then $\mathfrak{M}=\{\text{dist marg } \bar{\boldsymbol{X}}(T), T>1\}$ is relatively compact. Let μ be a limit point of $\mathfrak{M}$ as $T\to\infty$, and $D_1\in\mathscr{D}_0$ be such that marg $\bar{\boldsymbol{X}}(T)$ converges weakly to μ as $T\to\infty$ on D_1.

Let

$$\bar{X}_k^{\varepsilon}(Tt)=\frac{1}{\sqrt{V(T)}}\int_0^{Tt} X_k^{\varepsilon}(s)\,ds,\ 0<\varepsilon\leq\infty,\ T>1,\ 1\leq k<\infty,$$

and $m(1),\cdots, m(k)$ be as in (4.19). Then, as before, $S(\bar{X}^{\varepsilon}_{m(1)}(T,t),\cdots,\bar{X}^{\varepsilon}_{m(k)}(T,t))$ is a sum of terms like

$$\left(\frac{1}{\sqrt{V(T)}}\right)^k\int Q\{c^{\varepsilon}_{m(1)}(x_1)\cdots c^{\varepsilon}_{m(k)}(x_k)\mathscr{D}_{Tt}(\bar{x}_1)\cdots\mathscr{D}_{Tt}(\bar{x}_k)\}\,d^k\sigma,$$

of which the absolute value is, by 1-I and RV property of V, not greater than $\rho^{k/2}$, where $\rho=\sup_{1<T<\infty} V(Tt)/V(T)<\infty$ for every $t\geq 0$. This implies that

$$\sup_{\substack{0<\varepsilon\leq\infty\\ 1<T<\infty}} |E(\{\bar{S}_1^{\varepsilon}(T,t)\}^{p_1}\ldots\{\bar{S}_n^{\varepsilon}(T,t)\}^{p_n}|<\infty$$

for every $t\geq 0$, $n\geq 1$, and multi-index $(p_1,\cdots,p_n)$. So that $\{|\text{marg } \bar{S}_k^{\varepsilon}(T)|^p,\ 0<\varepsilon\leq\infty,\ 1<T<\infty\}$ is uniformly integrable, for any p, $1\leq k<\infty$, $p>0$, and $\mathscr{F}=\{\text{dist (marg } \bar{S}^{\varepsilon}(T),\cdots,\text{ marg } \bar{S}_n^{\varepsilon}(T)),\ T>1,\ 0<\varepsilon\leq\infty\}$ is relatively compact. Since by 4-II and regular variation of $V(T), V(\bar{S}_l(T,t)-\bar{S}_l^{\varepsilon}(T,t))=(V(Tt)/V(T))\,V(\Delta\bar{S}_l^{\varepsilon_0}(Tt))\to 0$, as $T\to\infty$, $t>0$, through a diagonal procedure, one can find a $D_2\in\mathscr{D}_0$, $D_2\subset D_1$, such that for every n both $\{\text{marg } \bar{S}_1^{\varepsilon_0}(T),\cdots,\text{ marg } \bar{S}_n^{\varepsilon_0}(T),\text{ marg } \bar{\boldsymbol{X}}(T)\}$ and $\{\text{marg } \bar{S}_1(T),\cdots,\text{ marg } \bar{S}_n(T),\text{ marg } \bar{\boldsymbol{X}}(T)\}$ converge in distribution to a same limit.

To show that $\mu=\text{dist marg } \boldsymbol{W}$, take $m(1),\cdots, m(p+1)$ $(p\geq 0)$, and proceed as in the paragraphs after (4.20) to have

$$S(\bar{X}^{\varepsilon_0}_{m(1)}(T, s(1)), \cdots, \bar{X}^{\varepsilon_0}_{m(p+1)}(T, s(p+1)) = \sum_Q J(Q),$$

$$J(Q) = \left(\frac{1}{\sqrt{V(T)}}\right)^{p+1} \int Q(c^{\varepsilon_0}_{m(1)}(x_1) \cdots c^{\varepsilon_0}_{m(p+1)}(x_{p+1})$$
$$\times \mathscr{D}_{Ts(1)}(\bar{x}_1) \cdots \mathscr{D}_{Ts(p+1)}(\bar{x}_{p+1})) d^r\sigma,$$

where $s(1)$ etc. are taken from $(t_1, \cdots, t_m)$. Then as in the passages (4.21)-(4.23),

$$|J(Q)| \leq c\left(\frac{1}{\sqrt{V(T)}}\right)^{p+1} T^{(p+1)\varepsilon_0}$$
$$\times \int |\mathscr{D}_{Ts(1)}(x_1) \cdots \mathscr{D}_{Ts(p)}(x_p) \mathscr{D}_{Ts(p+1)}(x_1 + \cdots + x_p)| dx_1 \cdots dx_p.$$
$$= c\left(\frac{1}{\sqrt{V(T)}}\right)^{p+1} T^{(p+1)\varepsilon_0} T \|\tilde{\Psi}\|_{L^1}$$

where $\tilde{\Psi}(x) = D_{s(1)}(x_1) \cdots D_{s(p)}(x_p) D_{s(p+1)}(x_1 + \cdots + x_p)$, $x = (x_1, \cdots, x_p)$, c is a constant depending on f. Define

$$f_n(x) = (\log 2(1+x^2))^n / (1+x^2)^{1/2} (-\infty < x < \infty), \quad n = 0, 1, \cdots.$$

Then after elementary calculus, this leads to $f_0(x) * f_n(x) \precsim f_{n+1}(x)$ on $(-\infty, \infty)$ for $0 \leq n < \infty$. The obvious relation $|D_{s(k)}(x)| \leq c(s(k)) f_0(x)$ $(1 \leq k \leq p+1)$ gives

$$\|\tilde{\Psi}\|_{L^1} \leq c(s(1), \cdots, s(p+1)) \int f_0^{p*}(x) f_0(x) dx < \infty,$$

where $c(a)$, $c(a, b, \cdots)$ denote constants respectively depending on a, $(a, b, \cdots)$. Arguing as in the proof of Theorem 3, by slow variation of h_0, one obtains

$$|J(Q)| \leq c(1/T^{(1-\delta)/2-\varepsilon_0})^{p+1} T, \quad 0 < \delta < 1 - 2\varepsilon_0,$$

whence

$$\lim_{T \to \infty} J(Q) = 0 \qquad \text{for } p \geq [2/(1-\delta-2\varepsilon_0)].$$

By Proposition 4-III, the concluding remark of the last paragraph, and that made directly after (4.24), dist $\{\text{marg } \bar{S}_1(T), \cdots, \text{marg } \bar{S}_n(T)\}$ converges weakly, as $T \to \infty$ on D_2, to a Gaussian distribution with zero mean.

Arguing as in the proof of Theorem 3, there is an $\boldsymbol{R}^m$-valued process $\{\boldsymbol{U}_l, 1 \leq l \leq \infty\}$, $\boldsymbol{U}_l = (U_l(t_1), \cdots, U_l(t_m))$, whose finite-dimensional marginal distributions are the limits of the corresponding ones of $\{\text{marg } \bar{S}_l(T), 1 \leq l$

$\leq\infty\}$, as $T\to\infty$ on D_2. Especially, $\{\boldsymbol{U}_l,\ 1\leq l<\infty\}$ is Gaussian with mean zero, and dist $\boldsymbol{U}_\infty=\mu$. As in (4.25)

$$\lim_{l\to\infty} Eg(|\boldsymbol{U}_l-\boldsymbol{U}_\infty|)=\lim_{l\to\infty}\lim_{\substack{T\to\infty\\ T\in D_2}} g(|\text{marg}\ \bar{\boldsymbol{S}}_l(T)\text{-marg}\ \bar{\boldsymbol{S}}_\infty(T)|)$$

$$\leq\lim_{l\to\infty}\overline{\lim_{T\to\infty}}\sum_{j=1}^m E(V(T)^{-1/2}\int_0^{Tt_j} S_l(s)ds-V(T)^{-1/2}\int_0^{Tt_j} X(s)ds)^2$$

$$\leq\sum_{j=1}^m\lim_{l\to\infty}\overline{\lim_{T\to\infty}}\ (V(Tt_j)/V(T))V(\int_0^{Tt_j} R_l(s)ds)V(Tt_j)^{-1}$$

$$\leq\sum_{j=1}^m t_j\lim_{l\to\infty}\overline{\lim_{T\to\infty}}\frac{1}{V(T)}V(\int_0^T R_l(s)ds)=0,$$

therefore $E|\boldsymbol{U}_l-\boldsymbol{U}_\infty|^2\to 0$, $\|U_l(t)-U_l(s)\|^2\to\|U_\infty(t)-U_\infty(s)\|^2$ $(l\to\infty)$, for $s,\ t\in(t_1,\dots,t_m)$. Similarly, from

$$\lim_{l\to\infty}\overline{\lim_{T\to\infty}}\bar{E}|\bar{\boldsymbol{S}}_l(T)-\bar{\boldsymbol{S}}_\infty(T)|^2\leq\sum_{j=1}^m\lim_{l\to\infty}\overline{\lim_{T\to\infty}}V(T)^{-1}V(\int_0^{Tt_j} R_l(s)ds)=0,$$

follows $\lim_{l\to\infty} K(l)=0$, where

$$K(l)=\overline{\lim_{\substack{T\to\infty\\ T\in D_2}}}\|\Delta\|^2,\ \Delta=(\bar{S}_l(T,t)-\bar{S}_l(T,s))-(\bar{S}_\infty(T,t)-\bar{S}_\infty(T,s)).$$

(B) implies that for $s<t$

$$\text{(4.32)}\qquad \lim_{T\to\infty}\|\bar{S}_\infty(T,t)-\bar{S}_\infty(T,s)\|^2=\lim_{T\to\infty} V(T(t-s))/V(T)=t-s.$$

The triangular inequality gives

$$\text{(4.33)}\qquad \lim_{T\to\infty}\|\bar{S}_\infty(T,t)-\bar{S}_\infty(T,s)\|-K(l)$$

$$\leq\lim_{\substack{T\to\infty\\ T\in D_2}}\|\bar{S}_l(T,t)-\bar{S}_l(T,s)\|\leq\lim_{T\to\infty}\|\bar{S}_\infty(T,t)-\bar{S}_\infty(T,s)\|+K(l).$$

On the other hand the uniform integrability of $\{|\text{marg}\ \bar{S}_l(T)|^p,\ T>1\}$ for arbitrary $l,\ p\geq 1$ implies that for any $l,\ p\geq 1$

$$\text{(4.34)}\qquad E(\bar{S}_l(T,t)-\bar{S}_l(T,s))^p\to E\,(U_l(t)-U_l(s))^p,\qquad (T\to\infty,\ T\in D_2).$$

Putting (4.32)-(4.34) together

$$\sqrt{t-s}-K(l)\leq\sqrt{E(U_l(t)-U_l(s))^2}\leq\sqrt{t-s}+K(l),$$

whence on making $l\to\infty$

(4.35) $\quad E(U_\infty(t)-U_\infty(s))^2=t-s.$

Take $s, t, u\in(t_1, \cdots, t_m)$, $s<t<u$, and put $\Delta_1=U_\infty(t)-U_\infty(s)$, $\Delta_2=U_\infty(u)-U_\infty(t)$. Then

$$\begin{aligned}u-s&=E(U_\infty(u)-U_\infty(s))^2=E\Delta_1^2+E\Delta_2^2+2E(\Delta_1\Delta_2)\\&=(u-t)+(t-s)+2E(\Delta_1\Delta_2),\end{aligned}$$

whence $E(\Delta_1\Delta_2)=0$, i. e. increment independence for the sequence $U_\infty(t_1)$, $\cdots$, $U_\infty(t_m)$. In other words

$$\mu=\text{dist marg } \boldsymbol{W},$$

which completes the proof of Step 1.

Step 2 (compactness). Fix a small a, $0<a<1$, to divide X into two parts, $X(t)=X^{(1)}(t)+X^{(2)}(t)$, such that

$$X^{(i)}(t)=\sum_{k\geq 1}\int \chi^{(i)}(\lambda)c_k(\lambda)e_k(\lambda, t)\,d^k\beta,\ (i=1,2),$$

where $\chi^{(1)}$, $\chi^{(2)}$ are respectively the indicators of $(\lambda\in\boldsymbol{R}^k:\ |\bar{\lambda}|>a)$, $(\lambda\in\boldsymbol{R}^k:\ |\bar{\lambda}|>a)$. Then

$$\begin{aligned}&X^{(i)}(T, t)=V(T)^{-1/2}\int_0^{Tt}X^{(i)}(s)\,ds=\sum_{k\geq 1}\int c_k^{(i)}(t, \lambda)\,d^k\beta,\\&c_k^{(i)}(t, \lambda)=\chi^{(i)}(\bar{\lambda})c_k(\lambda)\mathscr{D}_{Tt}(\bar{\lambda})/\sqrt{V(T)}.\end{aligned}$$

Fix an ε, $0<\varepsilon<1/2$, and let $\rho=\rho(T)$ be defined to satisfy $T^\rho=V(T)$. As will be easily seen $\rho(T)\to 1$, as $T\to\infty$. Notice that if we write $H(x)=\int_0^x\varphi(\lambda)\,d\lambda=x\dot{h}_0(x)$, $h_0(x)=c(x)s_0(x)$, then (c. f. 3- V , VI), $V(T)\sim\pi T\,c(1/T)s_0(1/T)$, $T\to\infty$. If we denote by $\varphi_k(\lambda)/2$ the spectral density of $X_k(t)$, then $\varphi_k(\lambda)/2=\varphi(|c_k|^2;\ \lambda)$ and

$$\|c_k^{(i)}(t, \cdot)\|_2^2=\int|c_k^{(i)}(t, \lambda)|^2 d_k\sigma=V(T)^{-1}\int\chi^{(i)}|c_k(\lambda)|^2D_{Tt}^2(\lambda)\,d^k\sigma.$$

Especially, there is a $T_0>1/a$ such that

$$\begin{aligned}\|c_k^{(2)}(t, \cdot)\|_2^2&=V(T)^{-1}(k\,!)^{-1}\int_a^\infty\varphi_k(\lambda)D_{Tt}^2(\lambda)\,d\lambda\\&=V(T)^{-1}(k\,!)^{-1}\int_a^\infty D_{Tt}(\lambda)|^\rho|D_{Tt}(\lambda)|^{2-\rho}\varphi_k(\lambda)\,d\lambda\\&\leq c_1\frac{(Tt)^\rho}{V(T)}(k\,!)^{-1}\int_a^\infty\varphi_k(\lambda)/\lambda^{2-\rho}d\lambda\leq c_2\frac{t^{1-\varepsilon}}{a^{2-\rho}k\,!}\int_{-\infty}^\infty\varphi_k(\lambda)\,d\lambda\end{aligned}$$

$$\leq c_2\|c_k\|_2^2 a^{-2}t^{1-\varepsilon},$$

therefore

(4.36) $$\Big(\sum_{k\geq 1} 3^{k/2}\sqrt{k!}\|c_k^{(2)}(t,\cdot)\|_2\Big)^4 \leq c_2^2 a^{-4}t^{2(1-\varepsilon)}\Big(\sum_{k\geq 1} 3^{k/2}\sqrt{k!}\|c_k\|_2\Big)^4,$$

for $T\geq T_0$. On the other hand

$$\|c_k^{(1)}(t,\cdot)\|_2^2 = V(T)^{-1}(k!)^{-1}\int_0^a \varphi_k(\lambda)D_{Tt}^2(\lambda)\,d\lambda$$
$$= Th_0(1/T)V(T)^{-1}(I_1+I_2),$$

where

$$I_1=\int_0^1 D_t^2(\lambda)\{h_0(\lambda/T)/h_0(1/T)\}h_k(\lambda/T)\,d\lambda,$$
$$I_2=\int_1^{aT} D_t^2(\lambda)\{h_0(\lambda/T)/h_0(1/T)\}h_k(\lambda/T)\,d\lambda,$$
$$h_k(\lambda)=\varphi_k(\lambda)/(k!\,h(\lambda)),\quad T\geq T_0.$$

First, T_0 being so large that by the integral representation of s_0

$$I_1\leq c_1 t^2\int_0^1 \lambda^{-\varepsilon}h_k(\lambda/T)\,d\lambda = c_1 t^2 T^{1-\varepsilon}\int_0^{1/T}\lambda^{-\varepsilon}h_k(\lambda)\,d\lambda.$$

But, since by integration by parts

$$\int_0^{1/T}\lambda^{-\varepsilon}h_k(\lambda)\,d\lambda \leq 2T^{\varepsilon-1}\sup_{0<x\leq 1/T} x^{-1}\int_0^x h_k(\lambda)\,d\lambda,$$

we have

(4.37) $$I_1\leq 2c_1 t^2(k!)^{-1}\xi_k^2(a).$$

Here notice that $\xi_k^2(a)$ is finite. Indeed, $c(x)$ in (B) satisfies that for an arbitrary $c_0>0$, there is $\alpha>0$ such that $c(x)\geq 1/\alpha$ on $(0,c_0)$. So that, having $\varphi_k(\lambda)\leq\varphi(\lambda)$, for $x\in(0,c_0)$ by integration by parts and the integral representation of s_0,

$$\int_0^x \varphi_k(\lambda)/h_0(\lambda)\,d\lambda \leq \alpha\int_0^x \varphi(\lambda)/s_0(\lambda)\,d\lambda$$
$$=\alpha\Big\{s_0^{-1}(x)\int_0^x\varphi(\lambda)\,d\lambda - \int_0^\lambda \frac{d}{d\lambda}s_0^{-1}(\lambda)\,d\lambda\int_0^\lambda\varphi(u)\,du\Big\}$$
$$\leq\alpha\Big(xc(x)+\int_0^x|\eta(\lambda)|c(\lambda)\,d\lambda\Big)\leq \text{const }x.$$

Second, again by the integral representation of s_0,

$$I_2 \leq c_2 \int_1^{aT} D_t^2(\lambda)\lambda^\varepsilon h_k(\lambda/T)\,d\lambda$$
$$\leq c_3 t^{1-\varepsilon} \int_t^{aTt} E(\lambda) h_k(\lambda/Tt)\,d\lambda,$$

where $E(\lambda)=\lambda^\varepsilon/(1+\lambda^2)$. By integration by parts

$$(4.38) \quad \int_t^{aTt} E(\lambda) h_k(\lambda/Tt)\,d\lambda \leq \int_0^{aTt} E(\lambda) h_k(\lambda/Tt)\,d\lambda$$
$$= E(aTt) \int_0^{aTt} h_k(u/Tt)\,du - \int_0^{aTt} E'(\lambda)\,d\lambda \int_0^\lambda h_k(u/Tt)\,du.$$

The first term is less than

$$(4.39) \quad \frac{(aTt)^{1+\varepsilon}}{1+(aTt)^2} a^{-1} \int_0^a h_k(u)\,du$$
$$\leq \Big(\sup_{0\leq x<\infty} \frac{x^{1+\varepsilon}}{1+x^2}\Big) a^{-1} \int_0^a h_k(\lambda)\,d\lambda.$$

To evaluate the second term of (4. 38), observe that since $\lambda/Tt \leq a$

$$\int_0^\lambda h_k(u/Tt)\,du \leq \lambda \sup_{0<x\leq a} x^{-1} \int_0^x h_k(u)\,du.$$

Then the second term is less than

$$(4.40) \quad \Big(2\int_0^\infty \frac{\lambda^{2+\varepsilon}}{(1+\lambda^2)^2}\,d\lambda\Big) \sup_{0<x\leq a} x^{-1} \int_0^x h_k(u)\,du.$$

Collecting (4. 39), (4. 40)

$$(4.41) \quad I_2 \leq c_4 t^{1-\varepsilon} (k\,!)^{-1} \zeta_k(\mathrm{a}).$$

Putting (4. 37), (4. 41) together

$$(4.42) \quad \Big(\sum_{k\geq 1} 3^{k/2}\sqrt{k\,!}\,\|c_k^{(1)}(\mathrm{t},\cdot)\|_2\Big)^4 \leq c_5\Big(\sum_{k\geq 1} 3^{k/2}\zeta_k(a)\Big)^4 t^{2(1-\varepsilon)}.$$

Due to 1-III and the condition (C) of Theorem 4, (4. 36), (4. 42) finally give

$$E(\bar{X}(Tt))^4 \leq 2^3\{E(X^{(1)}(T,t))^4 + E(X^{(2)}(T,t))^4\}$$
$$\leq c_6 t^{2(1-\varepsilon)}, \quad 0\leq t\leq 1,$$
$$c_6 = 8\{c_2^2 a^{-4}\Big(\sum_{k\geq 1} 3^{k/2}\sqrt{k\,!}\,\|c_k\|_2\Big)^4 + c_5\Big(\sum_{k\geq 1} 3^{k/2}\zeta_k(a)\Big)\} < \infty.$$

Since $2(1-\varepsilon)>1$, this completes the proof of Step 2.

Acknowledgement. Under the same framework as the present paper Giraitis-Surgailis [3] recently obtained a CLT for finite X. In some

respects, these authors' truncation conditions are weaker than (iv) of Theorem 3.

During the preparation of the present paper the author received the manuscript of a paper by Chambers and Slud [**1**]. The introduction of N_{2m} was motivated by the compactness conditions by these authors. In [**11**], [**12**]. in order to formulate compactness conditions for a periodogram limit theorem, the author made use of subclasses (M_{2m}, M'_{2m}) of N'_{2m} which are different from N_{2m}. Theorem 4 had been originally formulated under compactness conditions based on these subclasses and afterwards has been changed to use those based on N_{2m}, which are simpler than the original ones.

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