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On n-dimensional Lorentz manifolds admitting an isometry group of dimension $n(n-1)/2+1$ for $n\geqq 4$

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1. Introduction.

A connected n-dimensional Riemannian manifold admitting a connected closed isometry group of dimension $n(n-1)/2+1(n\geqq 4)$ was completely determined by Yano [8], Ishihara [1] and Obata [6] (cf. Kobayashi [2]). The result of Obata (Theorem 10 in [6]) is as follows: Let G be a connected Lie group of dimension r and H a compact subgroup of dimension $r-n$. Assume that $n(n-1)/2<r<n(n+1)/2$, $n\geqq 3$, $n\neq 4$ and G is almost effective on G/H as a transformation group. Then G is of dimension $n(n-1)/2+1$ and G/H is one of the spaces $C_{0}^{1}\times C_{+}^{n-1}$, $C_{0}^{1}\times C_{-}^{n-1}$, C_{0}^{n}, C_{-}^{n} as a Riemannian manifold. Here we denote by C_{+}^{m}, C_{-}^{m} and C_{0}^{m} an m-dimensional Riemannian manifold of positive and negative constant curvature and a locally flat Riemannian manifold respectively. We consider the classification problem of Lorentz manifolds. Each of the following examples is a connected n-dimensional Lorentz manifold M admitting a connected isometry group G of dimension $n(n-1)/2+1$.

EXAMPLE.

(i) $M=\boldsymbol{R}\times N$ with metric $-dt^{2}+ds_{N}^{2}$ and $G=\boldsymbol{R}\times I^{0}(N)$.

(ii) $M=\boldsymbol{S}^{1}\times N$ with metric $-d\theta^{2}+ds_{N}^{2}$ and $G=\boldsymbol{S}^{1}\times I^{0}(N)$.

(iii) $M=\boldsymbol{R}\times P^{n-1}$ with metric $-dt^{2}+ds_{P}^{2}$ and $G=\boldsymbol{R}\times I^{0}(P^{n-1})$.

(iv) $M=\boldsymbol{S}^{1}\times P^{n-1}$ with metric $-d\theta^{2}+ds_{P}^{2}$ and $G=\boldsymbol{S}^{1}\times I^{0}(P^{n-1})$.

(v) $M=U_{n}^{+}=\{(u_{1},\ldots,u_{n});u_{n}>0\}$ with metric $ds_{+}^{2}=(du_{1}^{2}+\ldots+du_{n-1}^{2}-du_{n}^{2})/(cu_{n})^{2}(c\neq 0)$ and $G=I^{0}(U_{n}^{+})$ (see Nomizu [5]).

(vi) $M=U_{n}^{-}=\{(u_{1},\ldots,u_{n});u_{n}>0\}$ with metric $ds_{-}^{2}=(-du_{1}^{2}+du_{2}^{2}+\ldots+du_{n}^{2})/(cu_{n})^{2}(c\neq 0)$ and $G=I^{0}(U_{n}^{-})$ (see Matsuda [3]).

Here N is a simply connected $(n-1)$-dimensional Riemannian manifold with metric ds_{N}^{2} of constant curvature and P^{n-1} is an $(n-1)$-dimensional real projective space with standard metric ds_{P}^{2}. A real line and a circle of certain radius are denoted by $\boldsymbol{R}$ and $\boldsymbol{S}^{1}$ respectively. $I^{0}(\cdot)$ denotes the identity component of the full isometry group of $(\cdot)$.

The purpose of this note is to prove the following theorem.

THEOREM. *Let M be a connected n-dimensional Lorentz manifold admitting a connected isometry group G of dimension $n(n-1)/2+1(n\geqq 4)$ whose isotropy subgroup is compact. Then M must be one of spaces* (i)~(v).

REMARK 1. This theorem was proved for the case $n\geqq 5$ (see Matsuda [4]). But the method in this paper partly differs from [4].

REMARK 2. The isotropy subgroup of G in the above example is compact except (vi).

REMARK 3. The spaces of (v) and (vi) are not geodesically complete.

2. Preliminaries.

Let $(M, <\ ,\ >)$ be a connected n-dimensional Lorentz manifold with signature $(-,+,\dots,+)$ $(n\geqq 2)$. Let G be a connected isometry group of $(M, <\ ,\ >)$ and H the isotropy subgroup of G at a point $o\in M$. Then the linear isotropy group $\tilde{H}=\{d\tau_h\,;\,h\in H\}$ acting on T_oM is a closed subgroup of $O(1, n-1)=\{A\in GL(n, \boldsymbol{R})\,;\,{}^tASA=S\}$ where S is the matrix

$$\begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{pmatrix}$$

Throughout this note, we assume that H is compact.

LEMMA 1. *Every compact subgroup of $O(1, n-1)$ is conjugate to a subgroup of $O(1)\times O(n-1)$ (cf. Wolf [7]). Especially if K is a compact subgroup of $O(1, n-1)$ whose dimension is $(n-1)(n-2)/2$, then K leaves invariant one and only one 1-dimensional subspace in an n-dimensional vector space (cf. Obata [6]).*

From Lemma 1, we can see that for $n(n+1)/2\geqq r>n(n-1)/2+1$ the full isometry group of M contains no subgroup of dimension r whose isotropy subgroup is compact. Furthermore, we can also have the following proposition from Lemma 1.

PROPOSITION. *If M admits a connected isometry group G of dimension $n(n-1)/2+1$, then G is transitive on M.*

Hereafter, let G be a connected isometry group of dimension $n(n-1)/2+1$. From Proposition, dim $H=(n-1)(n-2)/2$. Therefore, the linear isotropy group $\tilde{H}$ leaves one and only one 1-dimensional subspace T of T_oM which is timelike. Let e_0 be a unit timelike vector belonging to T.

LEMMA 2. *If M is time orientable, then the vector field ξ defined by $\xi(p):=d\tau_g(e_0)(\tau_g o=p, g\in G)$ is well-defined on M and G-invariant timelike vector field.*

PROOF. We will show that for each $p\in M$, $\xi(p)=d\tau_g(e_0)$ is independent of the choice of $g\in G$ such that $\tau_g o=p$.

Let $\tau_{g_1}o=\tau_{g_2}o=p(g_1, g_2\in G)$. G being connected, there exist curves $\tilde{g}_i$: $[0, 1]\to G$ such that $\tilde{g}_i(0)=$identity and $\tilde{g}_i(1)=g_i(i=1, 2)$. Set $c_i(t)=\tau_{\tilde{g}_i(t)}o$ $(i=1, 2)$. M being time orientable, there exists a unit timelike vector field X on M. Then we can see that

$$<X(c_i(t)),\ d\tau_{\tilde{g}_i(t)}\xi(o)>\neq 0$$

for any $t\in[0, 1]$. The map : $t\longrightarrow<X(c_i(t)),\ d\tau_{\tilde{g}_i(t)}\xi(0)>$ being continuous, if $<X(o), \xi(o)><0$(resp.>0), then $<X(p), d\tau_{g_i}\xi(o)><0$ (resp.>0). Thus $d\tau_{g_1}\xi(o)$ and $d\tau_{g_2}\xi(o)$ belong to the same connected component of the time cone in T_pM. On the other hand, let H_p be the isotropy subgroup of G at $p\in M$. Then $H_p=g_iHg_i^{-1}$ so that $d\tau_{g_i}\xi(o)$'s belong to the one and only one 1-dimensional subspace of T_pM which is invariant by the linear isotropy group H_p. Therefore $d\tau_{g_1}\xi(o)=d\tau_{g_2}\xi(o)$.

If M is time orientable, the 1-form ω can be defined by $\omega(X)=<X, \xi>$. Hereafter, we assume that $n\geqq 4$. Then the existence of linear maps A and B in the proof of the following Lemma 3 is guaranteed.

LEMMA 3. *ω is G-invariant closed form.*

PROOF. It is clear that ω is G-invariant. Let $\{\xi(o)=e_0, e_1, \ldots, e_{n-1}\}$ be a Lorentz basis of T_oM, i.e.,

$$<e_0, e_0>=-1,\ <e_0, e_j>=0,\ <e_j, e_j>=1,$$
$$<e_i, e_j>=0,\ (1\leqq i\neq j\leqq n-1).$$

We will prove that

$$(1)\quad d\omega(e_0, e_j)=0\ (1\leqq j\leqq n-1)$$

and

$$(2)\quad d\omega(e_i, e_j)=0\ (1\leqq i<j\leqq n-1).$$

For a fixed j, let $A: T_oM\longrightarrow T_oM$ be the linear map defined by

$$A(e_0)=e_0,\ A(e_j)=-e_j,\ A(e_k)=-e_k \text{ (for some } k\neq 0, j)$$
$$A(e_s)=e_s \text{ (for any } s\neq 0, j, k).$$

Then $A \in SO(1) \times SO(n-1)$ so that there exists $h \in H$ such that $d\tau_h = A$ on T_oM. Therefore, $d\omega(e_0, e_j) = d(\tau_h^* \omega)(e_0, e_j) = d\omega(A(e_0), A(e_j)) = -d\omega(e_0, e_j)$. Thus $d\omega(e_0, e_j) = 0$. For fixed i and j, we define the linear map $B : T_oM \longrightarrow T_oM$ by

$$B(e_0) = e_0,\ B(e_i) = e_j,\ B(e_j) = e_i,$$
$$B(e_k) = -e_k \text{ (for some } k \neq 0, i, j),$$
$$B(e_s) = e_s \text{ (for any } s \neq 0, j, k).$$

Then $B \in SO(1) \times SO(n-1)$ so that there exists $h \in H$ such that $d\tau_h = B$ on T_oM. Therefore $d\omega(e_i, e_j) = d(\tau_h^* \omega)(e_i, e_j) = d\omega(B(e_i), B(e_j)) = -d\omega(e_i, e_j)$ so that $d\omega(e_i, e_j) = 0$.

3. Proof of theorem.

In the first, we assume that M is simply connected (therefore M is time orientable). Since ω is closed by Lemma 3, there exists a differentiable function $f : M \longrightarrow \boldsymbol{R}$ such that $df = \omega$. Let $c_p(t)$ be an integral curve of ξ such that $c_p(0) = p$. Then we have easily that $f(c_p(t)) = -t + f(p)$.

LEMMA 4. *Each integral curve of ξ is a complete geodesic.*

PROOF. Let A be the linear map as in the proof of Lemma 3. Then there exists $h \in H$ such that $d\tau_h = A$ on T_oM. We have

$$< \nabla_\xi \xi, e_j > = < d\tau_h(\nabla_\xi \xi), d\tau_h(e_j) > = < \nabla_\xi \xi, -e_j >$$

so that we have $< \nabla_\xi \xi, e_j > = 0$. It is evident that $< \nabla_\xi \xi, \xi > = 0$. Therefore we have $\nabla_\xi \xi = 0$ at $o \in M$. Since ξ is G-invariant, $\nabla_\xi \xi$ vanishes on M so that each integral curve of ξ is geodesic. Furthermore, this geodesic is complete, because ξ is G-invariant.

From Lemma 4, we have $f(M) = \boldsymbol{R}$. Let $N := f^{-1}(0)$ $(0 \in \boldsymbol{R})$. Then N is a closed spacelike hypersurface of M. Let N_0 be a connected component of N.

LEMMA 5([4]). *$F : \boldsymbol{R} \times N_0 \to M$ defined by $F(t, x) := c_x(t) =$ $\mathrm{Exp}(t\xi(x))$ for $(t, x) \in \boldsymbol{R} \times N_0$ is onto diffeomorphism; furthermore, $N = N_0$.*

PROOF. Assume that $F(t, x) = F(t', x')$. We have $t = -f(c_x(t)) = -f(F(t, x)) = -f(F(t', x')) = t'$. Since $c_x(t) = c_{x'}(t')$ and $t = t'$, we have $x = x'$. Thus F is one to one. It is evident that F is differentiable. Set $M_0 := F(\boldsymbol{R} \times N_0)$. Then M_0 is open in M. It remains to be shown that M is closed in M. Suppose that $F(t_k, x_k) = p_k$ is a sequence approaching some point q in M. Let $\tilde{f} : \boldsymbol{R} \to \boldsymbol{R}$ be the function defined by $\tilde{f}(t) := f(F(t, x))$ for some $x \in N_0$. Then $\tilde{f}$ is independent of the choice $x \in N_0$, for $\tilde{f}(t) = -t$. Since $\tilde{f}^{-1}(f(p_k)) = t_k$, and $\tilde{f}^{-1}(f(p_k))$ approaches $\tilde{f}^{-1}(f(q))$, we have $t_k \to t_0 := \tilde{f}^{-1}(f(q))$ as $k \to \infty$. Letting $x_0 := c_q(-t_0) = \mathrm{Exp}(-t_0 \xi(q))$, we have $x_k = c_{p_k}(-t_k) = \mathrm{Exp}(-t_k \xi(p_k)) \to c_q(-t_0)$. Since N_0 is closed, x_0 belongs to N_0 so that $q = F(t_0, x_0)$ belongs to M_0. Thus $M = M_0$; furthermore, $N = N_0$.

REMARK 4. For each $a \in \boldsymbol{R}$, $f^{-1}(a)$ is a connected closed spacelike hypersurface of M.

LEMMA 6. *For each $a \in \boldsymbol{R}$, N and $f^{-1}(a)$ are rigid in M.*

PROOF. Since G acts transitively on M, for some point p of $f^{-1}(a)$ there exists $g \in G$ such that $\tau_g o = p (o \in N)$. Then $\tau_g N \subset f^{-1}(a)$. Because, for any $q \in \tau_g N$, there exists C^∞ curve $\tilde{c}: [0, 1] \to \tau_g N$ such that $\tilde{c}(0) = p$ and $\tilde{c}(1) = q$. Put $c := \tau_{g^{-1}} \tilde{c}$. Then c is C^∞ curve on N so that $f(c(s)) = 0$ for any $s \in [0, 1]$. We have

$$(df/ds)(\tilde{c}(s)) = < \nabla f, \dot{\tilde{c}}(s) > = < \xi(\tilde{c}(s)), \dot{\tilde{c}}(s) > =$$
$$< d\tau_g \xi(c(s)), d\tau_g \dot{c}(s) > = < \xi(c(s)), \dot{c}(s) > = (df/ds)(c(s)) = 0.$$

Therefore $f(\tau_g N) = f(p) = a$, that is, $\tau_g N \subset f^{-1}(a)$.

Since $f^{-1}(a)$ is connected and $\tau_g N$ is open and closed in $f^{-1}(a)$, we have $\tau_g N = f^{-1}(a)$.

LEMMA 7. *N is homogeneous Riemannian manifold.*

PROOF. For any p, $q \in N$, there exists $g \in G$ such that $\tau_g p = q$. By the same discussion as in the proof of Lemma 6, we can see that $\tau_g|_N$ is an isometric transformation of N.

Let $G' := \{g \in G ; \tau_g N = N\}$. Then G' is the Lie subgroup of G. We can verify that H is included in G' by the same discussion as in the proof of Lemma 6. G' acts effectively on N. In fact, if $g \in G'$ acts trivially on N, then $d\tau_g \xi(x) = \xi(x)$ $(x \in N)$ so that $d\tau_g = \mathrm{id.}$ on $T_x M = \boldsymbol{R}\{\xi(x)\} + T_x N$. Thus we have $g = \mathrm{id.}$. Furthermore we have $\dim G' = \dim N + \dim H = n(n-1)/2$. Therefore the simply connected $(n-1)$-dimensional homogeneous Riemannian manifold N admitting an isometry group G' of maximal dimension $n(n-1)/2$ is isometric to $\boldsymbol{S}^{n-1}$, $\boldsymbol{H}^{n-1}$ or $\boldsymbol{E}^{n-1}$.

LEMMA 8. *$\nabla_X \xi = -cX$ for any X such that $< X, \xi > = 0$ where c is a constant.*

PROOF. For $X \in T_o M$ such that $< X, \xi(o) > = 0$, $\nabla_X \xi(o)$ is expressed by

$$\nabla_X \xi(o) = c(X)X + b(X)X^{\perp}$$

for some $X^{\perp}$ such that $<X^{\perp}, \xi(o)> = 0 = <X, X^{\perp}>$ and for scalars $c(X)$, $b(X)$ depending on X. Because $<\nabla_X \xi, \xi> = 0$. Therefore $<\nabla_X \xi, X> = c(X) <X, X>$. Since the linear isometry group $\tilde{H}$ acts transitively on $U := \{Z \in T_o M ; <\xi(o), Z> = 0, <Z, Z> = 1\}$, c is constant on U. Furthermore we have $c(\alpha X) = c(X)$ for any non-zero $\alpha \in \boldsymbol{R}$. Thus we have $<\nabla_X \xi, X> = -c <X, X>$ for any X orthogonal to $\xi(o)$. Since M is homogeneous, we can see that $<\nabla_X \xi, X> = -c <X, X>$ for any X orthogonal to ξ. Therefore, after polarization, we have $<\nabla_X \xi, Y> = -c <X, Y>$ for any Y orthogonal to ξ. Thus we have $\nabla_X \xi = -cX$.

REMARK 5. Taking $-\xi$ instead of ξ (if necessary), we may assume that $c > 0$.

LEMMA 9([4]). *$F : (\boldsymbol{R} \times N, -dt^2 + \exp(-2ct) ds_N^2) \to (M, < , >)$ is isometry, where ds_N^2 is the metric of N.*

PROOF. Let $(V, \phi = (t_1, \dots, t_{n-1}))$ be a local coordinate around a point p in N. Then $(\boldsymbol{R} \times V, \mathrm{id} \times \phi = (t, t_1, \dots, t_{n-1}))$ is a local coordinate around (a, p) in $\boldsymbol{R} \times N$. Let $\tilde{V} := F(\boldsymbol{R} \times V)$ and define $\tilde{\phi} : \tilde{V} \to \boldsymbol{R}^n$ by $(\mathrm{id} \times \phi) \circ F^{-1}$. Then $(\tilde{V}, \tilde{\phi} = (x_0, x_1, \dots, x_{n-1}))$ is a local coordinate around $\tilde{p} = F(a, p)$ in M. We can see that $dF(\partial/\partial t) = \xi = \partial/\partial x_0$ and $dF(\partial/\partial t_i) = \partial/\partial x_i$ $(i = 1, \dots, n-1)$. We can also verify that $<\partial/\partial x_0, \partial/\partial x_i> = 0$ $(i = 1, \dots, n-1)$. Because

$$\begin{aligned} <\partial/\partial x_0, \partial/\partial x_i> &= <\xi, \partial/\partial x_i> = <\nabla f, \partial/\partial x_i> \\ &= dF(\partial/\partial t_i)(f) = (\partial f/\partial t_i)(F(t, x)) \\ &= (\partial/\partial t_i)(-t) = 0. \end{aligned}$$

Since $\partial/\partial x_0 <\partial/\partial x_j, \partial/\partial x_i> = -2c <\partial/\partial x_j, \partial/\partial x_i>$ by Lemma 8, we have $<\partial/\partial x_j, \partial/\partial x_i> = \exp(-2cx_0) g_{ji}(x_1, \dots, x_{n-1})$ for $i, j = 1, \dots, n-1$.

Thus we have $F^* < , > = -dt^2 + \exp(-2ct) ds_N^2$.

LEMMA 10. *If $N = \boldsymbol{S}^{n-1}$ or $\boldsymbol{H}^{n-1}$, then $c = 0$, i. e., the metric of $\boldsymbol{R} \times N$ is product metric.*

PROOF. Since $f^{-1}(a)$ $(a \in \boldsymbol{R})$ is isometric to N by Lemma 6, the scalar curvature S_a of $f^{-1}(a)$ coincides with the scalar curvature S_0 of N. The facts that $S_a = S_0$ is nonzero and $S_a = \exp(-2ca) S_0$ by Lemma 9 imply $c = 0$.

In the case $N = \boldsymbol{E}^{n-1}$ and $c = 0$, $(M, < , >)$ is isometric to $(\boldsymbol{R} \times \boldsymbol{E}^{n-1}, -dt^2 + ds_E^2)$ which is the Lorentz-Minkowski space. In the case $N = E^{n-1}$

and $c\neq 0$, $(M, <\ ,\ >)$ is isometric to $(\boldsymbol{R}\times\boldsymbol{E}^{n-1},\ -dt^{2}+\exp(-2ct)\sum_{j=1}^{n-1}dt_{j}^{2})$ which is isometric to $(U_{n}^{+},\ ds_{+}^{2})$ by the transformation

$$\begin{aligned}&\boldsymbol{R}\times\boldsymbol{E}^{n-1}\ni(t,\ t_{1},\ \ldots,\ t_{n-1})\\&\rightarrow(u_{1},\ \ldots,\ u_{n-1},\ u_{n})=(t_{1},\ \ldots,\ t_{n-1},\ e^{ct}/c)\in U_{n}^{+}.\end{aligned}$$

Thus if M is simply connected, then M is isometric to the space (i) or (v) in the Example.

To find non-simply connected M, we use the same procedure as in Kobayashi [2], p. 52. Thus we complete the proof of the theorem.

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