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On H-separable extensions of primitive rings

In memory of Professor Akira Hattori

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Introduction. Throughout this paper every ring will have the identity, and every subring of it will contain the identity of it. A ring is said to be strongly primitive if it has a faithful minimal left ideal. The structure of strongly primitive ring was researched in [1] and [2] by Nakayama and Azumaya. The aim of this paper is to give a necessary and sufficient condition for an H-separable extension ring A of a strongly primitive ring B to be strongly primitive. We will show that, if B is a strongly primitive ring with the socle $\mathfrak{z}$, and if A is an H-separable extension of B such that A is left (or right) B-finitely generated projective, then the necessary and sufficient condition for A to be strongly primitive is that $A\mathfrak{z}A\cap B=\mathfrak{z}$ holds (Theorem 1). This condition is a sufficient condition, if we assume that A is an H-separable extension of a strongly primitive ring B such that B is a left (or right) B-direct summand of A. Finally, we will consider the case where A is a left full linear ring with the center C, D is a simple C-subalgebra of A with $[D:C]<\infty$ and $B=V_A(D)$, the centralizer of D in A. In the above situation Nakayama and Azumaya obtained much more interesting results in [1] and [2]. In particular, they showed that B is also a left full linear ring, $V_A(B)=D$ and that the same inner Galois theory as in simple artinian ring holds in this case, too. In this paper we will show that $S=A\mathfrak{z}A$, $A\mathfrak{z}A\cap B=\mathfrak{z}$ and $S=Soc({}_BA)=Soc(A_B)=A\mathfrak{z}=\mathfrak{z}A$ hold if A and B are in the above situation, where S and $\mathfrak{z}$ are the socles of A and B, respectively (Theorem 2).

Preliminaries. First we recall some definitions. Let A be a ring. Hereafter we will call each two sided ideal of A, simply, an ideal of A. The socle of a left (resp. right) A-module M is the sum of all minimal A-submodules of M, and denoted by $Soc({}_AM)$ (resp. $Soc(M_A)$). A is said to be a left primitive ring if A has a faithful simple left A-module. A right primitive ring is similarly defined, and a both left and right primitive ring is called simply primitive ring. Now we put a stronger condition on A. A is said to be strongly primitive if A has a faithful minimal left ideal. In this case A has also a faithful minimal right ideal. Thus strong primitivity is left and right equivalent. Now let A be a strongly primitive ring. Then we have the following assertions;

(1) Every non zero left (or right) ideal of A is faithful

(2) All minimal left (or right) ideals of A are mutually isomorphic, and their sum is the smallest non zero ideal of A. Consequently we have $Soc({}_AA)=Soc(A_A)$.

(3) A left (or right) ideal of A is minimal if and only if it is generated by a primitive idempotent.

(4) Every left (or right) faithful A-module has a minimal submodule isomorphic to a minimal left (or right) ideal of A. Consequently all faithful simple left (or right) A-modules are mutually isomorphic

All of the above results are proved in [1] Theorems 1 and 2 without the assumption of the existance of the identity. In this paper we will use the above results freely.

H-separable extensions of strongly primitive rings.

The following proposition has already been shown in [6]

Proposition 1. *Let B be a strongly primitive ring, and A an H-separable extension of B such that B is a left (or right) B-direct summand of A. Then, A is a primitive ring, and $A\mathfrak{z}A$ is the smallest ideal of A, where $\mathfrak{z}$ is the socle of B.*

Proof. The former assertion is Proposition 3 [6], and the latter is shown in the proof of it.

Now we will show that the same assertion holds under the condition that A is left (or right) finitely generated projective over B in stead of the one that B is a left (or right) B-direct summand of A.

Proposition 2. *Let B be a strongly primitive ring with its socle $\mathfrak{z}$, and A an H-separable extension of B such that A is right B-finitely generated projective. Then, we have*

(1) *A is a primitive ring, and $A\mathfrak{z}A$ is the smallest ideal of A.*

(2) *Every simple right B-submodule of A is faithful, and $A\mathfrak{z}$ coincides with $Soc(A_B)$, the right B-socle of A.*

(3) *If furthermore A is strongly primitive, then $A\mathfrak{z}A$ coincides with the socle of A.*

Proof. For any ideal $\mathfrak{A}$ of A, we have $\mathfrak{A}=A(\mathfrak{A}\cap B)$ by Theorem 3.1 [5], since A is right B-finitely generated projective. Therefore if $\mathfrak{A}\neq\bar{0}$ we have $0\neq\mathfrak{A}\cap B\subset\mathfrak{z}$ by Theorem 1 [1]. Then we have $\mathfrak{A}\supset A\mathfrak{z}A$. Thus we see that $A\mathfrak{z}A$ is the smallest ideal of A. Let J be the radical of A, and suppose that $J \neq 0$. Then the above argument shows that $J \supset J \cap B \supset \mathfrak{z}$, which contradicts to the fact that J does not contain any non zero idempotents. Thus we have $J=0$. Then there exists a maximal left ideal L of A such that $\not\supset A\mathfrak{z}A$. Put $\mathfrak{m}=A/L$. If the annihilater $Ann({}_A\mathfrak{m})$ of $\mathfrak{m}$ is not zero, it must contain $A\mathfrak{z}A$, and we have $A\mathfrak{z}A=(A\mathfrak{z}A)A \subset L$, a contradiction. Thus we see that $Ann({}_A\mathfrak{m})=0$, and that $\mathfrak{m}$ is a faithful simple left A-module. Similarly, we can find a maximal right idght ideal I of A such that A/I is a faithful simple right A-module. Thus we have proved (1). Put $M=\{a \in A \mid a\mathfrak{z}=0\}$. M is an A-$V_A(B)$-submodule of A, and we have $M=A(M \cap B)$ by Theorem 3.1 [5]. But $M \cap B=0$, since $\mathfrak{z}$ is left B-faithful. Hence we have $M=0$. Now suppose that there exists a simple right B-submodule $\mathfrak{r}$ of A which is not faithful. Then $\mathfrak{r}\mathfrak{z}=0$, and we have $\mathfrak{r} \subset M \neq 0$, a contradiction. Let $\mathfrak{m}$ be a simple right B-submodule of A. Then $\mathfrak{m}$ is B-faithful by the above argument, and consequently, there exists a primitive idempotent e of B such that $eB \cong \mathfrak{m}$. Then we have immediately $\mathfrak{m}=\mathfrak{m}eB \subset A\mathfrak{z}$. and hence $Soc(A_B) \subseteq A\mathfrak{z}$. The converse inclusion is obvious, and we have $Soc(A_B)=A\mathfrak{z}$. Thus we have proved (2). (3) is clear by (1), because the socle of a strongly primitive ring is the smallest non zero ideal of it. Thus we have finished the proof of the theorem.

Furthermore, we can give a necessary and sufficient condition for an H-separable extension ring of a strongly primitive ring to be strongly primitive, as follows;

THEOREM 1. *Let B be a strongly primitive ring with the socle $\mathfrak{z}$, and A an H-separable extension of B. Suppose furthermore that A is left B-finitely generated projective. Then, A is also a strongly primitive ring, if and only if $A\mathfrak{z}A \cap B=\mathfrak{z}$.*

PROOF. First suppose that $A\mathfrak{z}A \cap B=\mathfrak{z}$. Since A is left B-finitely generated projective, we have $A\mathfrak{z}A=(A\mathfrak{z}A \cap B)A=\mathfrak{z}A$ by Theorem 3.1 [5]. Hence we have $A\mathfrak{z} \subset \mathfrak{z}A$. On the other hand, there exists a faithful simple left A-module $\mathfrak{m}$, and we have $\mathfrak{m}=A\mathfrak{z}\mathfrak{m} \subset \mathfrak{z}A\mathfrak{m}=\mathfrak{z}\mathfrak{m} \subset \mathfrak{m}$, and consequently, $\mathfrak{m}=\mathfrak{z}\mathfrak{m}$. But $\mathfrak{z}\mathfrak{m}$ is a sum, and consequently, a direct sum of faithful simple B-submodules which are isomorphic to some Be, where e is a primitive idempotent of B. Hence $\mathfrak{m}=\mathfrak{z}\mathfrak{m}$ is B-projective. Then $\mathfrak{m}$ is A-projective, since A is a separable extension of B. Therefore, there exists an A-split exact sequence $A \longrightarrow \mathfrak{m} \longrightarrow 0$, and we see that A has a faithful minimal left ideal isomorphic to $\mathfrak{m}$. Thus we have proved the 'if' part. Conversely suppose that A is strongly primitive, and let $b \in A\mathfrak{z}A \cap B$. Assume that $Bb \not\subset A\mathfrak{z}b$, and let p be the natural map of Ab to $A/A\mathfrak{z}b$. $Bb \not\subset A\mathfrak{z}b$ implies that

$p(Bb)\neq 0$, while $\mathfrak{z}p(Bb)=p(\mathfrak{z}b)=0$. On the other hand, Ab is completely reducible by Proposition 2 (3). Hence we can write $Ab=\oplus_{i=1}^{r}Ae_i$, where e_i's are primitive idempotents of A, and $0\neq Ab/A\mathfrak{z}b\cong\oplus Ae_{i_k}(1\leqq i_k\leqq r)$. Then $Ab/A\mathfrak{z}b$ is B-projective, since A is left B-projective by assumption. Therefore, there exists a non zere B-homomorphism q of $p(Bb)$ to B such that $qp(Bb)$ is annihilated by $\mathfrak{z}$. This is a contradiction, because every left ideal of B is faithful. Therefore, we have $Bb\subset A\mathfrak{z}b$, and $b\in A\mathfrak{z}\cap B\subset Soc(A_B)\cap B=\mathfrak{z}$. Thus we have $A\mathfrak{z}A\cap B\subset\mathfrak{z}$. The converse inclusion is obvious. Hence we have proved the 'only if' part, too.

By the same proof as Theorem 1 we can obtain the following two propositions;

PROPOSITION 3. *Let B be a strongly primitive ring with the socle $\mathfrak{z}$, and A an H-separable extension of B such that B is a right B-direct summand of A. Then if $A\mathfrak{z}A\cap B=\mathfrak{z}$, A is also a strongly primitive ring.*

PROOF. We have $A\mathfrak{z}A=(A\mathfrak{z}A\cap B)A$ by our assumption and Proposition 4.1 [5]. Then we can follow the same lines as the proof of the 'if' part of Theorem 1.

PROPOSITION 4. *Let A and B be strongly primitive rings with the socles S and $\mathfrak{z}$, respectively. If B is a subring of A such that A is left B-projective, then we have $S\cap B\subset\mathfrak{z}$.*

PROOF. For any $b\in S\cap B$, Ab is again completely reducible. Therefore, we can follow the same lines as the proof of the 'only if' part of Theorem 1.

A typical example of strongly primitive ring is a left full linear ring, that is, the ring of linear transformations of a left vector space over a division ring. In [1] Nakayama and Azumaya showed that, if A is a left full linear ring with its center C, and if D is a simple C-subalgebra of A such that $[D:C]<\infty$, then $V_A(D)$ is also a left full linear ring and $D=V_A(V_A(D))$ (Theorem 10 [1]). Now let A, D and C be as above, and put $B=V_A(D)$. Theorem 36.2 [2] shows that A has a right B-free basis consisting of $[D:C]$ elements of A. On the other hand, the author proved in [6] that A is an H-separable extension of B and has also a left free basis consisting of $[D:C]$ elements. Therefore we can apply Proposition 2 and Theorem 1, and have

THEOREM 2. *Let A be a left full linear ring with its center C, and D a simple C-subalgebra of A such that $[D:C]<\infty$. Let $B=V_A(D)$, and denote the socles of A and B by S and $\mathfrak{z}$, respectively. Then we have $S=$*

$A\mathfrak{z}A$ and $A\mathfrak{z}A\cap B=\mathfrak{z}$, $S=Soc({}_BA)=Soc(A_B)=A\mathfrak{z}=\mathfrak{z}A$.

Theorem 3 together with Proposition 2 (2) yields

COROLLARY. *Let A and B be as in Theorem 3. Then, every simple left (resp. right) ideal of A is a direct sum of mutually isomorphic faithful simple left (resp. right) B-submodules.*

REMARK. In Theorem 36. 2 [2] it is shown that, under the same conidtion as Theorem 2, every simple right ideal of A is a direct sum of faithful simple right B-submodules.

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