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On the energy decay of a weak solution of the M. H. D. equations in a three-dimensional exterior domain

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Introduction

Let O be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary ∂O. We set $\Omega=\boldsymbol{R}^3-O$. For simplicity, we assume that Ω is simply connected. In $Q:=\Omega\times(0,\infty)$, we consider the following magnetohydrodynamic (M. H. D.) equations;

$$
\text{(M. H. D.)}\quad
\begin{array}{ll}
\partial_t u-\Delta u+(u, \nabla)u+B\times \mathrm{rot}B+\nabla\pi=f & \text{in } Q,\\
\partial_t B-\Delta B+(u, \nabla)B-(B, \nabla)u=0 & \text{in } Q,\\
\mathrm{div}\, u=0,\ \mathrm{div}\, B=0 & \text{in } Q,\\
u=0,\ B\cdot\nu=0,\ \mathrm{rot}B\times\nu=0, & \text{on } \partial\Omega\times(0,\infty),\\
u|_{t=0}=u_0,\ B|_{t=0}=B_0. &
\end{array}
$$

Here $u=u(x, t)=(u^1(x, t), u^2(x, t), u^3(x, t))$, $B=B(x, t)=(B^1(x, t), B^2(x, t), B^3(x, t))$ and $\pi=\pi(x, t)$ denote respectively the unknown velocity field of the fluid, magnetic field and pressure of the fluid, $f=f(x, t)=(f^1(x, t), f^2(x, t), f^3(x, t))$ denotes the given external force, $u_0=u_0(x)=(u_0^1(x), u_0^2(x), u_0^3(x))$ and $B_0=B_0(x)=(B_0^1(x), B_0^2(x), B_0^3(x))$ denote the given initial data and ν denotes the unit outward normal on $\partial\Omega$.

Our problem reads as follows.

PROBLEM

Construct a weak solution $\{u, B\}$ *of* (M. H. D.) *on* $(0,\infty)$ *such that*

$$E(t):=(1/2)\int_\Omega(|u(x, t)|^2+|B(x, t)|^2)\,dx$$

tends to zero as $t\to\infty$.

In this paper, we solve this problem affimatively. To this end, we shall use the methods developed by Masuda [5] and Sohr [10] in the case of the Navier-Stokes equations.

As is shown by Masuda [5, Corollary 2], we shall show at first that if $\{u, B\}$ is a weak solution of (M. H. D.) such that $E(t)$ tends to some constant E as $t\to\infty$, then $E=0$. For such a weak solution, we shall construct the one satisfying the energy inequality of strong form (see Masuda [5, subsection 1.2 Remarks 3]). This procedure is due to Sohr [10].

1. Preliminary and Result

1.1 Definition of weak solution

Let us introduce some function spaces. $C_{0,\sigma}^{\infty}(\Omega)$ denotes the set of all C^{∞}-real vector functions $\phi=(\phi^1, \phi^2, \phi^3)$ with compact support in Ω such that div $\phi=0$. H is the completion of $C_{0,\sigma}^{\infty}(\Omega)$ with respect to the L^2-norm $\| \ \|$; $(\ , \ )$ denotes the L^2-inner product. The Hilbert space V_1 is the subspace of the Sobolev space $H_0^1(\Omega)^3$, consisting of all vector functions u in $H_0^1(\Omega)^3$ with div $u=0$. The Hilbert space V_2 is the subspace of the Sobolev space $H^1(\Omega)^3$, consisting of all vector functions B in $H^1(\Omega)^3$ with div $B=0$ in Ω and $B \cdot \nu=0$ on $\partial\Omega$.

If X is a Hilbert or Banach space, then $L^p(0, T; X)$, $1 \leq p < \infty$, denotes the set of all measurable functions $u(t)$ with values in X such that $\int_0^T \|u(t)\|_X^p \, dt < \infty$ ($\| \ \|_X$ is the norm of X). $L^{\infty}(0, T; X)$ denotes the set of all essentially bounded (in the norm of X) measurable functions of t with values in X. In the case of $X=L^r(\Omega)$, we denote by $\| \ \|_{r,p}$ and $\| \ \|_{r,\infty}$ the norms on $L^p(0, T; L^r(\Omega))$ and $L^{\infty}(0, T; L^r(\Omega))$, respectively.

Let $C^m([s, t]; X)$ denote the set of all X-valued m-times continuously differentiable functions of $\tau (s \leq \tau \leq t)$. For an interval I, $C_0^m(I; X)$ is the set of all X-valued m-times continuously differentiable functions on I with compact support in I. Throughout this paper, C denotes the positive constants which may change from line to line.

We define a weak solution of (M. H. D.) for $u_0 \in H$ and $B_0 \in H$ as follows:

Definition

Let $u_0 \in H$, $B_0 \in H$ and $f \in L^1(0, \infty; H)$.

A pair of measurable functions u and B on Q is called a weak solution of (M. H. D.) if

(i) $u \in L^{\infty}(0, \infty; H) \cap L_{loc}^2(0, \infty; V_1)$. $B \in L^{\infty}(0, \infty; H) \cap L_{loc}^2(0, \infty; V_2)$.

(ii) For any $\Phi \in C_0^1([0, \infty); V_1)$ and any $\Psi \in C_0^1([0, \infty); V_2)$, the equalities

$$\int_0^{\infty} \{-(u, \partial_t \Phi)+(\nabla u, \nabla \Phi)+((u, \nabla)u-(B, \nabla)B, \Phi)\} \, dt = (u_0, \Phi(0))+\int_0^{\infty}(f, \Phi) \, dt \tag{1.1}$$

$$\int_0^{\infty} \{-(B, \partial_t \Psi)+(\text{rot } B, \text{rot } \Psi)+((u, \nabla)B-(B, \nabla)u, \Psi)\} \, dt = (B_0, \Psi(0)) \tag{1.2}$$

are satisfied.

Concerning the definition of weak solutions of (M. H. D.), see Sermange and Temam [8].

The following lemma is essentially due to Serrin [9]. Hence we omit the proof.

LEMMA 1. 1

Let $\{u, B\}$ be a weak solution of (M. H. D.). After a suitable redefinition of $u(t)$ and $B(t)$ at a set of measure zero on $(0, \infty)$, we have that both u and B are weakly continuous in H as functions of t and that for any $s<t$,

$$\begin{aligned}&\int_s^t \{-(u, \partial_\tau \Phi)+(\nabla u, \nabla \Phi)+(u, \nabla)u-(B, \nabla)B, \Phi)\}\, d\tau \\ &=-(u(t), \Phi(t))+(u(s), \Phi(s))+\int_s^t (f, \Phi)\, d\tau \qquad (1.3)\\ &\int_s^t \{-(B, \partial_\tau \Psi)+(\operatorname{rot} B, \operatorname{rot} \Psi)+((u, \nabla)B-(B, \nabla)u, \Psi)\}\, d\tau \\ &=-(B(t), \Psi(t))+(B(s), \Psi(s)) \qquad (1.4)\end{aligned}$$

for every $\Phi \in C^1([s, t]; V_1)$ and every $\Psi \in C^1([s, t]; V_2)$.

1.2 Operators $A_{D(r)}$ and $A_{N(r)}$

Let H_r be the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in $\boldsymbol{L}^r(\Omega):=L^r(\Omega)^3 (r>1)$. As is well known, we have

$$\boldsymbol{L}^r(\Omega)=H_r \oplus G_r \text{ (direct sum)},$$

where $G_r=\{\nabla \pi \in \boldsymbol{L}^r(\Omega)\,;\, \pi \in L_{\mathrm{loc}}^r(\Omega)\}$.

Let P_r be the projection operator from $\boldsymbol{L}^r(\Omega)$ onto H_r along G_r. We define the operators $A_{D(r)}$ and $A_{N(r)}$ as follows:

$$\begin{aligned}&D(A_{D(r)})=H_r \cap \{u \in W^{2, r}(\Omega)\,;\, u|_{\partial\Omega}=0\}, \\ &A_{D(r)} u=-P_r \Delta u \text{ for } u \in D(A_{D(r)}), \\ &D(A_{N(r)})=H_r \cap \{B \in W^{2, r}(\Omega)\,;\, B \cdot \nu=0,\ \operatorname{rot} B \times \nu=0 \text{ on } \partial \Omega\}, \\ &A_{N(r)} B=-\Delta B \text{ for } B \in D(A_{N(r)}).\end{aligned}$$

Note that $A_{N(r)}$ maps $D(A_{N(r)})$ into H_r.

It follows from Miyakawa [6, 7] that both $-A_{D(r)}$ and $-A_{N(r)}$ generate the holmorphic semi-groups $e^{-tA_{D(r)}}$ and $e^{-tA_{N(r)}}$ in H_r. Moreover, we can define the fractional powers $\tilde{A}_{D(r)}^{\alpha}$ and $\tilde{A}_{N(r)}^{\alpha}$ of $\tilde{A}_{D(r)}:=1+A_{D(r)}$ and $\tilde{A}_{N(r)}:=1+A_{N(r)}$, respectively.

REMARK 1.2

We denote $A_{D(2)}$ and $A_{N(2)}$ simply by A_D and A_N, respectively. Let $a_D(\ ,\ )$ and $a_N(\ ,\ )$ be non-negative quadratic forms on V_1 and V_2 respectively defined by

$$a_D(u, v)=(\nabla u, \nabla v) \text{ for } u, v\in V_1$$

and

$$a_N(B, C)=(\mathrm{rot}\, B, \mathrm{rot}\, C) \text{ for } B, C\in V_2.$$

Then A_D and A_N coincide with the self-adjoint operators defined by $a_D(\ ,\ )$ and $a_N(\ ,\ )$, respectively. Hence we have

$$D(A_D^{1/2})=V_1,\ \|A_D^{1/2}u\|^2=\|\nabla u\|^2 \text{ for } u\in D(A_D^{1/2}), \tag{1.5}$$
$$D(A_N^{1/2})=V_2,\ \|A_N^{1/2}B\|^2=\|\mathrm{rot}\, B\| \text{ for } B\in D(A_N^{1/2}). \tag{1.6}$$

1.3 Result

We can now introduce the following assumptions.

Assumption 1

u_0 is in $H\cap D(\tilde{A}_{D(r_0)}^{1-1/r_0+\varepsilon})$ and B_0 is in $H\cap D(\tilde{A}_{N(r_0)}^{1-1/r_0+\varepsilon})$, where $r_0=5/4$ and $\varepsilon>0$.

Assumption 2

f is in $L^1(0,\infty;L^2(\Omega))\cap L^2(0,\infty;L^2(\Omega)\cap L^{r_0}(0,\infty;L^{r_0}(\Omega))$.

Our result reads:

THEOREM

Under the assumptions 1 *and* 2, *there exists a weak solution* $\{u, B\}$ *of* (*M. H. D.*) *such that*

$$E(t):=(1/2)(\|u(t)\|^2+\|B(t)\|^2)$$

tends to zero as $t\to\infty$.

We shall prove this theorem with the aid of the following two propositions.

PROPOSITION 1

Let u_0 *and* B_0 *be in* H *and let* f *be in* $L^1(0,\infty;L^2(\Omega))$. *Then any weak solution* $\{u, B\}$ *of* (*M. H. D.*) *with*

$$\int_0^\infty\|\nabla u(\tau)\|^2\,d\tau<\infty \text{ and } \int_0^\infty\|\mathrm{rot}\, B(\tau)\|^2\,d\tau<\infty \text{ satisfies}$$

$$\lim_{t\to\infty} \{\|(1+A_D)^{-1/4}u(t)\|+\|(1+A_N)^{-1/4}B(t)\|\}=0. \tag{W. D.}$$

PROPOSITION 2

Under the assumptions of Theorem, there is a weak solution $\{u, B\}$ of (M. H. D.) such that the energy inequality of strong form:

$$\|u(t)\|^2+\|B(t)\|^2+2\int_s^t(\|\nabla u(\tau)\|^2+\|\mathrm{rot}\ B(\tau)\|^2)\ d\tau$$
$$\leqq \|u(s)\|^2+\|B(s)\|^2+2\int_s^t(f(\tau),\ u(\tau))\ d\tau \tag{E. I. S.}$$

holds for almost all $s\geqq 0$, including $s=0$. and all $t>s$.

Propositions 1 and 2 are essentially due to Masuda [5, Theorem 4] and to Sohr [10], respectively.

1.4 Proof of Theorem

For a moment, we assume that the propositions 1 and 2 hold true. We follow the arguments developed by Sohr [10].

Let $\{u, B\}$ be the weak solution of (M. H. D.) costructed in Proposition 2. As is shown by Masuda [5, Corollary 1], it follows from (W. D.) that

$$\lim_{t\to\infty}\int_t^{t+1} E(\tau)\ d\tau=0. \tag{1.7}$$

On the other hand, by (E. I. S.) we have

$$E(t)\leqq E(s)+M_0\int_s^t \|f(\tau)\|\ d\tau \tag{1.8}$$

for almost all $s\geqq 0$, including $s=0$, and all $t>s$, where $M_0:=\sup_{\tau>0}\|u(\tau)\|$.

For $\varepsilon>0$, we choose $s_0=s_0(\varepsilon)$ such that $\int_{s_0}^\infty \|f(\tau)\|\ \ d\tau<\varepsilon/2M_0$. Moreover, we see that the measure of the set $\{\tau\geqq s_0\ ;\ E(\tau)<\varepsilon/2\}$ cannot be zero, since otherwise we have $\int_t^{t+1} E(\tau)\ d\tau\geqq\varepsilon/2$ for any $t\geqq s_0$ and this contradics (1.7).

Hence there is $s_1\geqq s_0$ such that $E(s_1)<\varepsilon/2$. It follows from (1.8) that

$$E(t)\leqq E(s_1)+M_0\int_s^\infty \|f(\tau)\|\ d\tau<\varepsilon/2+\varepsilon/2=\varepsilon$$

for all $t>s_1$. This completes the proof of Theorem.

We shall prove the propositions 1 and 2 in sections 2 and 3, respectively.

2. Proof of Proposition 1

In this section, we follow Masuda [5].

At the first step, we show that zero is not an eienvalue of A_D or A_N. In fact, by (1.5) it is easy to see that zero is not an eigenvalue of A_D. Suppose that $A_N B=0$ for $B\in D(A_N)$. Then by (1.6), rot $B=0$ in Ω. Since div $B=0$ in Ω and since $B\cdot\nu=0$ on $\partial\Omega$, it follows from the classical potential theory that there is a scalar function p with $p\in L^2_{\text{loc}}(\Omega)$, $\nabla p\in L^2(\Omega)$ and

$$\Delta p=0 \text{ in } \Omega,\ \partial p/\partial\nu=0 \text{ on } \partial\Omega$$

such that $B=\nabla p$.

According to Miyakawa [7, Lemma 1.4], such p must satisfy that $\nabla p=0$ and hence $B=0$. Thus zero is not an eigenvalue of A_N. At this stage, as is shown by Masuda [5], it suffices to show that there is a positive number C such that the inequality

$$\begin{aligned}
&\|(1+A_D)^{-1/4}u(t)\|^2+\|(1+A_N)^{-1/4}B(t)\|^2\\
&\leqq \|e^{-(t-s)A}(1+A_D)^{-1/4}u(s)\|^2+\|e^{-(t-s)A}(1+A_N)^{-1/4}B(s)\|^2\\
&C\int_s^t(\|\nabla u(\tau)\|^2+\|\text{rot } B(\tau)\|^2)\,d\tau+C\int_s^t\|f(\tau)\|\,d\tau \qquad (2.1)
\end{aligned}$$

holds for all $0\leqq s<t$.
In fact, we have

$$\lim_{t\to\infty}\|e^{-(t-s)A_D}(1+A_D)^{-1/4}u(s)\|^2=0,$$

$$\lim_{t\to\infty}\|e^{-(t-s)A_N}(1+A_N)^{-1/4}B(s)\|^2=0$$

for any $s\geqq 0$, since zero is not an eigenvalue of the non-negative self-adjoint operator A_D or A_N. Therefore letting t tend to infinity in (2.1), we get

$$\begin{aligned}
&\limsup_{t\to\infty}\|(1+A_D)^{-1/4}u(t)\|^2+\limsup_{t\to\infty}\|(1+A_N)^{-1/4}B(t)\|^2\\
&\leqq C\int_s^\infty(\|\nabla u(\tau)\|^2+\|\text{rot } B(\tau)\|^2)\,d\tau+C\int_s^\infty\|f(\tau)\|\,d\tau.
\end{aligned}$$

It follows from the assumptions of this proposition that the right hand side of the above inequality tends to zero as $s\to\infty$. Hence we obtain the desired result.

Now we shall prove (2.1).

Suppose that ρ is a C^∞ function in $\boldsymbol{R}^1$ with support in $|t|\leqq 1$ such that

$\rho(t) \geqq 0$, $\rho(t)=\rho(-t)$ and $\int_{-\infty}^{\infty} \rho(t)\, dt=1$. For $\varepsilon>0$ and $h>0$, we choose the test functions $\Phi=\Phi_{\varepsilon, h}$ and $\Psi=\Psi_{\varepsilon, h}$ in (1.3) and (1.4) as follows;

$$\Phi_{\varepsilon, h}(\tau)=U_{\varepsilon}(\tau) \int_{s}^{t} \rho_{h}(\tau-\sigma) U_{\varepsilon}(\sigma) u(\sigma)\, d\sigma,$$

$$\Psi_{\varepsilon, h}(\tau)=V_{\varepsilon}(\tau) \int_{s}^{t} \rho_{h}(\tau-\sigma) V_{\varepsilon}(\sigma) B(\sigma)\, d\sigma, \ (s \leqq \tau \leqq t)$$

where $\rho_{h}(\tau)=(1/h) \rho_{h}(\tau/h)$, $U_{\varepsilon}(\tau)=e^{-(t-s+\varepsilon) A_{D}}(1+A_{D})^{-1/4}$ and $V_{\varepsilon}(\tau)=e^{-(t-s+\varepsilon) A_{N}}(1+A_{N})^{-1/4}$. Then we have the followings:

(i) $\Phi_{\varepsilon, h} \in C^{1}([s, t]; V_{1}) \cap C([s, t]; D(A_{D}))$,
$\Psi_{\varepsilon, h} \in C^{1}([s, t]; V_{2}) \cap C([s, t]; D(A_{N}))$

and the equalities

$$\partial_{\tau} \Phi_{\varepsilon, h}(\tau)=A_{D} \Phi_{\varepsilon, h}(\tau)+U_{\varepsilon}(\tau) \int_{s}^{t} \partial_{\tau} \rho(\tau-\sigma) U_{\varepsilon}(\sigma) u(\sigma)\, d\sigma,$$

$$\partial_{\tau} \Psi_{\varepsilon, h}(\tau)=A_{N} \Psi_{\varepsilon, h}(\tau)+V_{\varepsilon}(\tau) \int_{s}^{t} \partial_{\tau} \rho(\tau-\sigma) V_{\varepsilon}(\sigma) B(\sigma)\, d\sigma.$$

hold.

(ii) There is a positive constant M_{1} such that the inequalities

$$\sup_{\tau>0} \|\Phi_{\varepsilon, h}(\tau)\| \leqq M_{1}, \ \sup_{\tau>0} \|\Psi_{\varepsilon, h}(\tau)\| \leqq M_{1},$$

$$\sup_{\tau>0} \|\Phi_{\varepsilon, h}(\tau)\|_{L^{3}(\Omega)} \leqq M_{1}, \ \sup_{\tau>0} \|\Phi_{\varepsilon, h}(\tau)\|_{L^{6}(\Omega)} \leqq M_{1},$$

$$\sup_{\tau>0} \|A_{N}^{1/2} \Psi_{\varepsilon, h}(\tau)\| \leqq M_{1}$$

hold for all $\varepsilon>0$ and all $h>0$.

(iii) There is a positive constant M_{2} such that the inequalities

$$\lim_{h \to 0} \sup \int_{s}^{t} \|\Phi_{\varepsilon, h}(\tau)\|_{L^{6}(\Omega)}^{2}\, d\tau \leqq M_{2} \int_{s}^{t} \|\nabla u(\tau)\|^{2}\, d\tau,$$

$$\lim_{h \to 0} \sup \int_{s}^{t} \|A_{N}^{1/2} \Psi_{\varepsilon, h}(\tau)\|^{2}\, d\tau \leqq M_{2} \int_{s}^{t} \|\operatorname{rot} B(\tau)\|^{2}\, d\tau$$

hold for all $\varepsilon>0$.

In fact, (i) can be seen casily. Since A_{D} and A_{N} are non-negative self-adjoint operators in H, we have $\|U_{\varepsilon}(\tau)\|_{B(H)} \leqq 1$ and $\|V_{\varepsilon}(\tau)\|_{B(H)} \leqq 1$ for all $\varepsilon>0$ and all $\tau \geqq 0$. Hence it follows

$$\sup_{\tau>0} \|\Phi_{\varepsilon, h}(\tau)\| \leqq \sup_{\tau>0} \|u(\tau)\| \text{ and } \sup_{\tau>0} \|\Psi_{\varepsilon, h}(\tau)\| \leqq \sup_{\tau>0} \|B(\tau)\|$$

for all $\varepsilon>0$ and all $h>0$.

Similarly, since $\|A_N^{1/2}(1+A_N)^{-1/2}\|_{B(H)}\leqq 1$, we have

$$\sup_{\tau>0}\|A_N^{1/2}\Psi_{\varepsilon,h}(\tau)\|\leqq\sup_{\tau>0}\|B(\tau)\| \text{ for all } \varepsilon>0 \text{ and all } h>0.$$

Moreover, by the Sobolev's imbedding theorem, we have $(1+A_D)^{-1/4}\phi\in L^3(\Omega)$ and $(1+A_N)^{-1/2}\phi\in L^6(\Omega)$ for all $\phi\in H$. It follows from the closed graph theorem that $(1+A_D)^{-1/4}\in B(H, L^3(\Omega))$ with bound C_1 and that $(1+A_D)^{-1/2}\in B(H, L^6(\Omega))$ with bound C_2. ($B(X, Y)$: the set of all bounded linear operators from X to Y) Then we have

$$\sup_{\tau>0}\|\Phi_{\varepsilon,h}(\tau)\|_{L^3(\Omega)}\leqq C_1^2\sup_{\tau>0}\|u(\tau)\| \text{ and}$$

$$\sup_{\tau>0}\|\Phi_{\varepsilon,h}(\tau)\|_{L^6(\Omega)}\leqq C_2\sup_{\tau>0}\|u(\tau)\|$$

for all $\varepsilon>0$ and $h>0$. Hence (ii) follows.

By the Sobolev inequality and (1.5), there is a constant C_3 independent of ε or h such that

$$\|\Phi_{\varepsilon,h}(\tau)\|^2_{L^6(\Omega)}\leqq C_3\int_s^t\rho_h(\tau-\sigma)\|A_D^{1/2}u(\sigma)\|^2\,d\sigma. \tag{2.2}$$

Integrating both sides of (2.2) in τ from s to t and then taking $h\to 0$, we have

$$\begin{aligned}&\lim_{h\to 0}\sup\int_s^t\|\Phi_{\varepsilon,h}(\tau)\|^2_{L^6(\Omega)}\,d\tau\\ &\leqq C_3\int_s^t\|A_D^{1/2}u(\tau)\|^2\,d\tau=C_3\int_s^t\|\nabla u(\tau)\|^2\,d\tau,\end{aligned}$$

since $\|\nabla u(\cdot)\|\in L^2(0,\infty)$. Similarly by (1.6), we have

$$\lim_{h\to 0}\sup\int_s^t\|A_N^{1/2}\Psi_{\varepsilon,h}(\tau)\|^2\,d\tau\leqq\int_s^t\|\mathrm{rot}\,B(\tau)\|^2\,d\tau.$$

Hence (iii) follows.

Now substituting $\Phi=\Phi_{\varepsilon,h}$ in (1.3) and $\Psi=\Psi_{\varepsilon,h}$ in (1.4) and then adding these equations, we have

$$\begin{aligned}&\int_s^t\{((u,\nabla)u+B\times\mathrm{rot}\,B,\Phi_{\varepsilon,h})+((u,\nabla)B-(B,\nabla)u,\Psi_{\varepsilon,h})\}\,d\tau\\ &=-(u(t),\Phi_{\varepsilon,h}(t))-(B(t),\Psi_{\varepsilon,h}(t))+(u(s),\Phi_{\varepsilon,h}(s))\\ &+(B(s),\Psi_{\varepsilon,h}(s))+\int_s^t(f,\Phi_{\varepsilon,h})\,d\tau,\end{aligned} \tag{2.3}$$

since the following identities hold:

$$\int_s^t \{-(u(\tau),\ \partial_\tau \Phi_{\varepsilon,h}(\tau))+(\nabla u(\tau),\ \nabla \Phi_{\varepsilon,h}(\tau))\}\ d\tau$$
$$=\int_s^t \{-(u(\tau),\ A_D \Phi_{\varepsilon,h}(\tau)+U_\varepsilon(\tau)\int_s^t \partial_\tau \rho_h(\tau-\sigma)U_\varepsilon(\sigma)u(\sigma)\ d\sigma)$$
$$+(u(\tau),\ A_D \Phi_{\varepsilon,h}(\tau))\}\ d\tau \qquad \text{(by (i))}$$
$$=-\int_s^t\int_s^t \partial_\tau \rho_h(\tau-\sigma)(U_\varepsilon(\tau)u(\tau),\ U_\varepsilon(\sigma)u(\sigma))\ d\sigma d\tau$$
$$=0 \text{ (by the symmetry of } \rho),$$
$$\int_s^t \{-(B(\tau),\ \partial_\tau \Psi_{\varepsilon,h}(\tau))+(\operatorname{rot} B(\tau),\ \operatorname{rot} \Psi_{\varepsilon,h}(\tau))\}\ d\tau$$
$$=\int_s^t \{-(B(\tau),\ A_N \Psi_{\varepsilon,h}(\tau)+V_\varepsilon(\tau)\int_s^t \partial_\tau \rho(\tau-\sigma)V_\varepsilon(\sigma)B(\sigma)\ d\sigma)$$
$$+(B(\tau),\ A_N \Psi_{\varepsilon,h}(\tau))\}\ d\tau$$
$$=-\int_s^t\int_s^t \partial_\tau \rho_h(\tau-\sigma)(V_\varepsilon(\tau)B(\tau),\ V_\varepsilon(\sigma)B(\sigma))\ d\sigma d\tau=0.$$

By the Hölder inequality, the Gagliardo-Nirenberg inequality (see, e. g., Tanabe [12, Capter 1 Lemma 1. 2. 1]) and Duvaut-Lions [2, Capter 7 Theorem 6. 1], we have

$$\left|\int_s^t ((u,\ \nabla)u+B\times \operatorname{rot} B,\ \Phi_{\varepsilon,h})\ d\tau\right|$$
$$\leqq \int_s^t (\|u\|_{L^6(\Omega)}\|\nabla u\|\ \|\Phi_{\varepsilon,h}\|_{L^3(\Omega)}+\|B\|_{L^3(\Omega)}\|\operatorname{rot} B\|\ \|\Phi_{\varepsilon,h}\|_{L^6(\Omega)})\ d\tau$$
$$\leqq \sup_{\tau>0}\|\Phi_{\varepsilon,h}(\tau)\|_{L^3(\Omega)}\int_s^t \|u\|_{L^6(\Omega)}\|\nabla u\|\ d\tau$$
$$+C\int_s^t \|B\|_{H^1(\Omega)}^{1/2}\ \|B\|^{1/2}\|\operatorname{rot} B\|\ \|\Phi_{\varepsilon,h}\|_{L^6(\Omega)}\ d\tau$$
$$\leqq C\sup_{\tau>0}\|\Phi_{\varepsilon,h}(\tau)\|_{L^3(\Omega)}\int_s^t \|\nabla u\|^2\ d\tau+C\sup_{\tau>0}\|B(\tau)\|\int_s^t \|\Phi_{\varepsilon,h}\|_{L^6(\Omega)}^2\ d\tau$$
$$+C(\sup_{\tau>0}\|\Phi_{\varepsilon,h}(\tau)\|_{L^6(\Omega)}+\sup_{\tau>0}\|B(\tau)\|)\int_s^t \|\operatorname{rot} B\|^2\ d\tau$$
$$\leqq 2CM_1\int_s^t (\|\nabla u\|^2+\|\operatorname{rot} B\|^2+\|\Phi_{\varepsilon,h}\|_{L^6(\Omega)}^2)\ d\tau \qquad \text{(by (ii))}$$

and

$$\left|\int_s^t ((u,\ \nabla)B-(B,\ \nabla)u,\ \Psi_{\varepsilon,h})\ d\tau\right|$$
$$=\left|\int_s^t (\operatorname{rot}\ (B\times u),\ \Psi_{\varepsilon,h})\ d\tau\right|=\left|\int_s^t (B\times u,\ \operatorname{rot} \Psi_{\varepsilon,h})\ d\tau\right|$$
$$\leqq \int_s^t \|B\|_{L^3(\Omega)}\|u\|_{L^6(\Omega)}\|\operatorname{rot} \Psi_{\varepsilon,h}\|\ d\tau$$
$$\leqq C\int_s^t \|B\|^{1/2}\|B\|_{H^1(\Omega)}^{1/2}\ \|\nabla u\|\ \|\operatorname{rot} \Psi_{\varepsilon,h}\|\ d\tau$$

$$\leqq C\int_s^t (\|\mathrm{rot}\ B\|+\|B\|)\|\nabla u\|\ \|\mathrm{rot}\ \Psi_{\varepsilon,h}\|\ d\tau$$

$$\leqq C\sup_{\tau>0}\|\mathrm{rot}\ \Psi_{\varepsilon,h}(\tau)\|\int_s^t (\|\mathrm{rot}\ B\|^2+\|\nabla u\|^2)\ d\tau$$

$$+C\sup_{\tau>0}\|B(\tau)\|\int_s^t (\|\nabla u\|^2+\|\mathrm{rot}\ \Psi_{\varepsilon,h}\|^2)\ d\tau$$

$$\leqq 2CM_1\int_s^t (\|\nabla u\|^2+\|\mathrm{rot}\ B\|^2+\|\mathrm{rot}\ \Psi_{\varepsilon,h}\|^2)\ d\tau, \qquad \text{(by (ii))}$$

where C is a positive constant independent of ε or h. Taking $h\to 0$ in the above inequalities, we have by (iii)

$$\limsup_{h\to 0}\left|\int_s^t ((u,\nabla)u+B\times\mathrm{rot}\ B,\ \Phi_{\varepsilon,h})\ d\tau\right| \leqq 4CM_1\int_s^t (\|\nabla u\|^2+\|\mathrm{rot}\ B\|^2)\ d\tau \tag{2.4}$$

and

$$\limsup_{h\to 0}\left|\int_s^t ((u,\nabla)B-(B,\nabla)u,\ \Psi_{\varepsilon,h})\ d\tau\right| \leqq 4CM_1\int_s^t (\|\nabla u\|^2+\|\mathrm{rot}\ B\|^2)\ d\tau \tag{2.5}$$

for all $\varepsilon>0$. Clearly the inequality

$$\left|\int_s^t (f,\Phi_{\varepsilon,h})\ d\tau\right|\leqq M_1\int_s^t \|f\|\ d\tau \tag{2.6}$$

holds for all $\varepsilon>0$ and all $h>0$.

Moreover, since $U_\varepsilon(\sigma)u(\sigma)$ and $V_\varepsilon(\sigma)B(\sigma)$ are continuous in $\sigma\in[s,t]$ in the weak topology of H, it follows that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\lim_{h\to 0}(u(t),\ \Phi_{\varepsilon,h}(t))\\ &=\lim_{\varepsilon\to 0}\lim_{h\to 0}\int_s^t \rho_h(\tau-\sigma)(U_\varepsilon(t)u(t),\ U_\varepsilon(\sigma)u(\sigma))\ d\sigma\\ &=\lim_{\varepsilon\to 0}\{(1/2)\|U_\varepsilon(t)u(t)\|^2\}\\ &=(1/2)\|(1+A_D)^{-1/4}u(t)\|^2\end{aligned} \tag{2.7}$$

and that

$$\lim_{\varepsilon\to 0}\lim_{h\to 0}(B(t),\ \Psi_{\varepsilon,h}(t))=(1/2)\|(1+A_N)^{-1/4}B(t)\|^2. \tag{2.8}$$

Similarly we have

$$\lim_{\varepsilon\to 0}\lim_{h\to 0}\ (u(s),\ \Phi_{\varepsilon,h}(s))=(1/2)\,\|e^{-(t-s)A_D}(1+A_D)^{-1/4}u(t)\|^2 \tag{2.9}$$

and

$$\lim_{\varepsilon\to 0}\lim_{h\to 0}\ (B(s),\ \Psi_{\varepsilon,h}(s))=(1/2)\,\|e^{-(t-s)A_N}(1+A_N)^{-1/4}B(s)\|^2. \tag{2.10}$$

Now letting $h\to 0$ and then $\varepsilon\to 0$ in (2. 3), we get the desired estimate (2. 1) by (2. 4)-(2. 10). This completes the proof of Proposition 1.

3. Proof of Proposition 2

In the case of the Navier-Stokes equations, Proposition 2 is due to Sohr [10]. Since our argument is parallel to that of [10], we shall give an outline of the proof. The L_r-estimates of the solution for linearlized equations of (M. H. D.) play an important role.

We approximate (M. H. D.) by the following initial-boundary value problem $(\text{A. P.})_k$, k being an arbitrary positive integer;

$$(\text{A. P.})_k\quad \begin{cases} \partial_t u-\Delta u+(J_k u,\ \nabla)u-(L_k B,\ \nabla)B+\nabla\pi^1=f \text{ in } Q,\\ \partial_t B-\Delta B+(J_k u,\ \nabla)B-(L_k B,\ \nabla)u+\nabla\pi^2=0 \text{ in } Q,\\ \operatorname{div} u=0,\ \operatorname{div} B=0 \qquad \text{in } Q,\\ u=0,\ B\cdot\nu=0,\ \operatorname{rot} B\times\nu=0,\ \text{on } \partial\Omega\times(0,\infty),\\ u|_{t=0}=J_k u_0,\ B|_{t=t}=L_k B_0, \end{cases}$$

where $J_k=(1+(1/k)\tilde{A}_D)^{-1}$ and $L_k=(1+(1/k)\tilde{A}_N)^{-1}$.
Note that $\tilde{A}_D=1+A_D$ and $\tilde{A}_N=1+A_N$.

Since $H^2(\Omega)\subset L^\infty(\Omega)$, $J_k u$ and $L_k B$ are in $L^\infty(\Omega)$ for all u and B in H; so there exists for each k a solution $\{u, B, \pi^1, \pi^2\}=\{u_k, B_k, \pi_k^1, \pi_k^2\}$ of $(\text{A. P.})_k$ satisfying the following properties:

(i) For any $T>0$,

$u\in L^2(0, T;\ D(A_D))\cap L^\infty(0,\infty;H)$, $u'\in L^2(0, T;H)$,
$\nabla\pi^1\in L^2(0, \mathrm{T};L^2(\Omega))$,
$B\in L^2(0, T;D(A_N))\cap L^\infty(0,\infty;H)$, $B'\in L^2(0, T;H)$,
$\nabla\pi^2\in L^2(0, T;L^2(\Omega))$;

(ii)

$$u'-\Delta u+(J_k u,\ \nabla)u-(L_k B,\ \nabla)B+\nabla\pi^1=f \text{ in } L^2(\Omega), \tag{3.1}$$
$$B'-\Delta B+(J_k u,\ \nabla)B-(L_k B,\ \nabla)u+\nabla\pi^2=0 \text{ in } L^2(\Omega) \tag{3.2}$$

for almost all $t\geqq 0$ with

$u(0)=J_k u_0,\ B(0)=L_k B_0$;

(iii) The inequality

$$\sup_{t\geq 0}\|u_k(t)\|^2+\sup_{t\geq 0}\|B_k(t)\|^2+\int_0^\infty(\|\nabla u_k(\tau)\|^2+\|\mathrm{rot}\ B_k(\tau)\|^2)\ d\tau$$
$$\leq\|a\|^2+2\|f\|_{L^1(0,\infty;\mathbf{L}^2)}(\|a\|^2+\|f\|^2_{L^1(0,\infty;\mathbf{L}^2)})\exp\|f\|_{L^1(0,\infty;\mathbf{L}^2)} \tag{3.3}$$

holds for all k.

By (3.3) there exist a subsequence of $\{u_k, B_k\}$, which we denote by $\{u_k, B_k\}$ for simplicity, and functions $u\in L^\infty(0,\infty;H)\cap L^2_{loc}(0,\infty;V_1)$ and $B\in L^\infty(0,\infty;H)\cap L^2_{loc}(0,\infty;V_2)$ such that

$$\begin{array}{lll} u_k\to u & \text{in } L^\infty(0,\infty;H) & \text{weakly-star,}\\ & \text{in } L^2(0,T;V_1) & \text{weakly,}\\ B_k\to B & \text{in } L^\infty(0,\infty;H) & \text{weakly-star,}\\ & \text{in } L^2(0,T;V_2) & \text{weakly,}\end{array} \tag{3.4}$$

for all $T>0$.

Moreover, we can choose a subsequence of $\{u_k, B_k\}$, which we denote by $\{u_k, B_k\}$ for simplicity, such that

$$\begin{array}{l} u_k\to u \text{ in } L^2(0,T;L^2(K)) \text{ strongly,}\\ B_k\to B \text{ in } L^2(0,T;L^2(K)) \text{ strongly}\end{array} \tag{3.5}$$

for all $T>0$ and all compact set K contained in Ω.

Indeed, for a complete orthonormal system $\{\phi_j\}_{j=1}^\infty$ ($\phi_j\in C^\infty_{0,\sigma}(\Omega)$) in H, we see that for each fixed j the families $\{(u_k(t),\phi_j)\}_{k=1}^\infty$ and $\{(B_k(t),\phi_j)\}_{k=1}^\infty$ respectively form uniformly bounded and equicontinuous ones of continuous functions on $[0,T]$ (see, e. g., Ladyzehenskaya [3, p. 175]). Hence by the Ascoli-Arzerâ theorem and the usual diagonal argument, there exist subsequences $u_{k_i}(t)$ and $B_{k_i}(t)$ of $u_k(t)$ and $B_k(t)$ which converge respectively to some $\bar{u}(t)$ and $\bar{B}(t)$ uniformly in $t\in[0,T]$ in the weak topology of H. For simplicity, we shall assume that the original sequences u_k and B_k converge respectively to u and B. Hence using the techniques of the Friedrichs inequality (Courant-Hilbert [1, p. 519]) and Duvaut-Lions [2, Chapter 7 Theorem 6.1], we have (3.5) by (3.3). See, e. g., Ladyzehenskaya [3, p. 176].

Now by (3.4) and (3.5), it is easy to see that $\{u, B\}$ is a weak solution of (M. H. D.).

To show that this $\{u, B\}$ is the desired solution, we need the following lemma.

LEMMA 3.1

For any $T>0$, we have the followings:

(i) *The sequence u_k and the sequence B_k remain in a bounded set of $L^{r_0}(0, T; W^{2,r_0}(\Omega))$. The sequence u'_k and the sequence B'_k remain in a bounded set of $L^{r_0}(0, T; L^{r_0}(\Omega))$.* $(r_0=5/4)$

(ii) *The sequence $\nabla\pi^1_k$ and the sequence $\nabla\pi^2_k$ remain in a bounded set of $L^{r_0}(0, T; L^{r_0}(\Omega))$.*

(iii) *The sequence π^1_k and the sequence π^2_k remain in a bounded set of $L^{r_0}(0, T; L^{3r_0/(3-r_0)}(\Omega))$.*

PROOF.

We can rewrite the equations (3.1) and (3.2) respectively as

$$u'_k-\Delta u_k+\nabla\pi^1_k=F^1_k \text{ in } L^2(\Omega), \tag{3.6}$$
$$B'_k-\Delta B_k+\nabla\pi^2_k=F^2_k \text{ in } L^2(\Omega), \tag{3.7}$$

where $F^1_k=f-(J_k u_k, \nabla)u_k+(L_k B_k, \nabla)B_k$ and $F^2_k=-(J_k u_k, \nabla)B_k+(L_k B_k, \nabla)u_k$. Then we have

$$\text{the sequence } F^1_k \text{ and the sequence } F^2_k \text{ are bounded in } L^{r_0}(0, T; L^{r_0}(\Omega)). \tag{3.8}$$

In fact, by the Hölder inequality, the Gagliardo-Nirenberg inequality (Tanabe [12, Chapter 1 Lemma 1.2.1]) and Duvaut-Lions [2, Chaptrer 7 Theorem 6.1], we get

$$\begin{aligned}
\|J_k u_k, \nabla)u_k\|_{5/4} &\leqq \|J_k u_k\|_{10/3}\|\nabla u_k\| \\
&\leqq C\|\nabla J_k u_k\|^{3(1/2-3/10)}\|J_k u_k\|^{1-3(1/2-3/10)}\|\nabla u_k\| \\
&\leqq C\|u_k\|^{2/5}\|\nabla u_k\|^{8/5}\leqq C\|u_k\|^{2/5}_{2,\infty}\ \|\nabla u_k\|^{8/5}, \\
\|(J_k u_k, \nabla\|_{5/4,5/4} &\leqq C\|u_k\|^{2/5}_{2,\infty}\ \left(\int_0^T\|\nabla u_k(t)\|^2\ dt\right)^{4/5} \\
&\leqq C\|u_k\|^{2/5}_{2,\infty}\ \|\nabla u_k\|^{8/5}_{2,2}, \\
\|(L_k B_k, \nabla)u_k\|_{5/4} &\leqq \|L_k B_k\|_{10/3}\|\nabla B_k\| \\
&\leqq C\|L_k B_k\|^{3(1/2-3/10)}_{H^1(\Omega)}\ \|L_k B_k\|^{1-3(1/2-3/10)}\|\nabla B_k\| \\
&\leqq C\|B_k\|^{2/5}_{2,\infty}\ (\|\mathrm{rot}\ B_k\|+\|B_k\|)^{8/5} \\
&\leqq C(\|B_k\|^2_{2,\infty}+\|B_k\|^{2/5}_{2,\infty}\|\mathrm{rot}\ B_k\|^{8/5}), \\
\|(L_k B_k, \nabla)B_k\|_{5/4,5/4} &\leqq C(\|B_k\|^2_{2,\infty}+\|B_k\|^{2/5}_{2,\infty}\|\mathrm{rot}\ B_k\|^{8/5}_{2,2}), \\
\|(J_k u_k, \nabla)B_k\|_{5/4} &\leqq \|J_k u_k\|_{10/3}\|\nabla B_k\| \\
&\leqq C\|\nabla J_k u_k\|^{3(1/2-3/10)}\|J_k u_k\|^{1-3(1/2-3/60)}\|\nabla B_k\| \\
&\leqq C\|u_k\|^{2/5}_{2,\infty}\|\nabla u_k\|^{3/5}\|\nabla B_k\|
\end{aligned}$$

$$\begin{aligned}
&\leqq C\|u_k\|_{2,\infty}^{2/5}\|\nabla u_k\|^{3/5}(\|\mathrm{rot}\ B_k\|+\|B_k\|)\\
&\leqq C(\|u_k\|_{2,\infty}^{2/5}\|\nabla u_k\|^{3/5}\|\mathrm{rot}\ B_k\|+\|u_k\|_{2,\infty}^{2/5}\|B_k\|_{2,\infty}\|\nabla u_k\|^{3/5}),
\end{aligned}$$

$$\begin{aligned}
&\|(J_k u_k,\ \nabla)B_k\|_{5/4,5/4}\\
&\leqq C\{\|u_k\|_{2,\infty}^{2/5}(\int_0^T\|\nabla u_k\|^{3/4}\|\mathrm{rot}\ B_k\|^{5/4}\ dt)^{4/5}\\
&+\|u_k\|_{2,\infty}^{2/5}\|B_k\|_{2,\infty}(\int_0^T\|\nabla u_k\|^{3/4}\ dt)^{4/5}\}\\
&\leqq C(\|u_k\|_{2,\infty}^{2/5}\|\nabla u_k\|_{2,2}^{3/5}\|\mathrm{rot}\ B_k\|_{2,2}+\\
&\qquad\|u_k\|_{2,\infty}^{2/5}\|B_k\|_{2,\infty}\|\nabla u_k\|_{2,2}^{3/5}),
\end{aligned}$$

$$\begin{aligned}
&\|(L_kB_k,\ \nabla)u_k\|\leqq\|L_kB_k\|_{10/3}\|\nabla u_k\|\\
&\leqq C\|L_kB_k\|_{H^1(\Omega)}^{3(1/2-3/10)}\ \|L_kB_k\|^{1-3(1/2-3/10)}\|\nabla u_k\|\\
&\leqq C\|B_k\|_{2,\infty}^{2/5}(\|\mathrm{rot}\ B_k\|+\|B_k\|)^{3/5}\|\nabla u_k\|,
\end{aligned}$$

$$\begin{aligned}
&\|(L_kB_k,\ \nabla)u_k\|_{4/5,4/5}\\
&\leqq C\|B_k\|_{2,\infty}^{2/5}\ \{\int_0^T(\|\mathrm{rot}\ B_k\|+\|B_k\|)^{3/4}\|\nabla u_k\|^{5/4}\ dt\}^{4/5}\\
&\leqq C\|B_k\|_{2,\infty}^{2/5}\|\nabla u_k\|_{2,2}(\|\mathrm{rot}\ B_k\|_{2,2}^2+\|B_k\|_{2,\infty}^2)^{3/10},
\end{aligned}$$

where C is a positive constant independent of k. Hence we obtain (3.8) by (3.3).

On the other hand, since $u_0\in D(\tilde{A}_{D(r_0)}^{1-1/r_0+\varepsilon})$ and since $B_0\in D(\tilde{A}_{N(r_0)}^{1-1/r_0+\varepsilon})$, (i) and (ii) follows from Solonnikov [1], p. 489 Corollary 2] and Ladyzehenskaya-Solonnikov-Ural'ceva [4, Theorem 10.4]. Now we shall show (iii). We set

$$g_k^1=-u_k'+\Delta u_k+F_k^1 \text{ and}$$
$$g_k^2=-B_k'+\Delta B_k+F_k^2.$$

Then for all $\phi\in C_0^\infty(\Omega\times[0,T])$ with div $\phi=0$, we have

$$\int_0^T\int_\Omega g_k^i(x,t)\cdot\phi(x,t)\ dxdt=0\ \ (i=1,2) \tag{3.9}$$

and the inequalities

$$\begin{aligned}
&|\int_0^T\int_\Omega g_k^i(x,t)\cdot\phi(x,t)\ dxdt|\\
&\leqq\int_0^T\|g_k^i(t)\|_{r_0}\|\phi(t)\|_{r_0^*}dt\\
&\leqq C\int_0^T\|g_k^i(t)\|_{r_0}\|\nabla\phi(t)\|_{3r_0/(4r_0-3)}\ dt\ \ \text{(by the Sobolev inequality)}\\
&\leqq C\|g_k^i\|_{r_0,r_0}\|\nabla\phi\|_{\{3r_0/(3-r_0)\}^*,\ r_0^*},
\end{aligned}$$

where C is a constant independent of k.
(For $r>1$, $r^*=r/(r-1)$.)
Hence we have by (i) that

the sequences $\{g_k^i\}_{k=1}^{\infty}$ $(i=1,2)$ are bounded in

$$L^{r_0}(0, T\,;\, W^{1,3r_0/(3-r_0)}(\Omega)^*).$$

(X^* denotes the dual space of X.)
Since ∇ is a bounded operator from $L^{r_0}(0, T\,;L^{3r_0/(3-r_0)}(\Omega))$ into $L^{r_0}(0, T\,;$ $W_0^{1,3r_0/(4r_0-3)}(\Omega)^*)$, it follows from (3.9) that there exist sequences $\{\bar{\pi}_k^i\}_{k=1}^{\infty}(i=1,2)$ bounded in $L^{r_0}(0, T\,;L^{3r_0/(3-r_0)}(\Omega))$ such that $g_k^i=\nabla\bar{\pi}_k^i$. Since we may assume that $\pi_k^i=\bar{\pi}_k^i$, we have (iii).

Let $\eta_m=\eta_m(x)$ $(m=1,2,\ldots)$ be a sequence of C^∞-functions in $\boldsymbol{R}^3$ such that $0\leqq\eta_m\leqq 1$, $|\nabla\eta_m|\leqq C$, C being a constant independent of m, and that $\eta_m(x)\to 1$, $\nabla\eta_m(x)\to 0$ for each $x\in\boldsymbol{R}^3$. Now take the inner products with $\eta_m u_k$ in (3.1) and $\eta_m B_k$ in (3.2) respectively, add the resulting equalities and then integrate in τ over $[s, t]$. Then, after integration by parts we get

$$\begin{aligned}&\int_\Omega \eta_m(|u_k(t)|^2+|B_k(t)|^2)\,dx+2\int_s^t\int_\Omega \eta_m(|\nabla u_k|^2+|\mathrm{rot}\ B_k|^2)\,dxd\tau\\ &=\int_\Omega \eta_m(|u_k(s)|^2+|B_k(s)|^2)\,dx+2\int_s^t\int_\Omega (f, \eta_m u_k)\,dxd\tau\\ &+2\int_s^t\int_\Omega \nabla\eta_m\ \textstyle\sum_{i=1}^3 R_k^i\ dxd\tau, \qquad (3.10)\end{aligned}$$

where $R_k^1=(1/2)\nabla|u_k|^2+B_k\times\mathrm{rot}\ B_k$,

$R_k^2=(u_k\cdot B_k)L_kB_k-(1/2)(|u_k|^2+|B_k|^2)J_ku_k$ and $R_k^3=\pi_k^1u_k+\pi_k^2B_k$.

Then it follows from (3.3) and Lemma 3.1 that $\{R_k^1\}_{k=1}^{\infty}$, $\{R_k^2\}_{k=1}^{\infty}$ and $\{R_k^3\}_{k=1}^{\infty}$ remain in a bounded set of $L^1(Q_T)^3\cap L^{5/4}(Q_T)^3$, $L^1(Q_T)^3\cap L^{10/9}$ $(Q_T)^3$ and $L^1(Q_T)^3\cap L^{29/30}(Q_T)^3$ respectively ($Q_T=\Omega\times(0, T)$). We set $q_1=5/4$, $q_2=10/9$ and $q_3=30/29$. For each $i=1, 2, 3$, there exist a subsequence of R_k^i, which we denote by R_k^i for simplicity, and functions $R^i\in L^1(Q_T)^3\cap L^{q_i}$ $(Q_T)^3$ such that

$$\lim_{k\to\infty}\int_s^t\int_\Omega \nabla\eta_m\ R_k^i\ dxd\tau=\int_s^t\int_\Omega \nabla\eta_m\ R^i\ dxd\tau\quad (i=1,2,3) \qquad (3.11)$$

hold for all m. Moreover, since $|\nabla\eta_m(x)R^i(x,\tau)|\leqq C|R^i(x,\tau)|$ $(i=1,2,3)$ for all $(x,\tau)\in Q_T$ and all m and since $|\nabla\eta_m(x)R^i(x,\tau)|\to 0$ $(i=1,2,3)$ for each $(x,\tau)\in Q_T$ as $m\to\infty$, it follows from the Lebesgue's dominated convergence theorem that the right hand side of (3.11) converges to zero as $m\to\infty$. Hence taking lim. inf in k and then letting $m\to\infty$ in (3.10), we see by (3.4) and (3.5) that $\{u, B\}$ satisfies (E. I. S.) for almost all $s\geqq 0$, including $s=0$, and all $t>s$. This completes the proof.

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Added in proof:

After this paper had been submitted, Professor Dr. Wolf von Wahl kindly informed the author the result of H. Sohr, W. von Wahl and M. Wiegner: Zur Asymptotik der Gleichungen von Navier–Stokes, Nachr. Akad. Wissenschaften Göttingen II. Math. Physikalische Klasse Jahrgang 1986, Nr. 3, 45–59. Then he pointed out that Assumption 1 might be weakened as follows:

$$u_0 \text{ and } B_0 \text{ are in } H \cap H_{r_0}, \text{ respectively.}$$

Under this assumption, we can obtain the same result of Theorem in the similar manner as the above paper.

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