Hokkaido Mathematical Journal Vol. 16 (1987), p. 135～144

GENERATING SUBSEMIGROUPS, ORDERS, AND A THEOREM OF GLICKSBERG

Shozo KOSHI* and Yuji TAKAHASHI*

(Received February 7, 1986, Revised September 12, 1986)

§ 1. Introduction

Let G be a locally compact Abelian group with dual group $\hat{G}$ and let m_G denote the Haar measure on G. Let $L^1(G)$ and $M(G)$ be the group algebra on G and the measure algebra on G, respectively. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures whose Fourier-Stieltjes transforms vanish off E. For μ in $M(G)$, μ_s denotes the singular part of μ with respect to m_G. For a closed subgroup H of G, $H^{\perp}$ means the annihilator of H. The symbols $\boldsymbol{Z}$, $\boldsymbol{Z}_+$, $\boldsymbol{R}$, $\boldsymbol{R}_+$, and $\boldsymbol{T}$ will denote the integers, the nonnegative integers, the real numbers, the nonnegative real numbers, and the circle group, respectively.

For a locally compact Abelian group with ordered dual group, a relation between the F. and M. Riesz theorem and the group structures was investigated in [3] and [5]: if G is a locally compact Abelian group with ordered dual group $\hat{G}$ and if P is a nondense order in $\hat{G}$ (i.e., P is a nondense subsemigroup of $\hat{G}$ such that $P\cup(-P)=\hat{G}$ and $P\cap(-P)=\{0\}$), then $M_P(G)\subset L^1(G)$ if and only if G is isomorphic to $\boldsymbol{R}\times\Delta$ or $\boldsymbol{T}\times\Delta$, where Δ is discrete. Moreover, there exists a nonzero measure $\mu\in M_P(G)$ which is singular with respect to m_G unless $G=\boldsymbol{R}\times\Delta$ or $\boldsymbol{T}\times\Delta$ for a discrete group Δ.

In his recent paper, I. Glicksberg showed that the above result holds in a more general setting:

THEOREM 1 ([2, Theorem 1]). *Let G be a locally compact Abelian group with dual group $\hat{G}$ and let S be a proper closed generating subsemigroup of $\hat{G}$ (i.e., S is a proper closed subsemigroup of $\hat{G}$ such that $S-S$ is dense in $\hat{G}$). Then*

(i) *there exists non-zero $\mu\in M_{S^c}(G)$ which is singular with respect to m_G unless $G=\boldsymbol{R}\times\Delta$ or $\boldsymbol{T}\times\Delta$* for a discrete group Δ,

(ii) if $G=\boldsymbol{R}\times\Delta$ or $\boldsymbol{T}\times\Delta$ for a discrete group Δ and if $\mu\in M_{S^c}(G)$, then

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education, Science and Culture.

μ is absolutely continuous with respect to m_G.

His proof for this theorem requires the theory of uniform algebra (the existence of analytic discs, Gleason part, etc.), and is complicated. In particular, the proof of assertion (ii) seems to be obscure. In this note, we first prove a theorem which gives a relation between generating subsemigroups and orders in an ordered locally compact Abelian group (Section 2). We next apply it to prove assertion (i) of Theorem 1. Also, we give a very simple proof of assertion (ii) of Theorem 1 by using the original F. and M. Riesz theorem (Section 3). Our treatment seems to be simpler and clearer than that of Glicksberg.

§ 2. Generating Subsemigroups and Orders

In this section we prove a theorem which gives a relation between generating subsemigroups and orders. Note that an Abelian group G is ordered if and only if G is torsion-free ([3]). Our theorem is the following:

THEOREM 2. *Let G be a torsion-free locally compact Abelian group and let S be a proper closed generating subsemigroup of G. Then there exist $x_0 \in G$ and a nondense order P in G such that P contains $x_0 + S$.*

To prove Theorem 2 we need a result due to Glicksberg. For completeness we include the proof.

LEMMA 1 ([1, Section 1.6]). *Let $n \in \boldsymbol{Z}_+$. Let F be a locally compact Abelian group which contains a compact open subgroup and let S be a proper closed generating subsemigroup of $\boldsymbol{R}^n \times F$ satisfying $S \cap (-S) = \phi$. Suppose that $\pi(S) = F/F_0$ for some compact open subgroup F_0 of F, where π denotes the natural homomorphism from $\boldsymbol{R}^n \times F$ onto F/F_0. Then there exists a continuous homomorphism ρ from $\boldsymbol{R}^n \times F$ to $\boldsymbol{R}$ which is nonnegative and nontrivial on S.*

PROOF. First note that we have $n \geq 1$ under our assumption. Indeed, suppose $n = 0$. Then we have $F \neq F_0$ and $F_0 \cap S = \phi$ because a closed subsemigroup of a compact group is a subgroup ([4, Theorem (9.16)]), S is proper, and $S \cap (-S) = \phi$. Let $x \in F \setminus F_0$. Since $\pi(S) = F/F_0$, there exists $x_0 \in F_0$ such that $x + x_0 \in S$; similarly there exists $x_0' \in F_0$ such that $-x + x_0' \in S$. Thus we have $x_0 + x_0' = x + x_0 + (-x) + x_0' \in F_0 \cap S$. But this is a contradiction.

We put $G = \boldsymbol{R}^n \times F$, $G_0 = \boldsymbol{R}^n \times F_0$, and $S_0 = G_0 \cap S$. Let α be the projection from G onto $\boldsymbol{R}^n$. Note that $G_0 \cap S \neq \phi$ and hence $\alpha(S_0) \neq \phi$ because $\pi(S) = F/F_0$. We now show that there exists a linear functional Ψ on $\boldsymbol{R}^n$ which is nonnegative and nontrivial on $\overline{\alpha(S_0)}$ (=the closure of $\alpha(S_0)$).

We first note that $\overline{\alpha(S_0)}\cap(-\overline{\alpha(S_0)})=\phi$. This can be easily verified by using the fact that $(\{\mathbf{0}\}\times F_0)\cap S=\phi$ and a compactness argument. Moreover we can show that

$$(1)\quad (\boldsymbol{R}_+\cdot\overline{\alpha(S_0)})\cap(-\boldsymbol{R}_+\cdot\overline{\alpha(S_0)})=\{\mathbf{0}\}.$$

Indeed, suppose $t\boldsymbol{r}=-t'\boldsymbol{r}'$ for $t,\ t'>0$ and $\boldsymbol{r},\ \boldsymbol{r}'\in\overline{\alpha(S_0)}$. Then $\boldsymbol{r}$ and $\boldsymbol{r}'$ are nonzero vectors because $\overline{\alpha(S_0)}\cap(-\overline{\alpha(S_0)})=\phi$. If t/t' is rational, then $m\boldsymbol{r}=-n\boldsymbol{r}'$ for some positive integers m and n. But this contradicts $\overline{\alpha(S_0)}\cap(-\overline{\alpha(S_0)})=\phi$. If t/t' is irrational, then $\overline{\alpha(S_0)}\cap\{\mathbf{0}\}$ contains $\{k\boldsymbol{r}-l(t/t')\boldsymbol{r}: k,\ l\in\boldsymbol{Z}_+\}$ which is dense in the line through $\mathbf{0}$ and $\boldsymbol{r}$. (Note that $\{k-l(t/t') : k,\ l\in\boldsymbol{Z}_+\}$ is dense in $\boldsymbol{R}$.) But this also contradicts $\overline{\alpha(S_0)}\cap(-\overline{\alpha(S_0)})=\phi$. Thus we have $(\boldsymbol{R}_+\cdot\overline{\alpha(S_0)})\cap(-\boldsymbol{R}_+\cdot\overline{\alpha(S_0)})=\{\mathbf{0}\}$. Let E be a linear subspace of $\boldsymbol{R}^n$ generated by a maximal independent set in $\overline{\alpha(S_0)}$. Then the interior of $\boldsymbol{R}_+\cdot\overline{\alpha(S_0)}$ in E is nonempty and so (1) implies $\boldsymbol{R}_+\cdot\overline{\alpha(S_0)})^-$ is proper in E. Thus $(\boldsymbol{R}_+\cdot\overline{\alpha(S_0)})^-$ is a proper closed convex cone in E and its interior is nonempty. Hence there exists a linear functional ψ on E which is nonnegative and nontrivial on $\overline{\alpha(S_0)}$. Let Ψ be an extension of ψ to $\boldsymbol{R}^n$. Then Ψ is a linear functional on $\boldsymbol{R}^n$ as asserted.

Now note that $S\cap((\boldsymbol{r},x)+G_0)\neq\phi$ for each $(\boldsymbol{r},x)\in G$ because $\pi(S)=F/F_0$. Thus, for each $(\boldsymbol{r},x)\in G$, we can define

$$h((\boldsymbol{r},\ x))=inf\{(\Psi\circ\alpha)\ ((\boldsymbol{r}',\ x'))=\Psi(\boldsymbol{r}') : (\boldsymbol{r}',\ x')\in S\cap((\boldsymbol{r},x)+G_0)\}.$$

Then we have

$$(2)\quad h((\mathbf{0},0))\geq 0$$

and

$$(3)\quad h((\boldsymbol{r},x)+(\boldsymbol{r}',x'))\leq h((\boldsymbol{r},x))+h((\boldsymbol{r}',x')).$$

Indeed, we have (2) because $\Psi\geq 0$ on $\overline{\alpha(S_0)}$. Since

$$S\cap((\boldsymbol{r},x)+G_0)+S\cap((\boldsymbol{r}',x')+G_0)\subset S\cap((\boldsymbol{r},x)+(\boldsymbol{r}',x')+G_0)$$

for all $(\boldsymbol{r},x)$ and $(\boldsymbol{r}',x')\in G$, we have (3). Note that $h((\boldsymbol{r},x))$ is finite for each $(\boldsymbol{r},x)\in G$ because

$$h((\boldsymbol{r},x))+h(-(\boldsymbol{r},x))\geq h((\mathbf{0},0))\geq 0$$

by (2) and (3).

Now put

$$\Sigma=\{(t,x)\in\boldsymbol{R}\times F : t>h(x)\ \ (=h((\mathbf{0},x)))\}.$$

Then Σ is a subsemigroup of $\boldsymbol{R}\times F$ by (3). Moreover we have $\Sigma\cap(-\Sigma)=\phi$. Indeed, let $(t, x)\in\Sigma\cap(-\Sigma)$. Then $t>h(x)$, $-t>h(-x)$, and so $0\leq h(0)\leq h(x)+h(-x)<0$ by (2) and (3). This is a contradiction. Applying Zorn's lemma, we can find a maximal subsemigroup Σ_0 among the subsemigroups Σ' of $\boldsymbol{R}\times F$ containing Σ with $\Sigma'\cap(-\Sigma')\subset\{0\}$. Then we can easily see that each element of Σ_0 other than 0 has infinite order, while $(\boldsymbol{R}\times F)\setminus(\Sigma_0\cup(-\Sigma_0))$ consists of elements of finite order. Since $\Sigma_0\cap((t, x)+\boldsymbol{R}\times F_0)\neq\phi$ for each $(t, x)\in\boldsymbol{R}\times F$, which we can easily see, we can define

$$H((t, x))=inf\{t'\in\boldsymbol{R}: (t', x')\in\Sigma_0\cap((t, x)+\boldsymbol{R}\times F_0)\}$$

for each $(t, x)\in\boldsymbol{R}\times F$. Since

$$\begin{aligned}&(\Sigma_0\cap((t, x)+\boldsymbol{R}\times F_0))+(\Sigma_0\cap((t', x')+\boldsymbol{R}\times F_0)\\&\subset\Sigma_0\cap((t, x)+(t', x')+\boldsymbol{R}\times F_0)\end{aligned}$$

for all (t, x) and $(t', x')\in\boldsymbol{R}\times F$, we have

$$(4)\quad H((t, x)+(t', x'))\leq H((t, x))+H((t', x')).$$

Note that $H((t, x))$ is finite for each $(t, x)\in\boldsymbol{R}\times F$. Indeed, by (4), it suffices to show that $H((0,0))$ is finite. Let $x_0\in F_0$ and $t<-h((\mathbf{0},0))$. Since $]h((\mathbf{0},0)),\infty[\times F_0\subset\Sigma\subset\Sigma_0$, $(t, x_0)\in(-\Sigma_0)$, and so $(t, x_0)\notin\Sigma_0$ because $\Sigma_0\cap(-\Sigma_0)=\{0\}$. Hence $(]-\infty, -h((\mathbf{0},0))\ [\times F_0)\cap\Sigma_0=\phi$. Thus we have $-h((\mathbf{0},0))\leq t$ for each $(t, x_0)\in(\boldsymbol{R}\times F_0)\cap\Sigma_0$. This implies $H((0,0))\geq-\mathrm{h}((\mathbf{0},0))$ and so $H((0,0))$ is finite. In particular we have $H((0,0))\geq 0$ by (4). We shall show that H is a continuous homomorphism from $\boldsymbol{R}\times F$ to $\boldsymbol{R}$. We first show that H is a homomorphism. Let $(t, x)\in\boldsymbol{R}\times F$. Then $H((t, x))-\epsilon\neq 0$ for all small positive numbers ϵ and therefore $(H((t, x))-\epsilon, x)$ has infinite order. Thus we have $(H((t, x))-\epsilon, x)\in\Sigma_0\cup(-\Sigma_0)$. But, in fact, we have $(H((t, x))-\epsilon, x)\in(-\Sigma_0)$ by the definition of H. Hence $-H((t, x))+\epsilon\geq H(-(t, x))$ and so $-H((t, x))\geq H(-(t, x))$. But then $0\leq H((0,0))\leq H((t, x))+H(-(t, x))\leq 0$ and therefore we have $H(-(t, x))=-H((t, x))$. Now let (t, x) and (t', x') be in $\boldsymbol{R}\times F$. Then

$$\begin{aligned}H((t, x))&=H((t, x)+(t', x')-(t', x'))\\&\leq H((t, x)+(t', x'))+H(-(t', x'))\\&=H((t, x)+(t', x'))-H((t', x')).\end{aligned}$$

By this inequality and (4), H is a homomorphism. To show that H is continuous, first note that $H^{-1}(\boldsymbol{R}_+)$ has interior because $\boldsymbol{R}\times F_0\subset H^{-1}(\boldsymbol{R}_+)$ and $\boldsymbol{R}\times F_0$ is open in $\boldsymbol{R}\times F$. Let (t_0, x_0) be an interior point of $H^{-1}(\boldsymbol{R}_+)$ and let U be a neighborhood of $(0,0)$ such that $(t_0, x_0)+U$ is included in $H^{-1}(\boldsymbol{R}_+)$. Choose a negative real number r_0 such that $H((t_0, x_0))+r_0<0$. We can easily see the following: if H is discontinuous at $(0,0)$, then, for each neighborhood V of $(0,0)$ and each $r<0$, there exists $(t, x)\in V$ such that $H((t, x))<r$. Thus we can find $(t, x)\in U$ such that $H((t, x))<r_0$. But then

$$\begin{aligned} H((t_0, x_0)+(t, x)) &= H((t_0, x_0))+H((t, x)) \\ &< H((t_0, x_0))+r_0<0. \end{aligned}$$

This is a contradiction. Thus H is continuous.

Now we define a continuous homomorphism ρ from G to $\boldsymbol{R}$ as follows:

$$\rho=\Psi\circ\alpha-H\circ\beta,$$

where β denotes the natural homomorphism from $G=\overbrace{\boldsymbol{R}\times\ldots\times\boldsymbol{R}}^{n-1}\times(\boldsymbol{R}\times F)$ onto $\boldsymbol{R}\times F$. It is easy to see that $h\geq H\circ\beta$ on G and $\Psi\circ\alpha\geq h$ on S. Thus ρ is nonnegative on S. Since $(\Psi\circ\alpha)((\boldsymbol{r}_0, x_0))>0$ for some $(\boldsymbol{r}_0, x_0)\in G_0\cap S$ and $(H\circ\beta)\ ((\boldsymbol{r}_0, x_0))=H((0,0))=0$, $\rho((\boldsymbol{r}_0, x_0))>0$. Thus ρ is nontrivial on S. This completes the proof.

Proof of Theorem 2. By the structure theorem of a locally compact Abelian group ([4, Theorem (24.30)]), G has the form $\boldsymbol{R}^n\times F$, where $n\in\boldsymbol{Z}_+$ and F contains a compact open subgroup. Let F_0 be a compact open subgroup of F and let π denote the natural homomorphism from $G=\boldsymbol{R}^n\times F$ onto F/F_0. Put $\tilde{S}=\pi(S)$. Then it is easy to see that $\tilde{S}$ is a closed generating subsemigroup of F/F_0. (Note that F/F_0 is discrete.) We examine two cases.

Case 1: $\tilde{S}\neq F/F_0$. Choose $\tilde{y}\in\tilde{S}$ such that $-\tilde{y}\notin\tilde{S}$. Such a choice of $\tilde{y}$ is possible because $\tilde{S}$ is proper and generating. It is easy to see that $\tilde{y}+\tilde{S}$ is a proper subsemigroup of F/F_0 and $(\tilde{y}+\tilde{S})\cap(-\tilde{y}-\tilde{S})=\phi$. We claim that there exists a subsemigroup $\tilde{Q}$ of F/F_0 such that

$$\tilde{Q}\cap(-\tilde{y}-\tilde{S})=\phi \text{ and } \tilde{Q}\cup(-\tilde{Q})=F/F_0.$$

To prove this claim, let $\sum$ denote the set of all subsemigroups of F/F_0 which don't intersect $(-\tilde{y}-\tilde{S})$. Then $\sum$ is nonempty because $\{0\}$ is included in $\sum$. We define the partially order on $\sum$ by inclusion relation of sets. It is easy to see that $\sum$ is inductively ordered. Hence, by Zorn's lemma, there exists a maximal element $\tilde{Q}$ of $\sum$. Of course $\tilde{Q}\cap(-\tilde{y}-\tilde{S})=\phi$. We also have $\tilde{Q}\cup(-\tilde{Q})=F/F_0$. To see this, suppose $\tilde{Q}\cup(-\tilde{Q})\neq F/F_0$. Then there exists $\tilde{z}\in F/F_0$ such that $\tilde{z}$ and $-\tilde{z}\notin\tilde{Q}$. Since $\tilde{Q}$ is maximal, there exist positive integers m and n and elements $\tilde{a}$ and $\tilde{b}$ in $\tilde{Q}$ such that

$$m\tilde{z}+\tilde{a}\in(-\tilde{y}-\tilde{S}) \text{ and } -n\tilde{z}+\tilde{b}\in(-\tilde{y}-\tilde{S}).$$

Then we have

$$n\tilde{a}+m\tilde{b}=n(m\tilde{z}+\tilde{a})+m(-n\tilde{z}+\tilde{b})\in(-\tilde{y}-\tilde{S}).$$

But this contradicts the fact that $\tilde{Q}\cap(-\tilde{y}-\tilde{S})=\phi$. Thus we have established the claim. Put

$$Q=\pi^{-1}(\tilde{Q}).$$

Then Q is a nondense subsemigroup of G and $Q\cup(-Q)=G$. Choose $x_0\in S\backslash(-S)$ such that $\pi(x_0)=\tilde{y}$. Then $(x_0+S)\cap(-x_0-S)=\phi$ and Q contains x_0+S. If $Q\cap(-Q)=\{0\}$, then we have only to put $P=Q$. Next we consider the case where $Q\cap(-Q)\neq\{0\}$. If $Q\cap(-x_0-S)=\phi$, choose any order Q_0 in $Q\cap(-Q)$ and put

$$P=(Q\backslash(-Q))\cup Q_0.$$

Then we can easily see that P is a nondense order in G and that P contains x_0+S. Finally, let $Q\cap(-x_0-S)\neq\phi$. Since $Q\cap(-x_0-S)$ and $(-Q)\cap(x_0+S)$ are subsemigroups and they are mutually disjoint, there exists an order Q_1 in $Q\cap(-Q)$ such that Q_1 contains $(-Q)\cap(x_0+S)$. (For the existence of such an order, see [3, (2.3) Lemma and (2.5) Theorem].) Now put

$$P=(Q\backslash(-Q))\cup Q_1.$$

Then we can easily see that P is a nondense order in G and contains x_0+S.

Case 2: $\tilde{S}=F/F_0$. Choose $s_0\in S$ such that $-s_0\notin S$. Then s_0+S is a proper closed generating subsemigroup of G and $(s_0+S)\cap(-s_0-S)=\phi$. We also have $\pi(s_0+S)=F/F_0$ because $\pi(S)=\tilde{S}=F/F_0$. By Lemma 1, there exists a proper closed subsemigroup P_0 of G such that P_0 contains s_0+S and $P_0\cup(-P_0)=G$. Now choose $t_0\in P_0\backslash(-P_0)$ and put $x_0=s_0+t_0$. Then it is easy to see that $(-P_0)\cap(x_0+S)=\phi$ and x_0+S is included in P_0. If $P_0\cap(-P_0)=\{0\}$, we have only to put $P=P_0$. If $P_0\cap(-P_0)\neq\{0\}$, choose any order Q_2 in $P_0\cap(-P_0)$ and put

$$P=(P_0\backslash(-P_0))\cup Q_2.$$

Then P is a nondense order in G and contains x_0+S. This completes the proof.

§ 3. Another Proof of a Theorem of Glicksberg

In this section we apply our theorem to give another proof of assertion (i) in Theorem 1. We also give a simple proof of assertion (ii) in Theorem 1 by using the original F. and M. Riesz theorem. To carry out the proofs, we need two Lemmas.

LEMMA 2. *Let G be a locally compact Abelian group and let H be an open subgroup of G. Let $\tilde{E}$ be a subset of $\hat{G}/H^{\perp}$ and put $E=\pi^{-1}(\tilde{E})$, where π denotes the natural homomorphism from $\hat{G}$ onto $\hat{G}/H^{\perp}$. If $\mu\in M_E(G)$, then $(\mu_{x+H})*\delta_{-x}\in M_{\tilde{E}}(H)$ for all $x\in G$, where μ_{x+H} denotes the restriction of μ to the coset $x+H$ and δ_{-x} denotes the Dirac measure at $-x$.*

PROOF. Since H is open, $H^{\perp}$ is a compact subgroup of $\hat{G}$. Let σ be the normalized Haar measure of $H^{\perp}$. Then its "inverse" Fourier transform $\check{\sigma}$ is nothing but the characteristic function of H. Therefore $\mu_H=\check{\sigma}\mu$ and

$$(5)\quad \hat{\mu}_H(\gamma_0)=\int_{H^{\perp}}\hat{\mu}(\gamma_0-\gamma)\,d\sigma(\gamma)$$

whenever $\mu\in M(G)$ and $\gamma_0\in\hat{G}$.

Now suppose $\mu\in M_E(G)$. Pick any $\gamma_0\in\hat{G}$ with $\gamma_0|H=\pi(\gamma_0)\notin\tilde{E}$. Then $\gamma_0+H^{\perp}\subset E^c$ and so $\hat{\mu}=0$ on $\gamma_0+H^{\perp}$. Hence

$$\hat{\mu}_H(\pi(\gamma_0))=\hat{\mu}_H(\gamma_0)=0$$

by (5) whenever $\pi(\gamma_0)\notin\tilde{E}$, that is, $\mu_H\in M_{\tilde{E}}(H)$. Since $M_E(G)$ is translation-invariant, it follows that $(\mu*\delta_x)_H\in M_{\tilde{E}}(H)$ for all $x\in G$, as desired.

LEMMA 3. *Let $G=\boldsymbol{R}\times K$ $(G=\boldsymbol{Z}\times K)$ with a compact group K and let S be a proper closed generating subsemigroup of G. Then S is contained in $\boldsymbol{R}_+\times K$ or $\boldsymbol{R}_-\times K$ (resp. $\boldsymbol{Z}_+\times K$ or $\boldsymbol{Z}_-\times K$), where $\boldsymbol{R}_-$ $(\boldsymbol{Z}_-)$ denotes the nonpositive real numbers (resp. the nonpositive integers).*

PROOF. We consider $G=\boldsymbol{R}\times K$. (We can also prove for $G=\boldsymbol{Z}\times K$ by the same argument.) Let π denote the projection from G onto $\boldsymbol{R}$. Then π is a closed mapping because K is compact. Hence $S_1=\pi(S)$ is a closed subsemigroup of $\boldsymbol{R}$. Also, since S is generating, S_1 is generating. Moreover, S_1 is proper. To see this, choose $x\in S$ such that $-x\notin S$. Then $-x\notin K+S$. Indeed, suppose that $-x=k+s$ for $k\in K$ and $s\in S$. Then $-k=s+x\in S\cap K$, and so $-s-x\in S\cap K$ because $S\cap K$ is a group ([4, Theorem (9.16)]). Hence $-x\in S+S\subset S$, but this is a contradiction. Thus we have $-x\notin K+S$. If $S_1=\boldsymbol{R}$, then

$$\boldsymbol{R}\times K=\pi^{-1}(S_1)=K+S.$$

But this is a contradiction. Thus S_1 is a proper closed generating subsemigroup of $\boldsymbol{R}$. Now suppose that S_1 contains both a positive number and a negative one. Then the following three cases have to be considered: (a) S_1

has both a positive minimum element and a negative maximum one, (b) S_1 has neither a positive minimum element nor a negative maximum one, and (c) neither (a) nor (b) holds. Clearly (c) is impossible. If (a) holds, then it is easy to see that $S_1=c\boldsymbol{Z}$ for $c=min\{S_1\cap(\boldsymbol{R}_+\backslash\{0\})\}$ but this contradicts the fact that S_1 generates $\boldsymbol{R}$. If (b) holds, then we can easily show that S_1 is dense in $\boldsymbol{R}$. Since S_1 is closed, we have $S_1=\boldsymbol{R}$. But this is a contradiction because S_1 is proper. Thus we have $S\subset\boldsymbol{R}_+\times K$ or $S\subset\boldsymbol{R}_-\times K$. This completes the proof.

PROOF OF ASSERTION (i) IN THEOREM 1. By [4, (25.32) (a) and Theorem (A.15)], there exists a divisible locally compact Abelian group G_0 such that G is an open subgroup of G_0. Thus $\hat{G}_0$ is torsion-free ([4, Theorem (24.23)]. Our assumption is that G is neither of the form $\boldsymbol{R}\times\Delta$ nor $\mathbf{T}\times\Delta$ for a discrete group Δ. We can easily see that G_0 is neither of the form $\boldsymbol{R}\times\Delta$ nor $\mathbf{T}\times\Delta$ for a discrete group Δ. Put

$$S_0=S\cup\{0\}.$$

Then clearly S_0 is a proper closed generating subsemigroup of $\hat{G}_0/G^{\perp}\cong\hat{G}$. Define

$$S_1=\pi^{-1}(S_0),$$

where π denotes the natural homomorphism from $\hat{G}_0$ onto $\hat{G}_0/G^{\perp}$. Then S_1 is a proper closed subsemigroup of $\hat{G}_0$. Noting that $0\in S_0$, we can easily see that S_1 is generating. Since $\hat{G}_0$ is torsion-free, by Theorem 2, there exist $\gamma_0\in\hat{G}_0$ and a nondense order P in $\hat{G}_0$ such that P contains γ_0+S_1. By our assumption and [3, (5.6) Theorem], there exists a non-zero measure ν in $M_{P^c}(G_0)$ which is singular with respect to m_{G_0}. Of course this measure ν is included in $M_{(\gamma_0+S_1)^c}(G_0)$. Now consider the measure $\bar{\gamma}_0\nu\in M(G_0)$. Then $\bar{\gamma}_0\nu$ is a measure in $M_{S_1^c}(G_0)$ which is singular with respect to m_{G_0}. Put

$$\mu=((\bar{\gamma}_0\nu)_{x_0+G})*\delta_{-x_0},$$

where x_0 is an element of G_0 such that $(\bar{\gamma}_0\nu)_{x_0+G}\neq 0$. Then, by Lemma 2, μ is included in $M_{S_0^c}(G)$ and therefore in $M_{S^c}(G)$. Moreover, noting that G is an open subgroup of G_0, we can see that μ is singular with respect to m_G. This completes the proof.

PROOF OF ASSERTION (ii) IN THEOREM 1. We may suppose that Δ is countable. Indeed, let μ be a measure in $M_{S^c}(G)$. Since μ is regular, there exists a countable subgroup Δ_0 of Δ such that $supp(\mu)\subset\boldsymbol{R}\times\Delta_0$. Put $G'=$

$\boldsymbol{R}\times\Delta_0$. Then we can regard μ as an element of $M(G')$. Let α denote the natural homomorphism $\hat{G}$ onto $\hat{G}/(G')^{\perp}$ and put $S'=\alpha(S)$. Then we can easily see that S' is a proper closed generating subsemigroup of $\hat{G}/(G')^{\perp}$. (Note that α is a closed mapping because $(G')^{\perp}$ is compact.) Moreover, we have $\mu\in M_{(S')^c}(G')$. If the assertion (ii) holds for Δ countable, then μ is absolutely continuous with respect to $m_{G'}$. Since G' is open in G, μ is absolutely continuous with respect to m_G.

Now we consider $G=\boldsymbol{R}\times\Delta$ with Δ countable. Let K denote the dual group of Δ. By Lemma 3, we may suppose that S is contained in $\boldsymbol{R}_+\times K$. Let μ be a measure in $M_{S^c}(G)$. Henceforth we write an element γ in $\hat{G}=\boldsymbol{R}\times K$ as $\gamma=(r, k)$, where $r\in\boldsymbol{R}$ and $k\in K$. We first show the following:

(6) $\hat{\mu}_s(n\gamma)=0$ for all $\gamma=(r, k)\in S$ with $r>0$ and $n\in\boldsymbol{Z}$.

This is proved as follows. Let $\gamma=(r, k)\in S$ with $r>0$ and put $\Lambda=\{n\gamma : n\in\boldsymbol{Z}\}$. Then it is obvious that Λ is isomorphic to $\boldsymbol{Z}$. Put $H=\Lambda^{\perp}$. Then H is countable. To see this, put $\Delta=\{d_j : j=1, 2, ,\ldots\}$ and choose $\{\beta_j\}_{j=1}^{\infty}\subset[0, 2\pi)$ such that $e^{i\beta_j}=(d_j, k)$ for each j. Define

$$H_j=\{(x, d_j)\in\boldsymbol{R}\times\Delta : ((x, d_j), (nr, nk))=1 \textit{ for all } n\in\boldsymbol{Z}\}.$$

If $(x, d_j)\in H_j$, then

$$\begin{aligned}1&=((x, d_j), (nr, nk))\\&=e^{ixnr}(d_j, nk)\\&=e^{ixnr}e^{i\beta_j n}\end{aligned}$$

for all $n\in\boldsymbol{Z}$, and so we have $x\in(2\pi\boldsymbol{Z}-\beta_j)/r$. Hence H_j is countable and therefore H is so because $H=\bigcup_{j=1}^{\infty}H_j$. Now let π be the natural homomorphism from G onto G/H. Then, since H is countable, we have

$$\pi(\nu)_S=\pi(\nu_S)$$

for each $\nu\in M(G)$. Since $\mu\in M_{S^c}(G)$, we have

$$(\pi(\mu))\hat{\ }(n\gamma)=\hat{\mu}(n\gamma)=0$$

for all $n>0$. Applying the F. and M. Riesz theorem to $\pi(\mu)\in M(G/H)$, we have $\pi(\mu)_S=0$, and therefore

$$\pi(\mu_S)=\pi(\mu)_S=0.$$

Thus (6) holds.

Next, let $\gamma\in S$ and $\gamma'=(r', k')\in S$ with $r'>0$ and define $\nu\in M(G)$ by

$\nu=\bar{\gamma}\mu$. Then $\nu\in M_{S^c}(G)$ and by (6) $\hat{\nu}_S(-\gamma')=0$. Hence we have

(7) $\hat{\mu}_S(\gamma-\gamma')=0$.

Since $\gamma'-\gamma=2\gamma'-(\gamma'+\gamma)$, we also have

(8) $\hat{\mu}_S(\gamma'-\gamma)=0$.

Using (7) and (8) , we can prove the following:

(9) $\hat{\mu}_S(\gamma)=0$ for all $\gamma\in\hat{G}\backslash(\{0\}\times K)$.

Indeed, let $\gamma=(r, k)\in\hat{G}\backslash(\{0\}\times K)$. Since S generates $\hat{G}$, there exist sequences $\{\gamma_n=(r_n, k_n)\}$ and $\{r'_n=(r'_n, k'_n)\}$ of elements in S such that $\{\gamma_n-\gamma'_n\}$ converges to γ. Then there exists a subsequence$\{n_j\}$ such that either r_{n_j} or r'_{n_j} is nonzero for all j, because $r\neq 0$. By (7) and (8), $\hat{\mu}_S(\gamma_{n_j}-\gamma'_{n_j})=0$ and therefore we have

$$\hat{\mu}_S(\gamma)=\lim_{j\to\infty}\hat{\mu}_S(\gamma_{n_j}-\gamma'_{n_j})=0.$$

Thus (9) holds.

Finally, since $supp(\hat{\mu}_S)\subset\{0\}\times K$ by (9) and K is compact, we have $\hat{\mu}_S\in L^1(\hat{G})$. Hence, by the inversion theorem, $\mu_S\in L^1(G)$, and so $\mu_S=0$. This completes the proof for $G=\boldsymbol{R}\times\Delta$. We can also prove for $G=\boldsymbol{T}\times\Delta$ by the same argument.

The authors would like to thank professor H. Yamaguchi for many helpful discussions and comments and also thank the referee for an improvement of proof of Lemma 2.

References

[1] I. GLICKSBERG: Spectra of invariant uniform and transform algebras, Trans. Amer. Math. Soc. 277 (1983), 381–396.

[2] I. GLICKSBERG: The strong conclusion of the F. and M. Riesz theorem on groups, Trans. Amer. Math. Soc. 285 (1984), 235–240.

[3] E. HEWITT and S. KOSHI: Orderings in locally compact Abelian groups and the theorem of F. and M. Riesz, Math. Proc. Cambridge Philos. Soc. 93 (1983), 441–457.

[4] E. HEWITT and K. A. ROSS: Abstract Harmonic Analysis, Vol. I, Berlin-Heidelberg-New York. Springer-Verlag, 1963.

[5] S. KOSHI and H. YAMAGUCHI: The F. and M. Riesz theorem and group structures, Hokkaido Math. J. 8 (1979), 294–299.

[6] Y. TAKAHASHI and H. YAMAGUCHI: On measures which are continuous by certain translation, Hokkaido Math. J. 13 (1984), 109–117.

[7] H. YAMAGUCHI: Remarks on Riesz sets, Hokkaido Math. J. 7 (1978), 328–335.

Department of Mathematics
Hokkaido University