Hokkaido Mathematical Journal Vol. 17 (1988), p. 345-353

Generalized variation and translation operator in some sequence spaces

J. MUSIELAK and A. WASZAK
(Received August 5, 1987, Revised February 23, 1988)

Abstract. There are defined and investigated some spaces of sequences provided with two-modular structure given by generalized variations and the translation operator. The results are applied to obtain an approximation theorem by means of translated sequences.

1. Let $x=(t_i)=(t_i)_{i=0}^{\infty}$ be a sequence of real numbers. We denote also $(x)_j=t_j$ for $j=0, 1, 2, \ldots$. We introduce two auxiliary notations: this of the Φ-variation of x and that of the sequential modulus of x.

1.1. Let X be the space of all real sequences and let Φ be a φ-function (see e.g. [4], 1.9). The Φ-variation $w_\Phi(x)$ of $x\in X$ is defined as

$$w_\Phi(x)=\sup_{(n_i)}\sum_{i=1}^{\infty}\Phi(|t_{n_i}-t_{n_{i-1}}|),$$

where the supremum runs through all increasing subsequences (n_i) of indices (see [2]). w_Φ is a pseudomodular in X defining the modular space

$$X_\Phi=X_{w_\Phi}=\{x\in X: w_\Phi(\lambda x)\longrightarrow 0 \text{ as } \lambda\longrightarrow 0_+\}$$

(see [7], [5] and also [8]). $\|\cdot\|_\Phi$ will denote the Luxemburg pseudonorm in X_Φ (see [4]). It is easily seen that $X_\Phi\subset c$, where c is the space of convergent sequences, and X_Φ is strongly modular complete and complete in the norm (see [2] and [5]).

1.2. Given any sequence $x=(t_i)_{i=0}^{\infty}$, we write

$$(\tau_m x)_j=\begin{cases} t_j & \text{for } j<m,\\ t_{m+j} & \text{for } j\geq m,\end{cases}$$

where $m, j=0, 1, 2, \ldots$ (see [3], also [4], 7.17). The sequence $\tau_m x=((\tau_m x)_j)_{j=0}^{\infty}$ is called the m-translation of the sequence x.

1.3. The sequential modulus of the sequence $x=(t_i)_{i=0}^{\infty}$ is defined as

$$\omega(x, r)=\sup_{m\geq r}\sup_i|(\tau_m x)_i-t_i|,$$

where $r=0, 1, 2, \ldots$. Obviously, we have

$$\omega(x, r)=\sup_{m\geq r}\sup_{i\geq m}|t_{m+i}-t_i|$$

for $r=0, 1, 2, \ldots$.

For example, taking $x=(a^i)_{i=0}^{\infty}$ with $0<a<1$ or $x=\left(\frac{1}{i+1}\right)_{i=0}^{\infty}$ or $x=\left(1+\frac{1}{2}+\cdots+\frac{1}{i+1}\right)_{i=0}^{\infty}$, then we have $\omega(x, r)=a^r(1-a^r)$ for $r\geq -\ln 2/\ln a$ or $\omega(x, r)=\frac{r}{(r+1)(2r+1)}$ for $r\geq 1$ or $\omega(x, r)=\ln 2$ for $r\geq 0$, respectively.

2. We shall consider two spaces of sequences $X(\Psi)$ and $X(\Phi, \Psi)$, defined by means of the sequential modulus and Φ-variation of the sequence.

2.1. Let Φ be a φ-function and let Ψ be a nonnegative, nondecreasing function of $u\geq 0$ such that $\Psi(u)\longrightarrow 0$ as $u\longrightarrow 0_+$. Then we write

$$X(\Psi)=\{x\in X : r\Psi(\omega(\lambda x, r))\longrightarrow 0 \text{ as } r\longrightarrow\infty \text{ for a } \lambda>0\},$$
$$X(\Phi, \Psi)=X_\Phi\cap X(\Psi).$$

Obviously, $X(\Psi)$ and $X(\Phi, \Psi)$ are vector spaces. If Ψ satisfies the condition (Δ_2) for small $u\geq 0$, then one may take fixed $\lambda=1$ in the definition of $X(\Psi)$.

2.2. We define now for every $x\in X$

$$\zeta(x)=\sup_r r\Psi(\omega(x, r)).$$

Obviously, ζ is a pseudomodular in X. The respective modular space will be denoted by X_ζ; we have $X(\Phi, \Psi)\subset X(\Psi)\subset X_\zeta$.

Let us remark that if Ψ is increasing and s-convex for $u\geq 0$ with some $0<s\leq 1$, then ζ is an s-convex pseudomodular in X and

$$\|x\|_\zeta^s=\sup_{r\geq 1}\left(\frac{\omega(x, r)}{\Psi^{-1}(1/r)}\right)^s,$$

where Ψ^{-1} is the inverse to Ψ, because

$$\|x\|_\zeta^s=\inf\left\{u>0 : \zeta\left(\frac{x}{u^{1/s}}\right)\leq 1\right\}$$
$$=\inf\left\{u>0 : \frac{\omega(x, r)}{u^{1/s}}\leq\Psi^{-1}\left(\frac{1}{r}\right) \text{ for all } r\geq 1\right\}.$$

For example, taking $x=(a^i)_{i=0}^{\infty}$, $0<a<1$ and both Φ, Ψ s-convex with $0<s\leq 1$, we have $w_\Phi(\lambda x)\leq \Phi(\lambda)(1-a^s)^{-1}$ for $\lambda>0$ and $r\Psi(\omega(\lambda x, r))\leq r\Psi(\lambda a^r)\leq r(a^s)^r\Psi(\lambda)\longrightarrow 0$ as $r\longrightarrow\infty$. Hence $x\in X(\Phi, \Psi)$.

2.3. Let $\bar{c}$ be the space of all sequences $x=(t_i)_{i=0}^{\infty}$ such that t_0 and t_1 are arbitrary and $t_i=t_{i+1}$ for $i=1, 2, \ldots$. Let Φ be a φ-function and let Ψ be a nonnegative, increasing function such that $\Psi(u)\longrightarrow 0$ as $u\longrightarrow 0_+$.

Then $w_\Phi(x)=\Phi(|t_1-t_0|)$ and $\omega(x, r)=0$, $r=0, 1, 2, \dots$ for $x\in\bar{c}$. Hence $\bar{c}$ is a vector subspace of $X(\Phi, \Psi)$ and $x\in\bar{c}$ is equivalent to $|x|_\zeta=0$, where $|\cdot|_\zeta$ is the F-pseudonorm generated by ζ (see [4], 1.5). Consequently, one may consider quotient space

$$\tilde{X}_\zeta=X_\zeta/\bar{c},\ \tilde{X}(\Psi)=X(\Psi)/\bar{c} \text{ and } \tilde{X}(\Phi, \Psi)=X(\Phi, \Psi)/\bar{c},$$

whose elements will be denoted by $\tilde{x}$, etc. Since $|x|_\zeta$ is constant in each of the classes $\tilde{x}$, we may define $|\tilde{x}|_\zeta=|x|_\zeta$, $x\in\tilde{x}$. In case if Ψ is s-convex, $0<s\leq 1$, we may define $\|\tilde{x}\|_\zeta^s=\|x\|_\zeta^s$, $x\in\tilde{x}$.

2.4. The following condition will be needed (see [4]):
(+) there exists a $u_0>0$ such that for every $\delta>0$ there is an $\eta>0$ satisfying the inequality $\Psi(\eta u)\leq\delta\Psi(u)$ for all $0\leq u\leq u_0$.

In particular, every s-convex φ-function Ψ, $0<s\leq 1$, satisfies (+). There are φ-functions Ψ not satisfying (+), for example

$$\Psi(u)=\begin{cases} 0 & \text{for } u=0,\\ \dfrac{1}{\sqrt{-\ln u}} & \text{for } 0<u\leq\dfrac{1}{e},\\ \text{arbitrary} & \text{for } u>\dfrac{1}{e}.\end{cases}$$

It is easily seen that (+) is equivalent to the following condition:
(++) for any $u_1>0$ and $\delta_1>0$ there is an $\eta_1>0$ such that $\Psi(\eta u)\leq\delta_1\Psi(u)$ for all $0\leq u\leq u_1$ and $0<\eta\leq\eta_1$.

2.5. THEOREM. *Let Ψ be an increasing, continuous function of $u\geq 0$, $\Psi(0)=0$, satisfying the condition 2.4(+). Then $\tilde{X}_\zeta$ and $\tilde{X}(\Psi)$ are Fréchet spaces with respect to the F-norm $|\cdot|_\zeta$.*

PROOF. Let $(\tilde{x}_n)$ be a Cauchy sequence in $\tilde{X}_\zeta$ and let $x_n\in\tilde{x}_n$, $x_n=(t_i^n)_{i=0}^\infty$ be such that $t_1^n=0$ for all n. Let an $\varepsilon>0$ be given and let Ψ^{-1} be the inverse to Ψ. There is an N such that $|x_p-x_q|_\zeta<\Psi(\varepsilon)$ for $p, q>N$. Hence there exists a u_ε, $0<u_\varepsilon<\Psi(\varepsilon)$, for which

$$r\Psi\left(\frac{\omega(x_p-x_q, r)}{u_\varepsilon}\right)\leq u_\varepsilon$$

for $p, q>N$ and $r=1, 2, \dots$, whence

$$\omega(x_p-x_q,\ r)\leq u_\varepsilon\Psi^{-1}\left(\frac{u_\varepsilon}{r}\right)\leq u_\varepsilon\cdot\varepsilon<\varepsilon\Psi(\varepsilon)$$

for $p, q>N$, $r\geq 1$. Thus

$$(*)\qquad |t^p_{m+i}-t^q_{m+i}-t^p_i+t^q_i|\le u_\varepsilon\Psi^{-1}\Big(\frac{u_\varepsilon}{r}\Big)<\varepsilon\Psi(\varepsilon)$$

for $p, q>N$, $i\ge m\ge r$. Taking $r=1$ and $m=1$ we obtain

$$|t^p_{i+1}-t^q_{i+1}|\le|t^p_i-t^q_i|+\varepsilon\Psi(\varepsilon)$$

for p, $q>N$, $i=1, 2, \ldots$. Hence, because $t^n_1=0$ for all n, we see that $(t^n_i)^\infty_{n=0}$ are Cauchy sequence for $i=1, 2, \ldots$. Let $t_i=\lim\limits_{n\to\infty}t^n_i$ for $i=1, 2, \ldots$, $t_0=0$, $x=(t_i)^\infty_{i=0}$. Taking $q\longrightarrow\infty$ in $(*)$, we have

$$(**)\qquad |t^p_{m+i}-t_{m+i}-t^p_i+t_i|\le u_\varepsilon\Psi^{-1}\Big(\frac{u_\varepsilon}{r}\Big)$$

for $p>N$, $i\ge m\ge r\ge 1$. Thus

$$r\Psi\Big(\frac{\omega(x_p-x, r)}{u_\varepsilon}\Big)\le u_\varepsilon$$

for $p>N$, $r\ge 1$. We shall see that this implies $x_p-x\in X_\zeta$ for large p, i. e. $x\in X_\zeta$. Indeed, let $u_\varepsilon>0$ and $p>N$ be fixed and let $\delta>0$ be arbitrary. Taking $\delta_1=\delta/u_\varepsilon$, $u_1=\Psi^{-1}(u_\varepsilon)$ and $u=\dfrac{\omega(x_p-x, r)}{u_\varepsilon}$ in 2.4(++), we obtain for $0<\lambda\le\dfrac{\eta_1}{u_\varepsilon}$

$$r\Psi(\lambda\omega(x_p-x, r))=r\Psi\Big(\lambda u_\varepsilon\frac{\omega(x_p-x, r)}{u_\varepsilon}\Big)\le\delta_1$$

uniformly with respect to r. Thus, $\zeta(\lambda(x_p-x))\longrightarrow 0$ as $\lambda\longrightarrow 0_+$, i. e. $x_p-x\in X_\zeta$. Moreover, $|x_p-x|_\zeta\le u_\varepsilon\le\Psi(\varepsilon)$ for $p>N$, i. e. $|x_p-x|_\zeta\longrightarrow 0$ as $p\longrightarrow\infty$. Thus, $\widetilde{X}_\zeta$ is complete.

We have still to show that $\widetilde{X}(\Psi)$ is closed in $\widetilde{X}_\zeta$ with respect to $|\cdot|_\zeta$. Let $\tilde{x}_p\longrightarrow\tilde{x}$ in $\widetilde{X}_\zeta$, $\tilde{x}_p\in\widetilde{X}(\Psi)$, and let $x_p\in\tilde{x}_p$, $x\in\tilde{x}$. Then for every $\lambda>0$,

$$r\Psi(\omega(\lambda(x_p-x), r))\longrightarrow 0 \text{ as } p\longrightarrow\infty$$

uniformly with respect to r. Let us fix $\lambda>0$ and $\varepsilon>0$. There is an index p_0 such that $r\,\Psi(2\omega(\lambda(x_p-x), r))<\frac{1}{2}\varepsilon$ for $p\ge p_0$ and all r. We may choose an r_0 such that $r\Psi(2\omega(\lambda x_{p_0}, r))<\frac{1}{2}\varepsilon$ for all $r\ge r_0$. Hence

$$r\Psi(\omega(\lambda x, r))\le r\Psi(2\omega(\lambda(x-x_{p_0}), r))+r\Psi(2\omega(\lambda x_{p_0}, r))<\varepsilon$$

for $r\ge r_0$. This shows that $x\in X(\Psi)$, i. e. $\tilde{x}\in\widetilde{X}(\Psi)$.

We are going now to express Theorem 2.5 replacing the F-norm $|\cdot|_{\zeta}$ by the pseudomodular ζ itself.

Let us remark that $\bar{c}=\{x\in X:\zeta(x)=0\}$. Moreover, $\zeta(x)=0$ implies $\zeta(2x)=0$ for all $x\in X$. By [1], 3.3, 3.2 and 3.5, $\tilde{\zeta}(\tilde{x})=\inf\{\zeta(y):y\in\tilde{x}\}$ is a modular in $\tilde{X}_{\zeta}$ and $\tilde{X}_{\zeta}=X_{\zeta}/\bar{c}=(X/\bar{c})_{\tilde{\zeta}}$.

Let us still recall that a sequence (x_n) is called ζ-Cauchy, if there is a $k>0$ such that for every $\varepsilon>0$ there exists an index N for which $\zeta(k(x_p-x_q))<\varepsilon$ for all $p,q>N$. A modular space is called ζ-complete, if every ζ-Cauchy sequence of its elements is ζ-convergent to an element of this space (see [5], 1.04).

2.6. THEOREM. *Let Ψ be an increasing, continuous function of $u\geq 0$, $\Psi(0)=0$, satisfying the condition* 2.4(+). *The spaces $\tilde{X}_{\zeta}$ and $\tilde{X}(\Psi)$ are $\tilde{\zeta}$-complete.*

Proof is similar to that of 2.5, and we give an outline only. Let $\tilde{x}_n\in\tilde{X}_{\zeta}$, $x_n=(t_i^n)_{i=0}^{\infty}\in\tilde{x}_n$, $t_1^n=0$ for all n, and let $(\tilde{x}_n)$ be $\tilde{\zeta}$-Cauchy in $\tilde{X}_{\zeta}$. For every $\varepsilon>0$ there exists an N such that $\tilde{\zeta}(2k(\tilde{x}_p-\tilde{x}_q))<\varepsilon$ for $p,q>N$, $k>0$ being fixed. There exists a $y\in 2k(\tilde{x}_p-\tilde{x}_q)$ such that $\zeta(y)<\varepsilon$. Let us remark that if $z_1,z_2\in X_{\zeta}$, $z_1-z_2\in\bar{c}$, then

$$\zeta(z_2)\leq\zeta(2z_1)+\zeta(2(z_2-z_1))=\zeta(2z_1).$$

Taking $z_1=\frac{1}{2}y$, $z_2k(x_p-x_q)$, we thus have $\zeta(k(x_p-x_q))\leq\zeta(y)<\varepsilon$ for $p,q>N$. Hence $\omega(k(x_p-x_q),r)<\Psi^{-1}(\varepsilon/r)$ for $p,q>N$ and all r. Arguing as in the proof of 2.5, we obtain inequalities (*) and (**) with right-hand side changed to $\frac{1}{k}\Psi^{-1}(\varepsilon/r)$, which gives $r\Psi(\omega(k(x_p-x),r))<\varepsilon$ for $p>N$ and every $r\geq 1$. This implies $x_p-x\in X_{\zeta}$, as in the proof of 2.5. Moreover, $x_p\in X_{\zeta}$, and so $x\in X_{\zeta}$. Further, $\zeta(k(x_p-x))\leq\varepsilon$ for $p>N$. This implies that (x_n) is ζ-convergent to x. We have $\tilde{\zeta}(k(\tilde{x}_p-\tilde{x}))=\inf\{\zeta(y):y\in k(\tilde{x}_p-\tilde{x})\}\leq\zeta(k(x_p-x))\leq\varepsilon$ for $p>N$. Thus, $(\tilde{x}_n)$ is $\tilde{\zeta}$-convergent to $\tilde{x}$. Consequently, $\tilde{X}_{\zeta}$ is $\tilde{\zeta}$-complete. Finally, we prove $\tilde{X}(\Psi)$ to be $\tilde{\zeta}$-closed in $\tilde{X}_{\zeta}$. Let $\tilde{x}_p\in\tilde{X}(\Psi)$, $\tilde{x}_p\xrightarrow{\tilde{\zeta}}\tilde{x}$. Then $\tilde{x}\in\tilde{X}_{\zeta}$ and $\tilde{\zeta}(2\lambda(\tilde{x}_p-\tilde{x}))\longrightarrow 0$ as $p\longrightarrow\infty$, for some $\lambda>0$. Arguing as in the first part of the proof we obtain that $\zeta(\lambda(x_p-x))\longrightarrow 0$ as $p\longrightarrow\infty$. This implies $x\in X(\Psi)$, i.e. $\tilde{x}\in\tilde{X}(\Psi)$, as in the proof of 2.5.

3. One may ask also the question, whether Theorems 2.5 and 2.6 remain true, if we replace the space $\tilde{X}(\Psi)$ by $\tilde{X}(\Phi,\Psi)$, or equivalently, whether $\tilde{X}(\Phi,\Psi)$ is a closed subspace of $\tilde{X}(\Psi)$ with respect to the F-norm $|\cdot|_{\zeta}$, or the modular $\tilde{\zeta}$. A negative answer to this question is provided by

the example $\Phi(u)=|u|$, $\Psi(u)=u^2$ and $x=(t_i)_{i=0}^{\infty}$, $x_n=(t_i^n)_{i=0}^{\infty}$, where $t_i=(-1)^i/(i+1)$, $t_i^n=t_i$ for $i\leq n$, $t_i^n=0$ for $i>n$. Obviously, $x_n\in X(\Phi,\Psi)$, $x\in X(\Psi)$, but $x\notin X_\Phi$. This negative answer leads to putting the same question in context of two-modular convergence in $\tilde{X}(\Phi,\Psi)$.

3.1. Let us recall the notion of two-modular convergence (γ-convergence), (see [6] or [4], p. 169). Let $<X, \zeta', \zeta>$ be a triple, where ζ' and ζ are two modulars in a vector space X. A set $K=\{x\in X_{\zeta'}: \zeta'(k_0x)\leq M_0\}$ with some k_0, $M_0>0$ is called a ζ'-ball. A sequence (x_n), $x_n\in X$ is called ζ'-bounded, if the sequence $(\varepsilon_n x_n)$ is ζ'-convergent to 0 for every sequence of numbers $\varepsilon_n\longrightarrow 0$. If (x_n) is ζ'-bounded, then $x_n\in K$, $n=1, 2, \ldots$, for some $k_0, M_0>0$ (see [6] or [4], 5.5). A sequence (x_n) is called γ-convergent to x, $x_n\xrightarrow{\gamma}x$, if (x_n) is ζ'-bounded and ζ-convergent to x. The two-modular space, i.e. the triple $<X, \zeta', \zeta>$ is called γ-complete, if for every fixed ζ'-ball K and every sequence (x_n), $x_n\in K$, which is ζ-Cauchy, there exists an element $x\in K$ such that $x_n\xrightarrow{\gamma}x$.

We are going now to investigate the two-modular space $<\tilde{X}(\Phi,\Psi), \tilde{w}_\Phi, \tilde{\zeta}>$, where $\tilde{w}_\Phi(\tilde{x})=\inf\{w_\Phi(y): y\in\tilde{x}\}$.

Let us remark that $\tilde{w}_\Phi(\tilde{x})=w_\Phi(\bar{x})$, where $x=(t_i)_{i=0}^{\infty}$, $\bar{x}=(\bar{t}_i)_{i=0}^{\infty}$, $\bar{t}_0=t_1$, $\bar{t}_i t_i$ for $i\geq 1$. Obviously, $\bar{x}\in\tilde{x}$, and so $\tilde{w}_\Phi(\tilde{x})\leq w_\Phi(\bar{x})$. Now, let $y=(s_i)_{i=0}^{\infty}\in\tilde{x}$, then $s_i-t_i=k$ for $i=1, 2, \ldots$ with some constant k. Denoting $\bar{y}=(\bar{s}_i)_{i=0}^{\infty}$, where $\bar{s}_0=t_1+k$, $\bar{s}_i\bar{t}_i+k$ for $i\geq 1$, we have $w_\Phi(y)\geq w_\Phi(\bar{y})=w_\Phi(\bar{x})$. This implies $\tilde{w}_\Phi(\tilde{x})\geq w_\Phi(\bar{x})$.

3.2. THEOREM. *Let Φ be a φ-function and let Ψ be an increasing, continuous function of $u\geq 0$, satisfying the condition 2.4(+) and such that $\Psi(0)=0$. Then the two-modular space*

$$<\tilde{X}(\Phi,\Psi),\ \tilde{w}_\Phi,\ \tilde{\zeta}>$$

is γ-complete.

PROOF. Let $\tilde{K}$ be a $\tilde{w}_\Phi$-ball in $\tilde{X}(\Phi,\Psi)$ and let $\tilde{x}_n\in\tilde{K}$ for $n=1, 2, \ldots$, $(\tilde{x}_n)$ be $\tilde{\zeta}$-Cauchy. By 2.6, $(\tilde{x}_n)$ is $\tilde{\zeta}$-convergent to an element $\tilde{x}\in\tilde{X}(\Psi)$. Hence $\tilde{x}_n\xrightarrow{\gamma}\tilde{x}$. We have to show that $\tilde{x}\in\tilde{K}$. It is easily seen that taking $x_n\in\tilde{x}_n$, $x_n\in X_\Phi$ in such a manner that the first two coordinates of x_n are the same, we have $w_\Phi(k_0x_n)\leq M_0$ for some $k_0, M_0>0$.

Thus, writing $x_n=(t_i)_{i=0}^{\infty}$, we have

$$\sum_{i=1}^{\infty}\Phi(k_0|t_{n_i}^p-t_{n_{i-1}}^p|)\leq M_0$$

for $p=1, 2, \ldots$ and any increasing sequence (n_i) of positive integers. Since $t_i^p\longrightarrow t_i$ as $p\longrightarrow\infty$, where $x=(t_i)_{i=0}^{\infty}$, we obtain easily

$$\sum_{i=1}^{\infty}\Phi(k_0|t_{n_i}-t_{n_{i-1}}|)\leq M_0$$

whence $w_\Phi(k_0x)\leq M_0$. Consequently, $\tilde{w}_\Phi(k_0\tilde{x})\leq M_0$, i. e. $\tilde{x}\in\tilde{K}$.

4. Let Φ be a φ-function and let Ψ be an increasing, continuous function for $u\geq 0$ such that $\Psi(0)=0$. We apply now the γ-convergence in $\tilde{X}(\Phi, \Psi)$ in order to obtain an approximation theorem by means of the m-translation, i. e. a result of the form $\tau_m x-x\longrightarrow 0$ in an Orlicz sequence space 1^Γ with a φ-function Γ satisfying the following condition:

(i) there exist positive constants a, b, u_0 such that $\Gamma(au)\leq b\Phi(u)\Psi(u)$ for $0\leq u\leq u_0$.

It is easily seen that (i) implies, that for every $u_1\geq 0$ there exists a $c>0$ such that $\Gamma(cu)\leq b\Phi(u)\Psi(u)$ for $0\leq u\leq u_1$; indeed, if $u_1\leq u_0$ we may take $c=a$, and if $u_1>u_0$, we may put $c=au_0(u_1)^{-1}$.

4. 1. LEMMA. *Let the assumptions of* 4 *be satisfied and let* $w_\Phi(\lambda x)<\infty$ *for a* $\lambda>0$. *Then*

$$\sum_{i=1}^{\infty}\Gamma(c\lambda|(\tau_r x)_i-(x)_i|)\leq br\Psi(\omega(\lambda x, r))w_\Phi(\lambda x)$$

for every $r\geq 0$.

PROOF. Since $x=(t_i)_{i=0}^{\infty}$ is bounded, so taking $u_1=2\lambda\sup_i|t_i|$, fixing r and choosing $m\geq r$ arbitrarily, we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}\Gamma(c\lambda|(\tau_m x)_i-(x)_i|)&=\sum_{i=m}^{\infty}\Gamma(c\lambda|t_{m+i}-t_i|)\\ &\leq b\Psi(\omega(\lambda x, r))\sum_{i=m}^{\infty}\Phi(\lambda|t_{m+i}-t_i|)\\ &\leq b\Psi(\omega(\lambda x, r))\sum_{k=1}^{\infty}\sum_{i=km}^{(k+1)m-1}\Phi(\lambda|t_{m+i}-t_i|)\\ &\leq b\Psi(\omega(\lambda x, r))\sum_{j=m}^{2m-1}\sum_{k=1}^{\infty}\Phi(\lambda|t_{km+j}-t_{(k-1)m+j}|)\\ &\leq b\Psi(\omega(\lambda x, r))mw_\Phi(\lambda x).\end{aligned}$$

Taking $m=r$, we get the required ineguality.

4. 2. THEOREM. *Let* Φ *and* Γ *be* φ*-functions and let* Ψ *be an increasing, continuous function for* $u\geq 0$, $\Psi(0)=0$, *such that* 4(i) *holds. Let* $x\in\tilde{x}\in\tilde{X}(\Phi, \Psi)$. *Then* $\tau_r x-x\in 1^\Gamma$ *for all* $r\geq 0$, *and* $\tau_r x-x\longrightarrow 0$ *in the sense of modular convergence in* 1^Γ.

PROOF. Since $x\in X(\Phi, \Psi)$, so $w_\Phi(\lambda x)<\infty$ and $r\Psi(\omega(\lambda x, r))\longrightarrow 0$ as

$r \longrightarrow \infty$ for sufficiently small $\lambda > 0$. By Lemma 4.1, $\tau_r x - x \in 1^{\Gamma}$ for all $r \geq 0$. Also, taking $r \longrightarrow \infty$ in the inequality of Lemma 4.1, we obtain $\tau_r x - x \longrightarrow 0$ in the sense of modular convergence in 1^{Γ}.

4.3. LEMMA. *Let* $x_n = (t_i^n)_{i=0}^{\infty} \in X_{\Phi}$, $t_0^n = 0$, *for* $n = 1, 2, \dots$, *and let* $x_n \in K$, *where* K *is a* w_{Φ}*-ball in* X_{Φ}. *Then there is a constant* $L > 0$ *such that* $|t_i^n| \leq L$ *for* $i = 0, 1, 2, \dots$ *and* $n = 1, 2, \dots$.

PROOF. Let $w_{\Phi}(k_0 x_n) \leq M_0$ for $n = 1, 2, \dots$ with some k_0, $M_0 > 0$, then $\Phi(k_0 |t_i^n|) = \Phi(k_0 |t_i^n - t_0^n|) \leq M_0$, and so $|t_i^n| \leq L$ for some $L > 0$, because $\Phi(u) \longrightarrow \infty$ as $u \longrightarrow \infty$.

4.4. THEOREM. *Let the same assumptions as in* 4.3 *be satisfied. Let* $\tilde{x}_n \in \tilde{X}(\Phi, \Psi)$, $\tilde{x}_n \xrightarrow{\gamma} 0$ *in* $< \tilde{X}(\Phi, \Psi), \tilde{w}_{\Phi}, \tilde{\zeta} >$ *as* $n \longrightarrow \infty$ *and* $x_n (t_i^n)_{i=0}^{\infty} \in \tilde{x}_n$, $t_0^n = 0, t_1^n = 0$ *for* $n = 1, 2, \dots$. *Then* $\tau_r x_n - x_n \longrightarrow 0$ *with respect to modular convergence in* 1^{Γ} *as* $n \longrightarrow \infty$, *uniformly for* $r \geq 0$.

PROOF. Since $\tilde{x}_n \xrightarrow{\gamma} 0$, so $\tilde{x}_n \in \tilde{K}$, where $\tilde{K}$ is a $\tilde{w}_{\Phi}$-ball. But $w_{\Phi}(k_0 \tilde{x}_n) \leq M_0$ with some $k_0, M_0 > 0$. By Lemma 4.3, $|t_i^n| \leq L$ for all i, n, with an $L > 0$. Let $u_1 = 2\lambda L$, $c = a u_0 / u_1$, where $0 < \lambda \leq k_0$. Then, by Lemma 4.1 we have

$$\sum_{i=0}^{\infty} \Gamma(c\lambda |(\tau_r x_n)_i - (x_n)_i|) \leq b\zeta(\lambda x_n) w_{\Phi}(\lambda x_n) \leq b\zeta(\lambda x_n) M_0.$$

By assumption there exists a $\lambda > 0$ such that for every $\varepsilon > 0$ there is an integer N for which $\tilde{\zeta}(2\lambda \tilde{x}_n) = \inf\{\zeta(y) : y \in 2\lambda \tilde{x}_n\} < \varepsilon$ for $n > N$. Hence there exist $y_n \in 2\lambda \tilde{x}_n$, $n > N$, such that $\zeta(y_n) < \varepsilon$. But $\frac{1}{2} y_n - \lambda x_n \in \bar{c}$. Arguing as in the proof of Theorem 2.6, with $z_1 = \frac{1}{2} y_n$, $z_2 = \lambda x_n$, we obtain $\zeta(\lambda x_n) \leq \zeta(y_n) < \varepsilon$ for $n > N$. Hence $\zeta(\lambda x_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

We are indebted to the refree for his kind remarks which enabled to improve the paper, especially in parts concerning the modular $\tilde{\zeta}$.

References

[1] T. M. JEDRYKA and J. MUSIELAK, Some remerks on F-modular spaces, Functiones et Approximatio 2 (1976), 83-100.

[2] J. MUSIELAK, Sequences of finite Φ-variation, Prace Matem. 6 (1961), 165-174 (in Polish).

[3] J. MUSIELAK, Modular approximation by a filtered family of linear operators, Analysis and Approximation, Preceed. Confer. Oberwolfach, August 9-16, 1980, Birkhäuser Verlag 1981, 99-110.

[4] J. MUSIELAK, Orlicz spaces and modular spaces, Lecture Notes in Math. 1034, Springer Verlag, Berlin-Heidelberg-New York-Tokyo 1983.

[5] J. MUSIELAK and W. ORLICZ, On modular spaces, Studia Math. 18 (1959), 49-65.
[6] J. MUSIELAK and A. WASZAK, On two-modular spaces, Commentationes Math. 23 (1983), 63-70.
[7] H. NAKANO, Topology and toptlogical linear spaces, Tokyo 1951.
[8] H. NAKANO, Generalized modular spaces, Studia Math. 31 (1968), 439-449.
[9] A. WASZAK, On convergence in some two-medular spaces, General Topology and its Relations to Modern Analysis and Algebra V, Heldermann Verlag, Berlin 1982, 674-682.

Institute of Mathematics,
Adam Mickiewicz University,