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Remarks on manifolds which admit locally free nilpotent Lie group actions

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0. Introduction

Let $\phi: G\times M \longrightarrow M$ be a smooth action of a connected Lie group G on a compact orientable manifold M. If for every point z of M the isotropy subgroup G_z is discrete, ϕ is said to be *locally free*. If the orbits of ϕ have codimension one, we call ϕ a *codimension one* action. Suppose that G is nilpotent and ϕ is a locally free codimension one action. Some dynamical properties of such an action ϕ and topological properties of M are stated in the paper [**HGM**]. We will consider this in detail. The object of this paper is to prove the following

THEOREM. *Let M be a connected closed orientable manifold. Suppose that M admits a locally free codimension one smooth action ϕ of a connected nilpotent Lie group G such that* i) *ϕ has no compact orbits and* ii) *the dimension of the commutator $[G, G]$ is one. Then M is homeomorphic to a nilmanifold i. e. the homogeneous space of a connected nilpotent Lie group.*

REMARK. (1) A compact nilmanifold always admits a locally free codimension one smooth action of a connected nilpotent Lie group which satisfies the above conditon i). (2) A Heisenberg group is a good example of a nilpotent Lie group which satisfies the above condition ii).

The theorem is a finer version of theorem (2.7) of [**HGM**] under the assumption ii).

Unless otherwise specified, we consider in the smooth (C^{∞}) category.

1. Unipotent flows on the space of lattices

Our method of proving the theorem is deeply concerned with characterization of a compact minimal set of a unipotent flow on the space of lattices. We describe it here.

Denote by $\mathscr{L}(k)$ the space of lattices in k-dimensional euclidean space $\boldsymbol{E}$ (cf. [C]). Fix a basis $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$ of $\boldsymbol{E}$. Then every element $\boldsymbol{b}$ of a lattice Λ has a expression $\boldsymbol{b}=\sum_i(\sum_j b_{ij}m_j)\boldsymbol{v}_i$ where m_j's are integers and (b_{ij}) is nonsingular matrix. Thus lattices can be represented as non-singular matrices, but this representation is not unique. If A and B are two matrix representations of the same lattice, the coefficients of $A^{-1}B$ are integers and its determinant is ± 1. That is, $\mathscr{L}(k)$ can be regarded as $GL(k, \boldsymbol{R})/\{\pm 1\} \times SL(k, \boldsymbol{Z})$.

Let $f: \boldsymbol{E} \longrightarrow \boldsymbol{E}$ be a nilpotent linear map, $\exp tf$ be the exponential of tf and $U_t: \mathscr{L}(k) \longrightarrow \mathscr{L}(k)$ be a map induced by $\exp tf$. We say the action $U: \boldsymbol{R} \times \mathscr{L}(k) \longrightarrow \mathscr{L}(k)$ the *unipotent flow* on $\mathscr{L}(k)$ defined by f. Recall that a *minimal set* of U is a nonempty, closed, U-invariant set which is also minimal with respect to these properties. Assume that there exists a compact minimal set $\mathscr{M}$. Then we obtain the following result for an element of $\mathscr{M}$.

(1.1) LEMMA. *If the dimension of* $\operatorname{Im}(f)$ *is one, then there exist a basis* $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$ *of* $\boldsymbol{E}$ *and an integer* $p (1 \leqq p \leqq k)$ *such that* $\boldsymbol{v}_2, \cdots, \boldsymbol{v}_k$ *spans* $\operatorname{Ker}(f)$, $f(\boldsymbol{v}_1) = \boldsymbol{v}_k$ *and to the basis* $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$ *every element of* $\mathscr{M}$ *is represented by a matrix* (a_{ij}) *which satisfies* $a_{ij} = 0$ *for* $1 \leqq i \leqq p$ *and* $p+1 \leqq j \leqq k$.

PROOF: Choose a basis $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$ such that $\boldsymbol{v}_2, \cdots, \boldsymbol{v}_k$ spans $\operatorname{Ker}(f)$ and $f(\boldsymbol{v}_1) = \boldsymbol{v}_k$ and fix an element Λ of $\mathscr{M}$. Let (b_{ij}) be a matrix representation of Λ. Without loss of generality, we may assume that $|\det(b_{ij})| = 1$ because $\det(U_t) = 1$. And we suppose that an element of $\boldsymbol{E}$ is expressed by its coefficient to the basis $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_k$, that is, $\boldsymbol{b} = \sum_i b_i \boldsymbol{v}_i$ is expressed to $(b_1, \cdots, b_k)$. Thus $U_t(\boldsymbol{b}) = (b_1, \cdots, b_{k-1}, tb_1 + b_k)$ where $\boldsymbol{b} = (b_1, \cdots, b_k)$ and $\boldsymbol{b}_1 = (b_{11}, \cdots, b_{k1}), \cdots, \boldsymbol{b}_k = (b_{1k}, \cdots, b_{kk})$ is a basis of Λ.

Assume that $b_{11}, \cdots, b_{1k}$ are independent over $\boldsymbol{Z}$ where $\boldsymbol{Z}$ is the ring of integers. We will show that this assumption contradicts to the compactness of $\mathscr{M}$. Since $\mathscr{M}$ is compact, there exists a ball $B(\varepsilon)$ centered at $\mathbf{O}$ with radius $\varepsilon > 0$ such that every element of $\mathscr{M}$ has no point in $B(\varepsilon)$ other than $\mathbf{O}$. For this ε and (b_{ij}), from [C, Theorem III, page 73] it follows that there exists a point $\boldsymbol{b} = m_1 \boldsymbol{b}_1 + \cdots + m_k \boldsymbol{b}_k = (b_1, \cdots, b_k)$ of Λ other than $\mathbf{O}$ such that $|b_j| \leqq \varepsilon k^{-1}$ for $1 \leqq j < k$ and $|b_k| < \varepsilon^{1-k} k^{k-1}$. Since $b_{11}, \cdots, b_{1k}$ are independent over $\boldsymbol{Z}$, $b_1 \neq 0$. It follows that there exists a real number t such that $U_t(\boldsymbol{b}) = (b_1, \cdots, b_{k-1}, 0)$ and therefore $\|U_t(\boldsymbol{b})\| < \varepsilon$. Since $U_t(\Lambda)$ is an element of $\mathscr{M}$, this induces a contradiction.

Since $b_{11}, \cdots, b_{1k}$ are not independent over $\boldsymbol{Z}$, it follows that there exist co-prime integers $m_1, \cdots, m_k$ such that $m_1 b_{11} + \cdots + m_k b_{1k} = 0$. If a point $\boldsymbol{b}$ of Λ is of the form $\boldsymbol{b} = u\boldsymbol{d}$ where u is real number and $\boldsymbol{d} = m_1 \boldsymbol{b}_1 + \cdots + m_k \boldsymbol{b}_k$, then u is an integer because m_j's are co-prime. From [C, Corollary 3, page 14] we see that there exists a basis $\boldsymbol{d}_1, \cdots, \boldsymbol{d}_k = \boldsymbol{d}$ of Λ, that is, there exists an another matrix representation (d_{ij}) of Λ such that $d_{1k}=0$. Again by [**C**, Theorem III, page 73] there exists a point $\boldsymbol{d}=m_1\boldsymbol{d}_1+\cdots+m_k\boldsymbol{d}_k=(d_1, \cdots, d_k)$ of Λ other than $\mathbf{O}$ such that $|d_j|\leqq \varepsilon k^{-1}$ for $1\leqq j<k$ and $|d_k|<\varepsilon^{1-k}k^{k-1}$.

(1) If $d_{11}, \cdots, d_{1k-1}$ are independent over $\boldsymbol{Z}$ and $d_1=0$, then $d_{ik}=0$ for $1\leqq i\leqq k-1$. In fact, ε can be chosen such as $\varepsilon k^{-1}<|d_{ik}|$ for $1\leqq i\leqq k-1$ with $d_{ik}\neq 0$ and $\boldsymbol{d}=m_k\boldsymbol{d}_k=(o, m_kd_{2k}, \cdots, m_kd_{kk})$ from the assumption.

(2) If $d_{11}, \cdots, d_{1k-1}$ are independent over $\boldsymbol{Z}$ and $d_1\neq 0$, by the same argument as to $b_{11}, \cdots, b_{1k}$, we see that this case does not occur.

(3) If $d_{11}, \cdots, d_{1k-1}$ are not independent over $\boldsymbol{Z}$, by the same argument as above we obtain new basis $\boldsymbol{f}_1, \cdots, \boldsymbol{f}_{k-1}, \boldsymbol{f}_k=\boldsymbol{d}_k$ of Λ such that $f_{1k-1}=f_{1k}=0$ where (f_{ij}) is its new matrix representation. If necessary, by selecting an another basis of $\boldsymbol{E}$, we can assume that $f_{ik}=0$ for $1\leqq i\leqq k-2$. Therefore by the same argument as in (1), we see that $d_{ik-1}=0$ for $1\leqq i\leqq k-2$.

In this way, this series of arguments and the minimality of $\mathscr{M}$ induces a complete proof of the lemma.

The following lemma is an easy consequence to lemma (1. 1).

(1. 2) LEMMA. *Under the same assumption as in lemma* (1. 1), *there exists a non-trivial proper subspace* $\boldsymbol{E}_1$ *of* $\boldsymbol{E}$ *such that* $\boldsymbol{E}_1$ *contains* Im(f) *and therefore is invariant under* U_t *and every element* Λ *of* $\mathscr{M}$ *is uniform in* $\boldsymbol{E}_1$, *i. e.,* $\Lambda\cap\boldsymbol{E}_1$ *is a lattice in* $\boldsymbol{E}_1$.

2. Proof of the theorem

Let $\phi: G\times M\longrightarrow M$ be a locally free smooth action of a connected Lie group G on a manifold M. The orbits of a locally free smooth action ϕ are leaves of a foliation. We call the foliation the *orbit foliation* of the action ϕ. In particular, if G is nilpotent, the orbit foliation is called a *nilfoliation* in [**HGM**]. We shall quote the necessary facts from those of [**HGM**].

We assume that M is a connected, orientable closed manifold, G is a connected, simply connected nilpotent Lie group such that dim $G=$dim $M-1$ and ϕ is a locally free smooth action of G on M. Denote by $\mathscr{F}_\phi$ the orbit foliation of ϕ. $\mathscr{F}_\phi$ is a codimension one nilfoliation. For $z\in M$ let G_z denote the isotropy subgroup of ϕ at z and $\hat{G}_z$ denote the *Malcev closure* of G_z in G (i. e. the unique closed connected subgroup $\hat{G}_z$ of G such that G_z is contained in $\hat{G}_z$ and the homogeneous space $\hat{G}_z/G_z$ is compact). The following two results are in [**HGM**].

(2. 1) LEMMA. *Supose that* $\mathscr{F}_\phi$ *has no compact leaves. Then all leaves are dense in* M.

(2.2) LEMMA. *Under the same assumption as above, $\hat{G}_z$ is a fixed normal subgroup of G which is independent of the choice of z.*

Let N denote a fixed normal subgroup as in lemma (2.2), that is, $N=\hat{G}_z$. Now assume that ϕ has no compact orbits (in other words, $\mathscr{F}_\phi$ has no compact leaves) and ϕ is not free. Then we obtain the following

(2.3) LEMMA. *If the dimension of $[G, G]$ is equal to one then there exists a non-trivial closed connected normal subgroup K of G such that K is contained in the center of G and $K/K\cap G_z$ is compact (in this case we shall say that G_z is uniform in K) for any $z\in M$.*

If $[N, N]$ is non-trivial, $[N, N]$ satisfies the conditions of the lemma. In fact a uniform subgroup of N is uniform in $[N, N]$ (cf. [**Ra**]) and $[N, N]$ is contained in the center of G because G is 2-step. Now we consider the case that N is abelian. Denote by $\mathscr{L}(k)$ the space of lattices in N where $k=\dim N$. Define a map $\chi: M\longrightarrow\mathscr{L}(k)$ by $\chi(z)=G_z$ and an action Ad: $G\times\mathscr{L}(k)\longrightarrow\mathscr{L}(k)$ by $\mathrm{Ad}(g, \Lambda)=g\Lambda g^{-1}$ where Λ is a lattice. Then the following lemma is easily proved.

(2.4) LEMMA. *χ and* Ad *are continuous and the following diagram*

$$\begin{tikzcd} G\times M \arrow[r, "\phi"] \arrow[d, "\mathrm{id}\times\chi"'] & M \arrow[d, "\chi"] \\ G\times\mathscr{L}(k) \arrow[r, "\mathrm{Ad}"] & \mathscr{L}(k) \end{tikzcd}$$

is commutative (where id: $G\longrightarrow G$ *is the identity map).*

PROOF OF LEMMA (2.3): We will apply the result of section one in order to prove the lemma. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{n}$ be the Lie algebra of N such that $\mathfrak{n}\subset\mathfrak{g}$. Since N is abelian, we can identify N with $\mathfrak{n}$. If $\mathfrak{n}$ is not contained in the center of $\mathfrak{g}$, we can choose a basis $X_1, \cdots, X_{n-k}, Y_1, \cdots, Y_k$ of $\mathfrak{g}$ such that $Y_1, \cdots, Y_k$ spans $\mathfrak{n}$, Y_k is a element of $[\mathfrak{g}, \mathfrak{g}]$, $[X_1, Y_1]=Y_k$ and $[X_1, Y_j]=0$ for $2\leqq j\leqq k$. Setting $f=\mathrm{ad}X_1|\mathfrak{n}$, from lemmas (1.2), (2.1) and (2.4), it follows that there exists a non-trivial ideal $\mathfrak{n}_1$ of $\mathfrak{g}$ such that $\mathfrak{n}_1\subsetneqq\mathfrak{n}$ and a lattice G_z is uniform in $\mathfrak{n}_1$ for all point z of M. And we obtain a commutative diagram

$$\begin{tikzcd} G\times M \arrow[r, "\phi"] \arrow[d, "\mathrm{id}\times\chi_1"'] & M \arrow[d, "\chi_1"] \\ G\times\mathscr{L}(k_1) \arrow[r, "\mathrm{Ad}_1"] & \mathscr{L}(k_1) \end{tikzcd}$$

where $k_1=\dim(\mathfrak{n}_1)$, $\mathscr{L}(k_1)$ is the space of lattices in $\mathfrak{n}_1$, a continuous map $\chi_1: M\longrightarrow\mathscr{L}(k_1)$ is defined by the equation $\chi_1(z)=G_z\cap\mathfrak{n}_1$ and Ad_1 is an obvious action. If $\mathfrak{n}_1$ is not contained in the center, we apply the same argument as above to $\mathfrak{n}_1$. In this way, we obtain a non-trivial ideal $\mathfrak{k}$ (therefore normal closed subgroup K) such that $\mathfrak{k}$ (resp. K) is contained in the center of $\mathfrak{g}$ (resp. G) and G_z is uniform in $\mathfrak{k}$ (resp. K).

By lemma (2. 3) we can prove the theorem.

(2. 5) THEOREM. *Let M be a connected closed orientable manifold. Suppose that M admits a locally free condimension one smooth action ϕ of a connected nilpotent Lie group G such that* i) *ϕ has no compact orbits and* ii) *the dimension of the commutator $[G, G]$ is noe. Then M is homeomorphic to a nilmanifold.*

PROOF: By considering the universal covering projection $p:\tilde{G}\longrightarrow G$, we obtain a locally free action $\tilde{\phi}$ of $\tilde{G}$ on M which is compatible with ϕ, that is, $\tilde{\phi}=\phi\circ(p\times\mathrm{id})$ (where $\mathrm{id}: M\longrightarrow M$ is the identity map). Therefore we may assume that G is always simply connected without loss of generality. If the action ϕ is free, all leaves of $\mathscr{F}_\phi$ are homeomorphic to $\boldsymbol{R}^n$ where $n=\dim G$ (in this case, G must be abelian, c. f. [**HGM**]). Therefore by [**JM**] and [**Ro**], M is homeomorphic to an $(n+1)$-torus T^{n+1}. When ϕ is not free, we will apply the follwing result deduced from [**N**] (see also [**Ra**]).

(2. 6) LEMMA. *If $p:E\longrightarrow B$ is a principal T^k-bundle and B is homeomorphic (resp. diffeomorphic) to a compact nilmanifold, then total space E is homeomorphic (resp. diffeomorphic) to a compact nilmanifold.*

According to lemma (2. 3), there exists a connected closed subgroup K which is contained in the center of G. Denote by ϕ_K the restriction of ϕ to $K\times M$. Then the orbit foliation of ϕ_K is without holonomy (cf. [**HGM**] or [**I**]) from lemma (2. 1) and all leaves are compact. It follows that there exists a smooth fiber bundle $p_1: M\longrightarrow M_1$ whose fibers are leaves of the orbit foliation of ϕ_K. Since K is contained in the center of G, the fibration p_1 is a principal T^k-bundle and ϕ induces a locally free codimension one smooth action ϕ_1 of G/K on M_1 such that ϕ_1 has no compact orbits. In the same way, if ϕ_1 is not free, we obtain p_2, M_2 and ϕ_2. Thus we obtain a series of fibrations

$$M=M_0\xrightarrow{p_1}M_1\xrightarrow{p_2}M_2\longrightarrow\cdots\xrightarrow{p_r}M_r,$$

such that each $p_i: M_{i-1}\longrightarrow M_i$ is a principal T^k-bundle, each M_i admits an induced action ϕ_i and the action ϕ_r on M_r is free. Since M_r is homeomorphic to an m-torus T^m, by lemma (2. 6), it follows that M is homeomorphic to a nilmanifold. This completes the proof of the theorem.

(2.7) COROLLARY. *If a connected closed orientable manifold M admits a locally free codimension one smooth action of a Heisenberg group such that all orbits are non-compact, then M is homeomorphic to a nilmanifold.*

References

[C] J. W. S. CASSELS: An Introduction to The Geometry of Numbers. Springer-Verlag, (1959).

[HGM] G. HECTOR, E. GHYS and Y. MORIYAMA: On codimension one nilfoliations and a theorem of Malcev, preprint (1987).

[I] H. IMANISHI: On the theorem of Denjoy-Sacksteder for codimension one foliations without holonomy, J. Math. Ktyoto Univ. 14-3 (1974), 607-634.

[JM] G. JOUBERT et R. MOSSU: Feuilletages sans holonomie d'une variété fermée, C. R. Acad. S. Paris, 270-A (1970), 507-509.

[N] K. NOMIZU: On the cohomology of compact homogeneous spaces of nilpotent Lie groups, Ann. of Math. 59 (1954), 531-538.

[P] J. PLANTE: Foliations with measure preserving holonomy, Ann. of Math. 102 (1975), 327-361.

[Ra] M. S. RAGUNATHAN: Discrete Subgroups of Lie Groups, Springer-Verlag (1972).

[Ro] H. ROSENBERG: Foliations by planes, Topology 7 (1968), 131-141.

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