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Spectral relations and unitary mixing in semifinite von Neumann algebras

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Abstract

Let $\mathscr{M}$ be a semifinite von Neumann algebra on a separable Hilbert space with a faithful normal semifinite trace τ on $\mathscr{M}$. Between τ-measurable operators, the condition of unitary mixing and other similar ones are characterized in terms of the spectral relations such as the (sub)majorization. Among other things, it is proved that, for positive x and y in $L^1(\mathscr{M};\ \tau)$, x is in the $\|\cdot\|_1$-closed convex hull of the unitary orbit of y if and only if the majorization $ex \prec ey$ holds for every central projection e.

Introduction

As a noncommutative measure space, let $(\mathscr{M}, \tau)$ be a pair of a semifinite von Neumann algebra $\mathscr{M}$ and a faithful normal semifinite trace τ on $\mathscr{M}$. The noncommutative integration theory (in the semifinite case) was initiated by Segal [27] and Dixmier [10] (also [24]), and the noncommutative probability theory was developed by Umegaki [32]. The concept of τ-measurable operators was introduced by Nelson [22]. The space $\tilde{\mathscr{M}}$ of τ-measurable operators affiliated with $\mathscr{M}$ gives a nice foundation for the noncommutative L^p-spaces $L^p(\mathscr{M};\ \tau)$. The notion of generalized s-numbers of τ-measurable operators extends the usual s-numbers of compact operators and the decreasing rearrangements of measurable functions. This notion has been studied in some contexts by several authors (see [12, 14, 25, 28, 33] for instance). Recently Fack and Kosaki [13] established an extensive and unified exposition on generalized s-numbers of τ-measurable operators.

Between positive selfadjoint x and y in $\tilde{\mathscr{M}}$, the spectral relations of majorization $x \prec y$, submajorization $x \prec_w y$, spectral dominance $x \lesssim y$ and spectral equivalence $x \approx y$ are defined by means of the generalized s-numbers of x and y. The precise definitions of these will be given in § 1 of this paper. The notions of majorization and submajorization have been extensively studied in theory of matrices (see Marshall and Olkin [21] and Ando [3]). We discussed in [15] those spectral relations in connection with doubly (sub)stochasic maps on $\mathscr{M}$. Furthermore, when $\mathscr{M}$ is a factor, we characterized them by the conditions of unitary (or contraction) mixing. For instance, we showed there that, for positive x and y in $L^1(\mathcal{M};\tau)$, the majorization $x\prec y$ (resp. the submajorization $x\prec_w y$) is equivalent to the condition of x being in the $\|\cdot\|_1$-closed convex hull of $\{uyu^*: u\in\mathcal{M}$ unitary$\}$ (resp. $\{aya^*: a\in\mathcal{M}, \|a\|\leqq 1\}$). This condition of unitary mixing was investigated by Alberti and Uhlmann [2] in more general setup. In particular if $\mathcal{M}$ is commutative, then the above condition is nothing but $x=y$ (resp. $x\leqq y$). When $\mathcal{M}$ is not a factor, it would be quite naturally expected that the above condition of unitary (contraction) mixing follows from the (sub) majorization restricted on each central projection in $\mathcal{M}$, namely $ex\prec ey$ ($ex\prec_w ey$) for all central projection e. The aim of this paper is to establish such results. Our main tool is the reduction of general $(\mathcal{M},\tau)$ to factor cases by theory of direct integral decompositions (cf. [11, 30]).

In this paper, let $(\mathcal{M},\tau)$ be as above. § 1 contains definitions and notations on generalized s-numbers of τ-measurable operators and the spectral relations raised above. Also, for the convenience of the reader, we summarize the results in [15] concerning characterizations of those spectral relations in case of $\mathcal{M}$ being a factor. Although the central decomposition of $\mathcal{M}$ into factors is enough for our main purpose, we assume in §§ 2, 3 that $\mathcal{M}$ is expressed as the direct integral $\mathcal{M}=\int_\Gamma^\oplus \mathcal{M}(\gamma)d\nu(\gamma)$ of a measurable field $\gamma\longrightarrow\mathcal{M}(\gamma)$ of von Neumann algebras on a standard σ-finite measure space (Γ,ν). Then τ is expressed as the direct integral $\tau=\int_\Gamma^\oplus \tau_\gamma d\nu(\gamma)$ of a unique (a. e.) measurable field $\gamma\longrightarrow\tau_\gamma$ of faithful normal semifinite traces τ_γ on $\mathcal{M}(\gamma)$. Moreover each $x\in\tilde{\mathcal{M}}$ is decomposed by the direct integral $x=\int_\Gamma^\oplus x(\gamma)d\nu(\gamma)$ of a unique (a. e.) measurable field $\gamma\longrightarrow x(\gamma)$ of $x(\gamma)\in\tilde{\mathcal{M}}(\gamma)$. We refer to [7, 20, 23] for direct integral decompositions of unbounded operators. In § 2, for τ-measurable operators x and y, the componentwise spectral relations such as $x(\gamma)\prec y(\gamma)$ a. e. are characterized by the corresponding spectral relations restricted on each projection in the diagonal algebra ($\cong L^\infty(\Gamma;\nu)$). In § 3, the conditions such as the unitary mixing are shown to be equivalent to their componentwise conditions. Finally in § 4, we assume that the representing Hilbert space for $\mathcal{M}$ is separable. Then we establish the main results by decomposing $\mathcal{M}$ into factors and by combining the theorems obtained in §§ 2, 3 with the results in the factor case.

1. Preliminaries

Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful normal semifinite trace τ. A densely-defined closed operator x affiliated with $\mathscr{M}$ is said to be *τ-measurable* if there is, for each $\delta>0$, a projection e in $\mathscr{M}$ such that $e\mathscr{H}\subseteq\mathscr{D}(x)$ and $\tau(1-e)<\delta$. We denote by $\tilde{\mathscr{M}}$ the set of all τ-measurable operators affiliated with $\mathscr{M}$. Then $\tilde{\mathscr{M}}$ becomes a complete Hausdorff topological $*$-algebra in the measure topology with respect to strong sum $\overline{x+y}$ and strong product $\overline{xy}$ (denoted simply by $x+y$ and xy in this paper). Here the measure topology on $\tilde{\mathscr{M}}$ is the linear topology given by the fundamental system $\{\mathscr{O}(\varepsilon,\delta):\varepsilon,\ \delta>0\}$ of neighborhoods of 0 where

$$\mathscr{O}(\varepsilon,\delta)=\{x\in\tilde{\mathscr{M}}:\ \|xe\|\leqq\varepsilon \text{ and } \tau(1-e)\leqq\delta \text{ for some projection } e \text{ in } \mathscr{M}\}.$$

For details on τ-measurable operators, see [22, 31]. For each subspace $\mathscr{L}$ of $\tilde{\mathscr{M}}$, the set of all selfadjoint (resp. positive selfadjoint) elements in $\mathscr{L}$ is denoted by $\mathscr{L}_{sa}$ (resp. $\mathscr{L}_+$). For $1\leqq p<\infty$, $L^p(\mathscr{M})=L^p(\mathscr{M};\tau)$ is the non-commutative L^p-space on $(\mathscr{M},\tau)$, that is, the Banach space consisting of all $x\in\tilde{\mathscr{M}}$ such that the norm $\|x\|_p=\tau(|x|^p)^{1/p}$ is finite (see [10, 22, 27]). Moreover let $\tilde{\mathfrak{S}}$ be the closure of $L^1(\mathscr{M})$ in $\tilde{\mathscr{M}}$ in the measure topology. Then $\tilde{\mathfrak{S}}$ includes all $L^p(\mathscr{M})$, $1\leqq p<\infty$. If τ is finite (i.e. $\tau(1)<\infty$), then $\tilde{\mathfrak{S}}=\tilde{\mathscr{M}}$ is the set of all densely-defined closed operators affiliated with $\mathscr{M}$.

For each $x\in\tilde{\mathscr{M}}_{sa}$ and each interval I in $\boldsymbol{R}$, let $e_I(x)$ denote the spectral projection of x corresponding to I. For $x\in\tilde{\mathscr{M}}$ and $t>0$, the *generalized s-number* $\mu_t(x)$ is defined by

$$\mu_t(x)=\inf\{s\geqq 0:\ \tau(e_{(s,\infty)}(|x|))\leqq t\}.$$

A complete exposition on generalized s-numbers is found in [13]. Furthermore, when $\tau(1)<\infty$, we define the *spectral scale* $\lambda_t(x)$ of $x\in\tilde{\mathscr{M}}_{sa}$ by

$$\lambda_t(x)=\inf\{s\in\boldsymbol{R}:\ \tau(e_{(s,\infty)}(x))\leqq t\},\quad 0<t<\tau(1).$$

Obviously, if $x\in\tilde{\mathscr{M}}_+$, then $\lambda_t(x)=\mu_t(x)$ for $t\in(0,\tau(1))$. The properties of spectral scales are analogous to those of generalized s-numbers (cf. [26], [15, § 6]).

In particular, let $\mathscr{M}$ be a factor of type I, namely, $\mathscr{M}=\boldsymbol{B}(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$. Then $\tilde{\mathscr{M}}$ is $\boldsymbol{B}(\mathscr{H})$ itself with the usual norm topology and $\tilde{\mathfrak{S}}$ is the algebra of all compact operators on $\mathscr{H}$. In this case, the generalized s-numbers $\mu_t(x)$ of a compact operator x mean the usual singular values of x arranged in decreasing order. When $\mathscr{M}$ is the $n\times n$ matrix algebra, the spectral scale $\lambda_t(x)$ of a Hermitian matrix x means the eigenvalues of x in decreasing order.

Next, let $\mathscr{M}$ be commutative, namely, $\mathscr{M}=L^\infty(\Omega;m)$ and $\tau(f)=$

$\int_\Omega f dm$ on a localizable measure space (Ω, m). Then $\tilde{\mathscr{M}}$ consists of all measurable functions on Ω bounded except on m-finite sets. For a real measurable function f on Ω, the *decreasing rearrangement* f^* of f (cf. [6, 29]) is given by

$$f^*(t)=\inf\{s\in\boldsymbol{R}: m(\{\omega\in\Omega: f(\omega)>s\})\leqq t\},\quad 0<t<m(\Omega).$$

If $f\in\tilde{\mathscr{M}}$, then $\mu_t(f)=|f|^*(t)$ for all $t\in(0, m(\Omega))$. If $m(\Omega)<\infty$ and f is real measurable on Ω, then $\lambda_t(f)=f^*(t)$ for all $t\in(0, m(\Omega))$.

Here we note that an $x\in\tilde{\mathscr{M}}$ belongs to $\tilde{\mathfrak{S}}$ if and only if $\tau(e_{(s,\infty)}(|x|))<\infty$ for all $s>0$, or equivalently $\lim_{t\to\infty}\mu_t(x)=0$ (see [13, Proposition 3.2] and [15, Proposition 1.3]). Also x belongs to the algebraic sum $L^1(\mathscr{M})+\mathscr{M}$ if and only if $\int_0^s\mu_t(x)dt<\infty$ for some (hence all) $s>0$ (see [15, Proposition 1.2]).

Several spectral relations between τ-measurable operators will be considered in this paper. For x, $y\in\tilde{\mathscr{M}}_+$, x is said to be *submajorized* by y, in notation $x\lessdot y$, if $\int_0^s\mu_t(x)dt\leqq\int_0^s\mu_t(y)dt$ for all $s>0$. Furthermore x is said to be *majorized by* y, in notation $x\prec y$, if $x\lessdot y$ and $\int_0^\infty\mu_t(x)dt=\int_0^\infty\mu_t(y)dt$ (i. e. $\tau(x)=\tau(y)$ permitting the value ∞). When $\tau(1)<\infty$, these are extended to x, $y\in\tilde{\mathscr{M}}_{sa}$ as follows: $x\lessdot y$ if $\int_0^s\lambda_t(x)dt\leqq\int_0^s\lambda_t(y)dt$ for all $s\in(0, \tau(1))$, and $x\prec y$ if $x\lessdot y$ and $\int_0^{\tau(1)}\lambda_t(x)dt=\int_0^{\tau(1)}\lambda_t(y)dt$ (i. e. $\tau(x)=\tau(y)$ permitting $\pm\infty$).

Two further spectral relations will be considered. For x, $y\in\tilde{\mathscr{M}}_+$, we say that x is *spectrally dominated* by y, in notation $x\lesssim y$, if $\mu_t(x)\leqq\mu_t(y)$ for all $t>0$, or equivalently if $\tau(e_{(s,\infty)}(x))\leqq\tau(e_{(s,\infty)}(y))$ for all $s\geqq0$. We say that x is *spectrally equivalent* to y, in notation $x\approx y$, if $x\lesssim y$ and $y\lesssim x$, namely $\mu_t(x)=\mu_t(y)$ for all $t>0$. When $\tau(1)<\infty$, $x\lesssim y$ and $x\approx y$ are defined for x, $y\in\tilde{\mathscr{M}}_{sa}$ with the use of $\lambda_t(\cdot)$ in place of $\mu_t(\cdot)$.

For each $y\in\tilde{\mathscr{M}}$, we define the subsets $U(y)$ and $C(y)$ of $\tilde{\mathscr{M}}$ by

$$U(y)=\{uyu^*: u\in U(\mathscr{M})\},$$
$$C(y)=\{aya^*: a\in\mathscr{M}_1\},$$

where $U(\mathscr{M})$ is the set of all unitaries in $\mathscr{M}$ and $\mathscr{M}_1=\{a\in\mathscr{M}: \|a\|\leqq1\}$. The convex hull of $U(y)$ (resp. $C(y)$) is denoted by conv $U(y)$ (resp. conv $C(y)$). For x, $y\in\tilde{\mathscr{M}}$, the condition $x\in U(y)$ means that x is *unitarily equivalent* to y in the exact sense. But it is suitable for our study to consider by [13, Lemma 4.1]. The same consideration works as well for $x, y\in L^1(\mathscr{M})_{sa}$ when $\tau(1)<\infty$. Hence the assertion (1) is obtained.

REMARK 1.4. In particular when $\mathscr{M}=\boldsymbol{B}(\mathscr{H})$, it is known (cf. [4, 9]) that, for each normal operators x and y on $\mathscr{H}$, x is in the norm-closure of $U(y)$ if and only if they have the same *crude multiplicity function*. Here the crude multiplicity function of x is the function on $\boldsymbol{C}$ which assigns the cardinal number $\inf_{r>0}[\text{rank } e_{D(\zeta,r)}(x)]$ to each $\zeta\in\boldsymbol{C}$, where $e_{D(\zeta,r)}(x)$ is the spectral projection of x corresponding to the disk $D(\zeta, r)$ of center ζ and radius r. The generalized s-numbers of a compact $x\in\boldsymbol{B}(\mathscr{H})_+$ have as much information as its crude multiplicity function. So Theorem 1.1(4) in case of $\mathscr{M}=\boldsymbol{B}(\mathscr{H})$ is regarded as a special case of the above result. However, as immediately seen, Theorem 1.1(4) fails to hold for general $x, y\in\boldsymbol{B}(\mathscr{H})_+$.

2. Spectral relations for direct integrals

Throughout this and next sections, let $\mathscr{M}$ be a semifinite von Neumann algebra on $\mathscr{H}$ expressed as the direct integral $\{\mathscr{M},\mathscr{H}\}=\int_\Gamma^\oplus\{\mathscr{M}(\gamma),\mathscr{H}(\gamma)\}d\nu(\gamma)$ of a measurable field $\gamma\longrightarrow\{\mathscr{M}(\gamma),\mathscr{H}(\gamma)\}$ of von Neumann algebras on a standard σ-finite measure space $(\Gamma, \mathscr{B}, \nu)$. We assume without loss of generality that $\mathscr{M}(\gamma)$ is semifinite for every $\gamma\in\Gamma$. Given a faithful normal semifinite trace τ on $\mathscr{M}$, there exists a unique (in the a. e. sense) measurable field $\gamma\longrightarrow\tau_\gamma$ of faithful normal semifinite traces τ_γ on $\mathscr{M}(\gamma)$ such that $\tau=\int_\Gamma^\oplus\tau_\gamma d\nu(\gamma)$ (see [11, §II.5]). For each $\gamma\in\Gamma$, 1_γ denotes the identity operator on $\mathscr{H}(\gamma)$. Let $\mathscr{A}$ be the diagonal algebra, that is, $\mathscr{A}=\int_\Gamma^\oplus\boldsymbol{C}1_\gamma d\nu(\gamma)\cong L^\infty(\Gamma ; \nu)$. The spaces $\tilde{\mathscr{M}}(\gamma)$, $\tilde{\mathfrak{S}}(\gamma)$ and $L^p(\mathscr{M}(\gamma))$ with respect to $(\mathscr{M}(\gamma), \tau_\gamma)$ are defined as well as $\tilde{\mathscr{M}}$, $\tilde{\mathfrak{S}}$ and $L^p(\mathscr{M})$ with respect to $(\mathscr{M}, \tau)$.

If x is a densely-defined closed operator on $\mathscr{H}$ affiliated with $\mathscr{M}$, then there exists a unique (a. e.) measurable field $\gamma\longrightarrow x(\gamma)$ of closed operators such that $x=\int_\Gamma^\oplus x(\gamma)d\nu(\gamma)$ (cf. [7, 20, 23]). For convenience, we give in the next lemma some elementary facts on direct integral decompositions of τ-measurable operators.

LEMMA 2.1. *Let* $x, y\in\tilde{\mathscr{M}}$, $x=\int_\Gamma^\oplus x(\gamma)d\nu(\gamma)$ *and* $y=\int_\Gamma^\oplus y(\gamma)d\nu(\gamma)$. *Then*:

(1) $\tau(e_{(s,\infty)}(|x|))=\int_\Gamma\tau_\gamma(e_{(s,\infty)}(|x(\gamma)|))d\nu(\gamma),\ s\geqq 0.$

that x is in the closure of $U(y)$ in the measure topology (or in the norm $\|\cdot\|_p$ when $y \in L^p(\mathcal{M})$). Also the condition of x being in the closure of $\mathrm{conv}\, U(y)$ means that x is *more unitarily mixed* than y. The similar conditions are considered with $C(y)$ in place of $U(y)$. Our main purpose is to characterize those conditions in terms of the spectral relations. This will be done in § 4. Such characterizations were obtained in [15] (also [18, 19]) when $\mathcal{M}$ is a factor. For convenience, we summarize them in the following:

THEOREM 1.1. *Assume that $\mathcal{M}$ is a semifinite factor. Then:*

(1) *If $x \in \tilde{\mathcal{M}}_+$ and $y \in L^1(\mathcal{M})_+$ (or $x \in \tilde{\mathcal{M}}_{sa}$ and $y \in L^1(\mathcal{M})_{sa}$ when $\tau(1) < \infty$), then $x \prec y$ if and only if x is in the $\|\cdot\|_1$-closure of* $\mathrm{conv}\, U(y)$.

(2) *If $x \in \tilde{\mathcal{M}}_+$ and $y \in (L^1(\mathcal{M}) + \mathcal{M})_+ \cap \tilde{\tilde{\mathfrak{S}}}$, then $x \underset{\cdot}{\prec} y$ if and only if x is in the closure of* $\mathrm{conv}\, C(y)$ *in the measure topology. If $x \in \tilde{\mathcal{M}}_+$ and $y \in L^p(\mathcal{M})_+$ with $1 \leq p < \infty$, then $x \underset{\cdot}{\prec} y$ if and only if x is in the $\|\cdot\|_p$-closure of* $\mathrm{conv}\, C(y)$.

(3) *If $x \in \tilde{\mathcal{M}}_+$ and $y \in (L^1(\mathcal{M}) + \mathcal{M})_+ \cap \tilde{\tilde{\mathfrak{S}}}$, then $x \lesssim y$ if and only if x is in the closure of $C(y)$ in the measure topology.*

(4) *If $x \in \tilde{\mathcal{M}}_+$ and $y \in \tilde{\tilde{\mathfrak{S}}}_+$ (or $x, y \in \tilde{\mathcal{M}}_{sa}$ when $\tau(1) < \infty$), then $x \approx y$ if and only if x is in the closure of $U(y)$ in the measure topology.*

REMARK 1.2. As to the above assertions (2) (the first part) and (3), we showed them in [15] in case of $x \in \tilde{\tilde{\mathfrak{S}}}_+$ and $y \in (L^1(\mathcal{M}) + \mathcal{M})_+$. But they hold in the above case too. In fact, let $x \in \tilde{\mathcal{M}}_+$ and $y \in (L^1(\mathcal{M}) + \mathcal{M})_+ \cap \tilde{\tilde{\mathfrak{S}}}$. Then $C(y)$ is included in $\tilde{\tilde{\mathfrak{S}}}$ (cf. [13, Lemma 2.5]) and hence so is the closure of $\mathrm{conv}\, C(y)$ in the measure topology. Also, if $x \underset{\cdot}{\prec} y$, then we get $x \in \tilde{\tilde{\mathfrak{S}}}$ (see [15, Proposition 2.1]). Hence each condition relevant to (2) or (3) implies $x \in \tilde{\tilde{\mathfrak{S}}}$.

REMARK 1.3. It is worth noting that Theorem 1.1(1) can be shown through *Ky Fan functional* instead of the direct proof in [15]. In our case, Ky Fan functional $K(x, \cdot)$ of $x \in L^1(\mathcal{M})$ $(= \mathcal{M}_*)$ is given by

$$K(x, a) = \sup_{u \in U(\mathcal{M})} \mathrm{Re}\, \tau(xu^* au), \quad a \in \mathcal{M}.$$

According to [2, Theorem 3-8], if $x, y \in L^1(\mathcal{M})_+$ and $\tau(x) = \tau(y)$, then x is in the $\|\cdot\|_1$-closure of $\mathrm{conv}\, U(y)$ if and only if $K(x, e) \leq K(y, e)$ for every projection e in $\mathcal{M}$. This is a consequence of the "Σ-property" of Ky Fan functional (see [1, 2]). When $\mathcal{M}$ is a factor, if $x \in L^1(\mathcal{M})_+$ and e is a projection in $\mathcal{M}$ with $\tau(e) < \infty$, then we have

$$\begin{aligned} K(x, e) &= \sup \{\tau(xe') : e' \text{ is a projection in } \mathcal{M} \text{ with } \tau(e') = \tau(e)\} \\ &= \int_0^{\tau(e)} \mu_t(x)\, dt \end{aligned}$$

(2) $x(\gamma)\in\tilde{\mathscr{M}}(\gamma)$ *a. e.*

(3) *The direct integral decompositions of* $x+y$ *and* xy *are* $x+y=\int_\Gamma^\oplus\{x(\gamma)+y(\gamma)\}d\nu(\gamma)$ *and* $xy=\int_\Gamma^\oplus x(\gamma)y(\gamma)d\nu(\gamma)$.

(4) *If* $x\in\tilde{\mathfrak{S}}$, *then* $x(\gamma)\in\tilde{\mathfrak{S}}(\gamma)$ *a. e.*

(5) *If* $x\in L^p(\mathscr{M})$ *where* $1\leqq p<\infty$, *then* $x(\gamma)\in L^p(\mathscr{M}(\gamma))$ *a. e. and* $\|x\|_p=\left[\int_\Gamma\|x(\gamma)\|_p^p d\nu(\gamma)\right]^{1/p}$.

PROOF. (1) and (5) are readily seen from [20, § 1]. It follows from (1) that $\lim_{s\to\infty}\tau_\gamma(e_{(s,\infty)}(|x(\gamma)|))=0$ a. e., showing (2). Let $e_n=e_{[0,n]}(|x|)\wedge e_{[0,n]}(|y|)\wedge e_{[0,n]}(|x+y|)$ and $e_n=\int_\Gamma^\oplus e_n(\gamma)d\nu(\gamma)$ for $n\geqq 1$. Then $\lim_{n\to\infty}\tau(1-e_n)=0$ and hence $\lim_{n\to\infty}\tau_\gamma(1-e_n(\gamma))=0$ a. e. Since

$$(x+y)e_n=\int_\Gamma^\oplus(x+y)(\gamma)e_n(\gamma)d\nu(\gamma)$$

and

$$xe_n+ye_n=\int_\Gamma^\oplus\{x(\gamma)e_n(\gamma)+y(\gamma)e_n(\gamma)\}d\nu(\gamma)$$

from the boundedness of xe_n and ye_n, we have

$$(x+y)(\gamma)e_n(\gamma)=x(\gamma)e_n(\gamma)+y(\gamma)e_n(\gamma)\ \text{a. e.}$$

By taking the limits of this both sides in the measure topology, we obtain the first part of (3). We can show the second part similarly by considering $e_n'(xy)e_n'=(e_n'x)(ye_n')$ where $e_n'=e_{[0,n]}(|x^*|)\wedge e_{[0,n]}(|y|)\wedge e_{[0,n]}(|xy|)$. If $x\in\tilde{\mathfrak{S}}$, then it follows from (1) that $\tau_\gamma(e_{(s,\infty)}(|x(\gamma)|))<\infty$ a. e. for all $s>0$. Hence (4) is obtained. □

Let $\tilde{\Gamma}=\Gamma\times(0,\infty)$ and $\tilde{\nu}$ be the product measure of ν and the Lebesgue measure on $(0,\infty)$. The Lebesgue measure of a Borel subset S of $(0,\infty)$ is denoted by $|S|$. A function on $\tilde{\Gamma}$ is considered to be defined except on some $\tilde{\nu}$-null set. For each $x\in\tilde{\mathscr{M}}$ with $x=\int_\Gamma^\oplus x(\gamma)d\nu(\gamma)$, in view of Lemma 2. 1(2), we define a function $\Psi(x):\tilde{\Gamma}\longrightarrow[0,\infty)$ by

$$\Psi(x)(\gamma,t)=\mu_t(x(\gamma)),\quad(\gamma,t)\in\tilde{\Gamma}.$$

When $\tau(1)<\infty$ and hence $\tau_\gamma(1_\gamma)<\infty$ a. e., for each $x\in\tilde{\mathscr{M}}_{sa}$ we define a function $\Phi:\tilde{\Gamma}_1\longrightarrow\boldsymbol{R}$ by

$$\Phi(x)(\gamma,t)=\lambda_t(x(\gamma)),\quad(\gamma,t)\in\tilde{\Gamma}_1,$$

where $\tilde{\Gamma}_1=\{(\gamma,t)\in\tilde{\Gamma}:0<t<\tau_\gamma(1_\gamma),\ \gamma\in\Gamma\}$, a measurable subset of $\tilde{\Gamma}$ with

$\tilde{\nu}(\tilde{\Gamma}_1)=\tau(1)$. The next lemma will be very important in later discussions.

LEMMA 2.2. (1) *If* $x\in\tilde{\mathscr{M}}$, *then* $\Psi(x)$ *is* $\tilde{\nu}$*-measurable and* $\mu_t(x)=\Psi(x)^*(t)$, *the decreasing rearrangement of* $\Psi(x)$, *for all* $t>0$.

(2) *If* $\tau(1)<\infty$ *and* $x\in\tilde{\mathscr{M}}_{sa}$, *then* $\Phi(x)$ *is* $\tilde{\nu}$*-measurable and* $\lambda_t(x)=\Phi(x)^*(t)$ *for all* $t\in(0,\tau(1))$.

PROOF. (1) Let $f(\gamma, s)=\tau_\gamma(e_{(s,\infty)}(|x(\gamma)|))$ for $(\gamma, s)\in\Gamma\times[0,\infty)$. For each integers $n\geqq 1$ and $1\leqq k\leqq n2^n$, let $\{E_{j,k}^{(n)}: 0\leqq j\leqq n2^n\}$ be a measurable partition of Γ given by

$$E_{j,k}^{(n)}=\{\gamma\in\Gamma: j/2^n\leqq f(\gamma, k/2^n)<(j+1)/2^n\},\quad 0\leqq j\leqq n2^n-1,$$
$$E_{n2^n,k}^{(n)}=\{\gamma\in\Gamma: f(\gamma, k/2^n)\geqq n\}.$$

Define $f_n: \Gamma\times[0,\infty)\longrightarrow[0,\infty)$ by

$$f_n(\gamma, s)=\begin{cases} j/2^n & \text{on } E_{j,k}^{(n)}\times[(k-1)/2^n, k/2^n),\ 0\leqq j\leqq n2^n,\ 1\leqq k\leqq n2^n,\\ 0 & \text{on } \Gamma\times[n,\infty),\end{cases}$$

and $g_n: [0,\infty)\longrightarrow[0,\infty]$ by $g_n(s)=\int_\Gamma f_n(\gamma, s)d\nu(\gamma)$. Let $\{E_i^{(n)}\}_i$ be the refinement of partitions $\{E_{j,k}^{(n)}\}_j$, $1\leqq k\leqq n2^n$. Then f_n and g_n are written as follows:

$$f_n(\gamma, s)=\sum_{k=1}^{n2^n}\sum_i \alpha_{i,k}^{(n)}\chi_{E_i^{(n)}\times[(k-1)/2^n, k/2^n)}(\gamma, s),$$
$$g_n(s)=\sum_{k=1}^{n2^n}\beta_k^{(n)}\chi_{[(k-1)/2^n, k/2^n)}(s),$$

where $\alpha_{i,1}^{(n)}\geqq\dots\geqq\alpha_{i,n2^n}^{(n)}\geqq 0$, $\beta_k^{(n)}=\sum_i\alpha_{i,k}^{(n)}\nu(E_i^{(n)})$, and χ_S denotes the characteristic function of a set S. Now let $F_n: \tilde{\Gamma}\longrightarrow[0,\infty)$ and $G_n: (0,\infty)\longrightarrow[0,\infty)$ be given by

$$\begin{aligned}F_n(\gamma, r)&=\inf\{s\geqq 0: f_n(\gamma, s)\leqq r\}\\ &=\begin{cases}(k-1)/2^n & \text{on } E_i^{(n)}\times[\alpha_{i,k}^{(n)}, \alpha_{i,k-1}^{(n)}),\ 1\leqq k\leqq n2^n,\\ n & \text{on } E_i^{(n)}\times(0,\alpha_{i,n2^n}^{(n)}),\end{cases}\\ G_n(t)&=\inf\{s\geqq 0: g_n(s)\leqq t\}\\ &=\sum_{k=1}^{n2^n}(k-1)/2^n\chi_{[\beta_k^{(n)},\beta_{k-1}^{(n)})}(t)+n\chi_{(0,\beta_{n2^n}^{(n)})}(t),\end{aligned}$$

with conventions $\alpha_{i,0}^{(n)}=\beta_0^{(n)}=\infty$. Since

$$\begin{aligned}\tilde{\nu}(\{(\gamma, r)\in\tilde{\Gamma}: F_n(\gamma, r)\geqq k/2^n\})&=\tilde{\nu}(\bigcup_i[E_i^{(n)}\times(0,\alpha_{i,k}^{(n)})])\\ &=\beta_k^{(n)}=|\{t>0: G_n(t)\geqq k/2^n\}|,\end{aligned}$$

it follows that $F_n^*(t)=G_n(t)$ for all $t>0$. Since $f_n(\gamma, s)\nearrow f(\gamma, s)$ for all (γ, s)

$\in\Gamma\times[0,\infty)$, we get $F_n(\gamma, r)\nearrow\Psi(x)(\gamma, r)$ for all $(\gamma, r)\in\tilde{\Gamma}$, so that $\Psi(x)$ is $\tilde{\nu}$-measurable and $F_n^*(t)\nearrow\Psi(x)^*(t)$, $t>0$. Furthermore we get $g_n(s)\nearrow\tau(e_{(s,\infty)}(|x|))$ for all $s\geqq 0$ by Lemma 2.1(1) and the monotone convergence theorem, so that $G_n(t)\nearrow\mu_t(x)$, $t>0$. Therefore $\mu_t(x)=\Psi(x)^*(t)$.

(2) It may be assumed that $\tau_\gamma(1_\gamma)<\infty$ and $x(\gamma)\in\tilde{\mathscr{M}}(\gamma)_{sa}$ for every $\gamma\in\Gamma$. Letting $f(\gamma, s)=\tau_\gamma(e_{(s,\infty)}(x(\gamma)))$ for $(\gamma, s)\in\Gamma\times\boldsymbol{R}$, we define for each $n\geqq 1$

$$E_{j,k}^{(n)}=\{\gamma\in\Gamma: j/2^n\leqq f(\gamma, k/2^n)<(j+1)/2^n\},\ j\geqq 0,\ -n2^n\leqq k\leqq n2^n,$$

$$f_n(\gamma, s)=\begin{cases} j/2^n & \text{on } E_{j,-n2^n}^{(n)}\times(-\infty, -n),\ j\geqq 0,\\ j/2^n & \text{on } E_{j,k}^{(n)}\times[(k-1)/2^n, k/2^n),\ j\geqq 0,\ -n2^n+1\leqq k\leqq n2^n,\\ 0 & \text{on } \Gamma\times[n,\infty),\end{cases}$$

and $g_n(s)=\int_\Gamma f_n(\gamma, s)d\nu(\gamma)$, $s\in\boldsymbol{R}$. Then $f_n(\gamma, s)\nearrow f(\gamma, s)$ for all $(\gamma, s)\in\Gamma\times\boldsymbol{R}$ and $g_n(s)\nearrow\tau(e_{(s,\infty)}(x))$ for all $s\in\boldsymbol{R}$. Now we can proceed as in the proof of (1). So the details may be omitted. □

In the sequel of this section, we shall discuss the spectral relations described in § 1 between x, $y\in\tilde{\mathscr{M}}_+$ (or x, $y\in\tilde{\mathscr{M}}_{sa}$) under the direct integral decompositions $x=\int_\Gamma^\oplus x(\gamma)d\nu(\gamma)$ and $y=\int_\Gamma^\oplus y(\gamma)d\nu(\gamma)$.

THEOREM 2.3. *For every x, $y\in\tilde{\mathscr{M}}_+$, the following conditions are equivalent*:

(i) *$ex\prec ey$ for every projection e in $\mathscr{A}$*;

(ii) *$ax\prec ay$ for every $a\in\mathscr{A}_+$*;

(iii) *$x(\gamma)\prec y(\gamma)$ a. e.*

PROOF. (iii)$\Longrightarrow$(ii). For $a=\int_\Gamma^\oplus\alpha(\gamma)1_\gamma d\nu(\gamma)$ with $\alpha\in L^\infty(\Gamma;\nu)_+$, (iii) implies that $(ax)(\gamma)=\alpha(\gamma)x(\gamma)\prec\alpha(\gamma)y(\gamma)=(ay)(\gamma)$ a.e. So it is enough to show the case $a=1$. By Lemma 2.2(1), we observe (cf. [29, p. 202]) that

$$\int_0^r\mu_s(x)dt=\sup\{\int_H\Psi(x)(\gamma, t)d\tilde{\nu}(\gamma, t):\ H\subseteq\tilde{\Gamma}\text{ measurable, }\tilde{\nu}(H)=r\}$$

for every $r>0$. For a measurable subset H of $\tilde{\Gamma}$ with $\tilde{\nu}(H)=r$, let $H'=\{(\gamma, t)\in\tilde{\Gamma}: 0<t<|H_\gamma|,\ \gamma\in\Gamma\}$ where H_γ is the γ-section of H. Then

$$\int_H\Psi(x)(\gamma, t)d\tilde{\nu}(\gamma, t)=\int_\Gamma\int_{H_\gamma}\mu_t(x(\gamma))dtd\nu(\gamma)$$
$$\leqq\int_\Gamma\int_0^{|H_\gamma|}\mu_t(y(\gamma))dtd\nu(\gamma)\quad\text{(by (iii))}$$

$$= \int_{H'} \Psi(y)(\gamma, t) d\tilde{\nu}(\gamma, t)$$
$$\leqq \int_0^r \mu_s(y) ds$$

by $\tilde{\nu}(H') = r$ and Lemma 2.2(1). Hence $\int_0^r \mu_s(x) ds \leqq \int_0^r \mu_s(y) ds$ for every $r > 0$, so that $x < y$ because $\tau(x) = \tau(y)$ follows from (iii).

(ii)$\Longrightarrow$(i) is trivial.

(i)$\Longrightarrow$(iii). For every $E \in \mathscr{B}$, letting $e = \int_\Gamma^\oplus \chi_E(\gamma) 1_\gamma d\nu(\gamma)$, we have

$$\int_E \tau_\gamma(x(\gamma)) d\nu(\gamma) = \tau(ex) = \tau(ey) = \int_E \tau_\gamma(y(\gamma)) d\nu(\gamma),$$

so that $\tau_\gamma(x(\gamma)) = \tau_\gamma(y(\gamma))$ a.e. For $n \geqq 1$, define $F_n : \tilde{\Gamma} \longrightarrow [0, \infty)$ by

$$F_n(\gamma, t) = \begin{cases} 1/2^n & \text{if } 0 \leqq \Psi(y)(\gamma, t) \leqq 1/2^n, \\ j/2^n & \text{if } (j-1)/2^n < \Psi(y)(\gamma, t) \leqq j/2^n, \ j \geqq 2. \end{cases}$$

Since $\int_0^r F_n(\gamma, t) dt \searrow \int_0^r \Psi(y)(\gamma, t) dt$ for each $0 < r < \infty$, it suffices to show that $\int_0^r \Psi(x)(\gamma, t) dt \leqq \int_0^r F_n(\gamma, t) dt$ a.e. for every $n \geqq 1$ and every rational number $r > 0$. If this does not hold, then

$$\nu(\{\gamma \in \Gamma : \int_0^r \Psi(x)(\gamma, t) dt > \int_0^r F_n(\gamma, t) dt\}) > 0$$

for some $n \geqq 1$ and some $r > 0$. From the countability of values of F_n, we can choose an $E \in \mathscr{B}$ with $0 < \nu(E) < \infty$ such that $F_n(\cdot, r)$ is constant on E and $\int_0^r \Psi(x)(\gamma, t) dt > \int_0^r F_n(\gamma, t) dt$ for all $\gamma \in E$. Letting $e = \int_\Gamma^\oplus \chi_E(\gamma) 1_\gamma d\nu(\gamma)$, we have by Lemma 2.2(1)

$$\begin{aligned} \int_0^{\nu(E)r} \mu_s(ex) ds &\geqq \int_{E \times (0,r)} \Psi(ex)(\gamma, t) d\tilde{\nu}(\gamma, t) \\ &= \int_E \int_0^r \Psi(x)(\gamma, t) dt d\nu(\gamma) \\ &> \int_E \int_0^r F_n(\gamma, t) dt d\nu(\gamma) \\ &= \int_0^{\nu(E)r} (\chi_E F_n)^*(s) ds, \end{aligned}$$

because $F_n(\gamma, t)$ is non-increasing in t and $F_n(\cdot, r)$ is constant on E. Since $\chi_E F_n \geqq \chi_E \Psi(y) = \Psi(ey)$, we get $\int_0^{\nu(E)r} \mu_s(ex) ds > \int_0^{\nu(E)r} \mu_s(ey) ds$, contradicting (i). $\square$

THEOREM 2.4. *Let $x, y \in \tilde{\mathscr{M}}_+$. Then the following conditions* (i) —(iii) *are equivalent*:

(i) $ex \precsim ey$ *for every projection* e *in* $\mathscr{A}$;

(ii) $ax \precsim ay$ *for every* $a \in \mathscr{A}_+$;

(iii) $x(\gamma) \precsim y(\gamma)$ *a. e.*

Furthermore if $y \in L^1(\mathscr{M}) + \mathscr{M}$, *then the above conditions are equivalent to the following*:

(iv) $\tau((x-a)_+) \leqq \tau((y-a)_+)$ *for every* $a \in \mathscr{A}_+$.

PROOF. The equivalence of (i)—(iii) can be seen in the proof of Theorem 2.3.

(iii)$\Longrightarrow$(iv). For $a = \int_\Gamma^\oplus \alpha(\gamma) 1_\gamma d\nu(\gamma)$ with $\alpha \in L^\infty(\Gamma; \nu)_+$,

$$\tau((x-a)_+) = \int_\Gamma \tau_\gamma((x(\gamma) - \alpha(\gamma))_+) d\nu(\gamma)$$
$$\leqq \int_\Gamma \tau_\gamma((y(\gamma) - \alpha(\gamma))_+) d\nu(\gamma) = \tau((y-a)_+)$$

by (iii) and [15, Proposition 2.3].

(iv)$\Longrightarrow$(i). Assume $y \in L^1(\mathscr{M}) + \mathscr{M}$ and hence $\lim_{n\to\infty} \tau((y-n)_+) = 0$. Let e be a projection in $\mathscr{A}$ and $r > 0$. For each $n \geqq 1$, letting $a = re + n(1-e)$, we have

$$\tau((ex-r)_+) \leqq \tau((x-a)_+) \leqq \tau((y-a)_+)$$
$$\leqq \tau((ey-r)_+) + \tau((y-n)_+)$$

from $(x-a)_+ = (ex-r)_+ + ((1-e)x-n)_+$ and the same for $(y-a)_+$. Hence $\tau((ex-r)_+) \leqq \tau((ey-r)_+)$ for all $r > 0$, so that $ex \precsim ey$ by [15, Proposition 2.3]. □

THEOREM 2.5. *Assume* $\tau(1) < \infty$ *and let* $x, y \in \tilde{\mathscr{M}}_{sa}$. *If* $y_+ \in L^1(\mathscr{M})$, *then the following conditions* (i)—(iii) *are equivalent*:

(i) $ex \prec ey$ *for every projection* e *in* $\mathscr{A}$;

(ii) $ax \prec ay$ *for every* $a \in \mathscr{A}_+$;

(iii) $x(\gamma) \prec y(\gamma)$ *a. e.*

Furthermore if $y \in L^1(\mathscr{M})$, *then the above conditions are equivalent to the following*:

(iv) $\tau(|x-a|) \leqq \tau(|y-a|)$ *for every* $a \in \mathscr{A}_{sa}$.

PROOF. The equivalence of (i)—(iii) can be shown in a manner analogous to the proof of Theorem 2.3 with the use of $\Phi(x)$, $\Phi(y)$ (Lemma 2.2(2)) in place of $\Psi(x)$, $\Psi(y)$. Here we note only that either of (i) and (iii) implies $x_+ \in L^1(\mathscr{M})$ as well as $y_+ \in L^1(\mathscr{M})$ and hence $\Phi(x)_+$, $\Phi(y)_+ \in$

$L^1(\tilde{\Gamma}_1;\ \tilde{\nu})$ by Lemma 2.2(2). This guarantees the use of Fubini's theorem to $\Phi(x)$, $\Phi(y)$.

We now assume $y\in L^1(\mathcal{M})$ and show the equivalence of (iii) and (iv). First note that either of (iii) and (iv) implies $x\in L^1(\mathcal{M})$.

(iii)$\Longrightarrow$(iv). For $a=\int_\Gamma^\oplus \alpha(\gamma)1_\gamma d\nu(\gamma)$ with $\alpha\in L^\infty(\Gamma;\ \nu)_{sa}$,

$$\begin{aligned}\tau(|x-a|)&=\int_\Gamma \tau_\gamma(|x(\gamma)-\alpha(\gamma)|)\,d\nu(\gamma)\\&\leqq\int_\Gamma \tau_\gamma(|y(\gamma)-\alpha(\gamma)|)\,d\nu(\gamma)=\tau(|y-a|)\end{aligned}$$

by (iii) and [16, Proposition 1.3].

(iv)$\Longrightarrow$(iii). Let $E_0=\{\gamma\in\Gamma:\ \tau_\gamma(x(\gamma))>\tau_\gamma(y(\gamma))\}$. Suppose $\nu(E_0)>0$. Let $e_0=\int_\Gamma^\oplus \chi_{E_0}(\gamma)1_\gamma d\nu(\gamma)$ and $a_n=-ne_0+n(1-e_0)$. Because we have

$$\lim_{n\to\infty}\tau(|x-a_n|-|a_n|)=\int_{E_0}\tau_\gamma(x(\gamma))\,d\nu(\gamma)-\int_{\Gamma\setminus E_0}\tau_\gamma(x(\gamma))\,d\nu(\gamma)$$

and

$$\lim_{n\to\infty}\tau(|y-a_n|-|a_n|)=\int_{E_0}\tau_\gamma(y(\gamma))\,d\nu(\gamma)-\int_{\Gamma\setminus E_0}\tau_\gamma(y(\gamma))\,d\nu(\gamma)$$

by Lebesgue's dominated convergence theorem, there is an n such that $\tau(|x-a_n|-|a_n|)>\tau(|y-a_n|-|a_n|)$ and hence $\tau(|x-a_n|)>\tau(|y-a_n|)$, contradicting (iv). Therefore $\tau_\gamma(x(\gamma))\leqq\tau_\gamma(y(\gamma))$ a.e. To show (iii), it suffices by [16, Proposition 1.3] to obtain $\tau_\gamma(|x(\gamma)-r|)\leqq\tau_\gamma(|y(\gamma)-r|)$ a.e. for every rational number r. If this does not hold, then there are an $r\in\boldsymbol{R}$ and an $E\in\mathscr{B}$ with $\nu(E)>0$ such that $\tau_\gamma(|x(\gamma)-r|)>\tau_\gamma(|y(\gamma)-r|)$ for all $\gamma\in E$. Letting $a=re+n(1-e)$ where $e=\int_\Gamma^\oplus \chi_E(\gamma)1_\gamma d\nu(\gamma)$, we have

$$\begin{aligned}0&<\int_E \tau_\gamma(|x(\gamma)-r|)\,d\nu(\gamma)-\int_E \tau_\gamma(|y(\gamma)-r|)\,d\nu(\gamma)\\&=\tau(|x-a|)-\int_{\Gamma\setminus E}\tau_\gamma(|x(\gamma)-n|)\,d\nu(\gamma)\\&\qquad-\tau(|y-a|)+\int_{\Gamma\setminus E}\tau_\gamma(|y(\gamma)-n|)\,d\nu(\gamma)\\&\leqq\int_{\Gamma\setminus E}\tau_\gamma(n-|x(\gamma)-n|)\,d\nu(\gamma)-\int_{\Gamma\setminus E}\tau_\gamma(n-|y(\gamma)-n|)\,d\nu(\gamma)\end{aligned}$$

by (iv). This last expression converges as $n\longrightarrow\infty$ to

$$\int_{\Gamma\setminus E}\tau_\gamma(x(\gamma))\,d\nu(\gamma)-\int_{\Gamma\setminus E}\tau_\gamma(y(\gamma))\,d\nu(\gamma)\leqq 0,$$

a contradiction. □

THEOREM 2.6. *For every* $x, y \in \tilde{\mathscr{M}}_+$ (*or* $x, y \in \tilde{\mathscr{M}}_{sa}$ *when* $\tau(1) < \infty$), *the following conditions are equivalent*:

(i) $ex \lesssim ey$ (*resp.* $ex \approx ey$) *for every projection* e *in* $\mathscr{A}$;

(ii) $ax \lesssim ay$ (*resp.* $ax \approx ay$) *for every* $a \in \mathscr{A}'_+$;

(iii) $x(\gamma) \lesssim y(\gamma)$ (*resp.* $x(\gamma) \approx y(\gamma)$) *a. e.*

PROOF. We shall establish the equivalence for the relation $\lesssim$. This obviously shows the equivalence for $\approx$. Let $x, y \in \tilde{\mathscr{M}}_+$ in the following proof. The proof for $x, y \in \tilde{\mathscr{M}}_{sa}$ when $\tau(1) < \infty$ is analogous.

(iii)$\Longrightarrow$(ii). The case $a=1$ is enough. Since (iii) implies $\Psi(x)(\gamma, t) \leqq \Psi(y)(\gamma, t)$ $\tilde{\nu}$-a. e., we have by Lemma 2.2(1)

$$\mu_s(x) = \Psi(x)^*(s) \leqq \Psi(y)^*(s) = \mu_s(y), \quad s > 0,$$

namely $x \lesssim y$.

(i)$\Longrightarrow$(iii). For $n \geqq 1$, let F_n be defined just as in the proof (i)$\Longrightarrow$(iii) of Theorem 2.3. We need to show that $\Psi(x)(\gamma, r) \leqq F_n(\gamma, r)$ a. e. for every $n \geqq 1$ and every rational number $r > 0$. Suppose on the contrary that

$$\nu(\{\gamma \in \Gamma : \Psi(x)(\gamma, r) > F_n(\gamma, r)\}) > 0$$

for some $n \geqq 1$ and some $r > 0$. Then we can choose an $E \in \mathscr{B}$ with $0 < \nu(E) < \infty$ such that $F_n(\cdot, r)$ is constant $(=c)$ on E and $\Psi(x)(\gamma, r) > c$ for all $\gamma \in E$. Now let $e = \int_\Gamma^\oplus \chi_E(\gamma) 1_\gamma d\nu(\gamma)$ and $s = \nu(E) r$. Since

$$\tilde{\nu}(\{(\gamma, t) \in \tilde{\Gamma} : \chi_E(\gamma)\Psi(x)(\gamma, t) > c\}) > \tilde{\nu}(E \times (0, r)) = s,$$

we get $\mu_s(ex) = (\chi_E \Psi(x))^*(s) > c$. On the other hand, since

$$\tilde{\nu}(\{(\gamma, t) \in \tilde{\Gamma} : \chi_E(\gamma) F_n(\gamma, t) > c\}) \leqq s,$$

we get $\mu_s(ey) = (\chi_E \Psi(y))^*(s) \leqq (\chi_E F_n)^*(s) \leqq c$. Hence $\mu_s(ex) > \mu_s(ey)$, contradicting (i). □

3. Unitary mixing for direct integrals

In this section, we shall discuss the conditions such as the unitary mixing for τ-measurable operators under direct integral decompositions. Let $(\mathscr{M}, \tau)$ be decomposed by the direct integrals $\{\mathscr{M}, \mathscr{H}\} = \int_\Gamma^\oplus \{\mathscr{M}(\gamma), \mathscr{H}(\gamma)\} d\nu(\gamma)$ and $\tau = \int_\Gamma^\oplus \tau_\gamma d\nu(\gamma)$ as in the previous section. We begin with some technical lemmas.

LEMMA 3.1. (1) *There exists a sequence* $\{a_n\}$ *in* $\mathscr{M}_1$ *such that* $\{a_n(\gamma) :$

$n\geqq 1\}$ is strongly dense in $\mathscr{M}(\gamma)_1$ for a. e. $\gamma\in\Gamma$.*

(2) *There exists a sequence $\{u_n\}$ in $U(\mathscr{M})$ such that $\{u_n(\gamma): n\geqq 1\}$ is strongly* dense in $U(\mathscr{M}(\gamma))$ for a. e. $\gamma\in\Gamma$.*

Before proving the lemma, we recall the notion of measurable multifunctions (i. e. set-valued functions). Let $\mathfrak{X}$ be a complete separable metric space and $\mathscr{B}(\mathfrak{X})$ the Borel σ-field on $\mathfrak{X}$. The family of all nonempty closed subsets of $\mathfrak{X}$ is denoted by $\mathscr{K}(\mathfrak{X})$. A multifunction H from Γ into $\mathscr{K}(\mathfrak{X})$ is said to be *measurable* (or *$\mathscr{B}$-measurable*) if $\{\gamma\in\Gamma: H(\gamma)\cap O\neq\emptyset\}$ belongs to $\mathscr{B}$ for each open subset O of $\mathfrak{X}$, or equivalently if there is a sequence $\{h_n\}$ of measurable functions $h_n:\Gamma\longrightarrow\mathfrak{X}$ such that $H(\gamma)$ is the closure of $\{h_n(\gamma)\}$ for every $\gamma\in\Gamma$. If $H:\Gamma\longrightarrow\mathscr{K}(\mathfrak{X})$ is measurable, then the *graph* $\mathrm{Gr}(H)=\{(\gamma, x)\in\Gamma\times\mathfrak{X}: x\in H(\gamma)\}$ of H belongs to $\mathscr{B}\otimes\mathscr{B}(\mathfrak{X})$. Also the converse holds when $(\Gamma,\mathscr{B},\nu)$ is complete. The proofs of these facts are found in [5, 17].

PROOF OF LEMMA 3. 1. The assertion (1) is readily seen from the definition of measurable field of von Neumann algebras and Kaplansky density theorem. To show (2), we may assume (cf. [30, Theorem IV. 8. 13]) that $\gamma\longrightarrow\mathscr{H}(\gamma)$ is a constant field $\mathscr{H}_0$ (a fixed separable Hilbert space). Then $\boldsymbol{B}(\mathscr{H}_0)_1$ becomes a complete separable metric space in the strong* topology, and a function $a:\Gamma\longrightarrow\boldsymbol{B}(\mathscr{H}_0)_1$ is measurable if and only if $\gamma\longrightarrow a(\gamma)$ is a measurable field of operators. Hence (1) shows that the multifunction $\mathscr{M}(\cdot)_1:\Gamma\longrightarrow\mathscr{K}(\boldsymbol{B}(\mathscr{H}_0)_1)$ is $\bar{\mathscr{B}}$-measurable where $\bar{\mathscr{B}}$ is the ν-completion of $\mathscr{B}$. Because the graph of the multifunction $U(\mathscr{M}(\cdot)):\Gamma\longrightarrow\mathscr{K}(\boldsymbol{B}(\mathscr{H}_0)_1)$ is

$$\begin{aligned}\mathrm{Gr}(U(\mathscr{M}(\cdot))&=\mathrm{Gr}(\mathscr{M}(\cdot)_1)\cap[\Gamma\times U(\boldsymbol{B}(\mathscr{H}_0))]\\&\in\bar{\mathscr{B}}\otimes\mathscr{B}(\boldsymbol{B}(\mathscr{H}_0)_1),\end{aligned}$$

there is a sequence $\{\bar{u}_n\}$ of $\bar{\mathscr{B}}$-measurable functions $\bar{u}_n:\Gamma\longrightarrow\boldsymbol{B}(\mathscr{H}_0)_1$ such that $U(\mathscr{M}(\gamma))$ is the strong* closure of $\{\bar{u}_n(\gamma)\}$ for every $\gamma\in\Gamma$. Taking $\mathscr{B}$-measurable functions $u_n:\Gamma\longrightarrow\boldsymbol{B}(\mathscr{H}_0)_1$ such that $u_n(\gamma)=\bar{u}_n(\gamma)$ a. e. and denoting $\int_\Gamma^{\oplus}u_n(\gamma)d\nu(\gamma)\in U(\mathscr{M})$ by the same u_n, we obtain (2). □

LEMMA 3. 2. *Let $\{a_n\}$ be a sequence in $\mathscr{M}_1$ such that $a_n\longrightarrow a$ strongly. Then:*

(1) *$a_nya_n^*\longrightarrow aya^*$ in the measure topology for all $y\in\tilde{\mathfrak{S}}$.*

(2) *$\|a_nya_n^*-aya^*\|_p\longrightarrow 0$ for all $y\in L^p(\mathscr{M})$ where $1\leqq p<\infty$.*

PROOF. Let $y\in\tilde{\mathfrak{S}}$. Since $L^2(\mathscr{M})$ is dense in $\tilde{\mathfrak{S}}$ in the measure topology, for each ε, $\delta>0$ there exists a $z\in L^2(\mathscr{M})$ such that $\mu_\delta(y-z)<\varepsilon$ (cf. [13, Lemma 3. 1]). By [13, Lemma 2. 5],

$$\mu_{4\delta}(a_n y a_n^* - a y a^*) \leqq \mu_\delta(a_n(y-z)a_n^*) + \mu_\delta((a_n - a) z a_n^*) \\ + \mu_\delta(a z (a_n - a)^*) + \mu_\delta(a(z-y)a^*) \\ < 2\varepsilon + \mu_\delta((a_n - a)z) + \mu_\delta((a_n - a)z^*).$$

Noting that $L^2(\mathscr{M})$ is the standard Hilbert space for $\mathscr{M}$, we get $\|(a_n - a)z\|_2 \longrightarrow 0$ and $\|(a_n - a)z^*\|_2 \longrightarrow 0$. Hence

$$\limsup_{n\to\infty} \mu_{4\delta}(a_n y a_n^* - a y a^*) \leqq 2\varepsilon,$$

which shows (1). (2) follows from (1) and [13, Theorem 3.6]. □

We note that Lemma 3.2(1) fails to hold for general $y \in \tilde{\mathscr{M}}$ (consider $y=1$ in case of $\mathscr{M} = \boldsymbol{B}(\mathscr{H})$).

We are now ready to obtain the next theorem.

THEOREM 3.3. *Let $x, y \in \tilde{\mathfrak{S}}$. Then the following conditions are equivalent*:

(i) *x is in the closure of* conv $U(y)$ (*resp.* conv $C(y)$) *in the measure topology*;

(ii) *for a. e. $\gamma \in \Gamma$, $x(\gamma)$ is in the closure of* conv $U(y(\gamma))$ (*resp.* conv $C(y(\gamma))$) *in the measure topology.*

PROOF. We shall establish the equivalence of (i) and (ii) concerning conv $U(\cdot)$. The proof of the equivalence concerning conv $C(\cdot)$ is the same in view of Lemma 3.1.

(i)$\Longrightarrow$(ii). This implication holds for $x, y \in \tilde{\mathscr{M}}$. Let $\{x_n\}$ be a sequence in conv $U(y)$ such that $x_n \longrightarrow x$ in the measure topology. Since $x_n(\gamma) \in$ conv $U(y(\gamma))$ a. e., we need to show that, for a. e. $\gamma \in \Gamma$, $x(\gamma)$ is in the closure of $\{x_n(\gamma)\}$ in the measure topology. Suppose the contrary. Then, by [13, Lemma 3.1], there are an $r>0$, an $\varepsilon>0$ and an $E \in \mathscr{B}$ with $\nu(E)>0$ such that

$$\inf_{n\geqq 1} \mu_r(x_n(\gamma) - x(\gamma)) \geqq \varepsilon, \quad \gamma \in E,$$

which shows by Lemma 2.1(3) that

$$\Psi(x_n - x)(\gamma, t) = \mu_t(x_n(\gamma) - x(\gamma)) \geqq \varepsilon$$

for all $(\gamma, t) \in E \times (0, r)$ and $n \geqq 1$. This leads us to a contradiction, because we have by Lemma 2.2(1)

$$\lim_{n\to\infty} \Psi(x_n - x)^*(s) = \lim_{n\to\infty} \mu_s(x_n - x) = 0, \quad s>0.$$

(ii)$\Longrightarrow$(i). Let $\{u_n\}$ be a sequence in $U(\mathscr{M})$ as in Lemma 3.1(2). Let $(\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots)$, $j=1, 2, \ldots$, be an enumeration of all sequences $(\alpha_1, \alpha_2, \ldots)$

of rational numbers $\alpha_n \geqq 0$ with $\alpha_n = 0$ except for a finite number of n and $\sum_n \alpha_n = 1$. For $j \geqq 1$, define $x_j \in \mathrm{conv}\ U(y)$ by $x_j = \sum_n \alpha_n^{(j)} u_n y u_n^*$. Then (ii) implies by Lemmas 2.1(4) and 3.2(1) that, for a.e. $\gamma \in \Gamma$, $x(\gamma)$ is in the closure of $\{x_j(\gamma) : j \geqq 1\}$ in the measure topology, namely

$$\inf_{j \geqq 1} \Psi(x_j - x)(\gamma, t) = \inf_{j \geqq 1} \mu_t(x_j(\gamma) - x(\gamma)) = 0, \quad t > 0.$$

For each ε, $\delta > 0$, taking a measurable function $\xi : \Gamma \longrightarrow (0, \infty)$ with $\int_\Gamma \xi d\nu < \delta$ in view of the σ-finiteness of (Γ, ν), we obtain a measurable partition $\{E_j : j \geqq 1\}$ of Γ such that $\Psi(x_j - x)(\gamma, \xi(\gamma)) < \varepsilon$ for a.e. $\gamma \in E_j$, $j \geqq 1$. For $k \geqq 1$, define $z_k \in \tilde{\mathscr{M}}$ by $z_k(\gamma) = \sum_{j=1}^k \chi_{E_j'}(\gamma) x_j(\gamma)$ where $E_1' = E_1 \cup [\cup_{j>k} E_j]$ and $E_j' = E_j$, $2 \leqq j \leqq k$. Since

$$z_k(\gamma) = \sum_{n_1, \cdots, n_k} (\alpha_{n_1}^{(1)} \dots \alpha_{n_k}^{(k)}) \sum_{j=1}^k \chi_{E_j'}(\gamma) u_{n_j}(\gamma) y(\gamma) u_{n_j}(\gamma)^*,$$

we get $z_k \in \mathrm{conv}\ U(y)$. Furthermore it follows that

$$\begin{aligned}
&\tilde{\nu}(\{(\gamma, t) \in \tilde{\Gamma} : \Psi(z_k - x)(\gamma, t) > \varepsilon\}) \\
&\leqq \tilde{\nu}(\bigcup_{j=1}^k \{(\gamma, t) \in \tilde{\Gamma} : 0 < t < \xi(\gamma),\ \gamma \in E_j\} \\
&\qquad \cup \{(\gamma, t) \in \tilde{\Gamma} : \Psi(x_1 - x)(\gamma, t) > \varepsilon,\ \gamma \in \bigcup_{j>k} E_j\}) \\
&\leqq \int_\Gamma \xi d\nu + \tilde{\nu}(\{(\gamma, t) \in \tilde{\Gamma} : \Psi(x_1 - x)(\gamma, t) > \varepsilon,\ \gamma \in \bigcup_{j>k} E_j\})
\end{aligned}$$

The second term of this last expression converges to 0 as $k \longrightarrow \infty$, because we have

$$\tilde{\nu}(\{(\gamma, t) \in \tilde{\Gamma} : \Psi(x_1 - x)(\gamma, t) > \varepsilon\}) < \infty$$

by $x_1 - x \in \tilde{\mathfrak{S}}$ and Lemma 2.2(1). Hence $\mu_\delta(z_k - x) = \Psi(z_k - x)^*(\delta) \leqq \varepsilon$ for sufficiently large k. Therefore (i) is obtained. □

THEOREM 3.4. *Let $x \in \tilde{\mathscr{M}}$ and $y \in L^p(\mathscr{M})$ where $1 \leqq p < \infty$. Then the following conditions are equivalent*:

(i) *x is in the $\|\cdot\|_p$-closure of* conv $U(y)$ (*resp.* conv $C(y)$);

(ii) *for a.e. $\gamma \in \Gamma$, $x(\gamma)$ is in the $\|\cdot\|_p$-closure of* conv $U(y(\gamma))$ (*resp.* conv $C(y(\gamma))$).

PROOF. We shall show the equivalence concerning conv $U(\cdot)$. (i) $\Longrightarrow$ (ii) is readily seen from Lemma 2.1(5) and the argument of extracting an a.e. convergent subsequence. Conversely assume (ii). Let $\{u_n\}$ be as in Lemma 3.1(2) and $z \in L^q(\mathscr{M})$ where $1/p + 1/q = 1$. Given a measurable function $\xi : \Gamma \longrightarrow (0, \infty)$, by Lemmas 2.1(5) and 3.2(2), there exists a measurable partition $\{E_n\}$ of Γ such that

$$\mathrm{Re}\ \tau_\gamma(x(\gamma)z(\gamma)) \leqq \mathrm{Re}\ \tau_\gamma(u_n(\gamma)y(\gamma)u_n(\gamma)^* z(\gamma)) + \xi(\gamma)$$

for a. e. $\gamma \in E_n$, $n \geqq 1$. Hence

$$\mathrm{Re}\ \tau(xz) \leqq \mathrm{Re}\ \tau(uyu^* z) + \int_\Gamma \xi d\nu$$

where $u \in U(\mathscr{M})$ is defined by $u(\gamma) = \sum_{n=1}^\infty \chi_{E_n}(\gamma) u_n(\gamma)$. Since $\int_\Gamma \xi d\nu$ can be arbitrarily small, we have

$$\mathrm{Re}\ \tau(xz) \leqq \sup_{u \in U(\mathscr{M})} \mathrm{Re}\ \tau(uyu^* z),$$

which shows (i). □

THEOREM 3.5. *Let $x \in \tilde{\mathscr{M}}$ and $y \in \tilde{\mathfrak{S}}$. Then the following conditions are equivalent*:

(i) *x is in the closure of $U(y)$ (resp. $C(y)$) in the measure topology*;

(ii) *for a. e. $\gamma \in \Gamma$, $x(\gamma)$ is in the closure of $U(y(\gamma))$ (resp. $C(y(\gamma))$) in the measure topology.*

Furthermore if $y \in L^p(\mathscr{M})$ where $1 \leqq p < \infty$, then the above conditions are equivalent to those where the closure in the measure topology is replaced by the $\|\cdot\|_p$-closure.

PROOF. First note that (ii), as well as (i), implies $x \in \tilde{\mathfrak{S}}$ by Lemma 2.2(1). So the proof of the first part can be done in a manner similar to that of Theorem 3.3. Next let $y \in L^p(\mathscr{M})$, $1 \leqq p < \infty$. Since $\mu_t(z) \leqq \mu_t(y)$ for every $z \in C(y)$, it follows from [13, Theorem 3.6] that the $\|\cdot\|_p$-closure of each subset of $C(y)$ coincides with its closure in the measure topology. Hence the second part is verified. □

REMARK 3.6. In connection with Theorems 3.4 and 3.5, we see also that if $y \in L^p(\mathscr{M})$ and $1 < p < \infty$, then the $\|\cdot\|_p$-closure of each convex subset of conv $C(y)$ coincides with its closure in the measure topology (cf. the proof of [15, Theorem 2.5(3)]). However this is not the case when $p=1$. In fact, let $\{e_n\}$ be a sequence of mutually orthogonal one dimensional projections in $\mathscr{M} = \boldsymbol{B}(\mathscr{H})$. Then $k^{-1}\sum_{n=1}^k e_n \in \mathrm{conv}\ U(e_1)$ and $\|k^{-1}\sum_{n=1}^k e_n\| = k^{-1} \longrightarrow 0$, but 0 is not in the $\|\cdot\|_1$-closure of conv $U(e_1)$.

4. Main results

In this section, we shall establish the main results by connecting the spectral relations discussed in § 2 with the corresponding conditions discussed in § 3.

In the theorems below, we assume that $\mathscr{M}$ is a semifinite von Neumann algebra on a "separable" Hilbert space $\mathscr{H}$ with a faithful normal semifinite trace τ. So there exists a measurable field $\gamma \longrightarrow \{\mathscr{M}(\gamma), \mathscr{H}(\gamma)\}$ of factors on a standard σ-finite measure space $(\Gamma, \mathscr{B}, \nu)$ such that $\{\mathscr{M}, \mathscr{H}\} = \int_\Gamma^\oplus \{\mathscr{M}(\gamma), \mathscr{H}(\gamma)\} d\nu(\gamma)$ (see [30, Theorem IV. 8.21]). In this case, the diagonal algebra is the center $\mathfrak{Z} = \mathscr{M} \cap \mathscr{M}'$.

The next Theorem 4.1 is now immediate from Theorems 2.3, 2.5 and 3.4 together with Theorem 1.1(1), and Theorem 4.2 is so from Theorems 2.4, 3.3 and 3.4 together with Theorem 1.1(2) (see also Remark 1.2 and Lemma 2.1).

THEOREM 4.1. *If $x \in \tilde{\mathscr{M}}_+$ and $y \in L^1(\mathscr{M})_+$, then the following conditions* (i) *and* (ii) *are equivalent*:

(i) *x is in the $\|\cdot\|_1$-closure of* conv $U(y)$;

(ii) *$ex \prec ey$ for every projection e in $\mathfrak{Z}$.*

Assume $\tau(1) < \infty$. If $x \in \tilde{\mathscr{M}}_{sa}$ and $y \in L^1(\mathscr{M})_{sa}$, then the above (i), (ii) *and the following* (iii) *are equivalent*:

(iii) *$\tau(|x-a|) \leqq \tau(|y-a|)$ for every $a \in \mathfrak{Z}_{sa}$.*

THEOREM 4.2. *Let $x \in \tilde{\mathscr{M}}_+$ and $y \in (L^1(\mathscr{M}) + \mathscr{M})_+ \cap \tilde{\mathfrak{S}}$, Then the following conditions* (i)—(iii) *are equivalent*:

(i) *x is in the closure of* conv $C(y)$ *in the measure topology*;

(ii) *$ex \underset{\cdot}{\prec} ey$ for every projection e in $\mathfrak{Z}$*;

(iii) *$\tau((x-a)_+) \leqq \tau((y-a)_+)$ for every $a \in \mathfrak{Z}_+$.*

Furthermore if $y \in L^p(\mathscr{M})$ where $1 \leqq p < \infty$, then the above conditions are equivalent to the following:

(iv) *x is in the $\|\cdot\|_p$-closure of* conv $C(y)$.

THEOREM 4.3. *Let $x \in \tilde{\mathscr{M}}_+$ and $y \in (L^1(\mathscr{M}) + \mathscr{M})_+ \cap \tilde{\mathfrak{S}}$, Then the following conditions are equivalent*:

(i) *x is in the closure of $C(y)$ in the measure topology*;

(ii) *$ex \lesssim ey$ for every projection e in $\mathfrak{Z}$*;

(iii) *$e_{(s,\infty)}(x) \precsim e_{(s,\infty)}(y)$ in the Murray-von Neumann sense for every $s \geqq 0$.*

PROOF. The equivalence of (i) and (ii) follows from Theorems 2.6 and 3.5 together with Theorem 1.1(3).

(iii)$\Longrightarrow$(ii). For every projection e in $\mathfrak{Z}$, we have by (iii)

$$e_{(s,\infty)}(ex) = ee_{(s,\infty)}(x) \precsim ee_{(s,\infty)}(y) = e_{(s,\infty)}(ey), \quad s \geqq 0.$$

Hence $\tau(e_{(s,\infty)}(ex)) \leqq \tau(e_{(s,\infty)}(ey))$ for all $s \geqq 0$, so that $ex \lesssim ey$.

(ii)$\Longrightarrow$(iii). For each $s \geqq 0$, because (ii) implies by Lemma 2.1(1) that

$$\int_E \tau_\gamma(e_{(s,\infty)}(x(\gamma)))\,d\nu(\gamma) \leqq \int_E \tau_\gamma(e_{(s,\infty)}(y(\gamma)))\,d\nu(\gamma), \quad E\in\mathscr{B},$$

there is a $\Gamma_0\in\mathscr{B}$ with $\nu(\Gamma\backslash\Gamma_0)=0$ such that $\tau_\gamma(e_{(s,\infty)}(x(\gamma)))\leqq\tau_\gamma(e_{(s,\infty)}(y(\gamma)))$ for all $\gamma\in\Gamma_0$. Since each $\mathscr{M}(\gamma)$ is a factor, we have $e_{(s,\infty)}(x(\gamma))\precsim e_{(s,\infty)}(y(\gamma))$, $\gamma\in\Gamma_0$. For $\gamma\in\Gamma_0$, define

$$H(\gamma)=\{v\in\mathscr{M}(\gamma)_1:\ v^*v=e_{(s,\infty)}(x(\gamma)),\ vv^*\leqq e_{(s,\infty)}(y(\gamma))\}$$

which is a nonempty strongly* closed subset of $\mathscr{M}(\gamma)_1$. Also let $H(\gamma)=\{0\}$ for $\gamma\in\Gamma\backslash\Gamma_0$. To show the existence of a measurable field $\gamma\longrightarrow w(\gamma)$ of operators such that $w(\gamma)\in H(\gamma)$ a. e., we may assume that $\gamma\longrightarrow\mathscr{H}(\gamma)$ is a constant field $\mathscr{H}_0$. Then $\mathscr{M}(\cdot)_1:\Gamma\longrightarrow\mathscr{K}(\boldsymbol{B}(\mathscr{H}_0)_1)$ is $\bar{\mathscr{B}}$-measurable by the proof of Lemma 3. 1. In view of the strong* continuity of $v\longrightarrow v^*v$ and $v\longrightarrow vv^*$ on $\boldsymbol{B}(\mathscr{H}_0)_1$, it is readily seen that the graph of the multifunction H: $\Gamma\longrightarrow\mathscr{K}(\boldsymbol{B}(\mathscr{H}_0)_1)$ belongs to $\bar{\mathscr{B}}\otimes\mathscr{B}(\boldsymbol{B}(\mathscr{H}_0)_1)$. Hence there is a measurable function $w:\Gamma\longrightarrow\boldsymbol{B}(\mathscr{H}_0)_1$ such that $w(\gamma)\in H(\gamma)$ a. e. This w gives a desired measurable field of operators. Denoting $\int_\Gamma^\oplus w(\gamma)\,d\nu(\gamma)\in\mathscr{M}$ by the same w, we get $w^*w=e_{(s,\infty)}(x)$ and $ww^*\leqq e_{(s,\infty)}(y)$, implying $e_{(s,\infty)}(x)\precsim e_{(s,\infty)}(y)$. □

The final theorem follows from Theorems 2. 6 and 3. 5 together with Theorem 1. 1(4) and from the argument as in the above proof.

THEOREM 4. 4. *Let $x\in\tilde{\mathscr{M}}_+$ and $y\in\tilde{\mathfrak{S}}_+$ (or x, $y\in\tilde{\mathscr{M}}_{sa}$ when $\tau(1)<\infty$). Then the following conditions are equivalent*:

(i) *x is in the closure of $U(y)$ in the measure topology*;

(ii) *$ex\approx ey$ for every projection e in $\mathfrak{Z}$*;

(iii) *$e_{(s,\infty)}(x)\sim e_{(s,\infty)}(y)$ in the Murray-von Neumann sense for every $s\geqq 0$.*

In conclusion, Theorem 1. 1(1)—(4) are reduced to the special cases of Theorems 4. 1—4. 4 respectively, while the separability of $\mathscr{H}$ is not assumed in Theorem 1. 1.

We close this paper with the following remarks.

REMARK 4. 5. Even when the representing Hilbert space $\mathscr{H}$ is not separable, it is not difficult to show directly (i)$\Longrightarrow$(ii) of each of Theorems 4. 1—4. 4. To consider the converse, let x, $y\in\tilde{\mathfrak{S}}_+$ and $e_0=e_{(0,\infty)}(x)\vee e_{(0,\infty)}(y)$ which is a σ-finite projection. Let $\mathscr{N}$ be the von Neumann subalgebra of $e_0\mathscr{M}e_0$ generated by the spectral projections of x any y. Then $\mathscr{N}$ is faithfully represented on a separable Hilbert space and τ restricted on $\mathscr{N}$ is semifinite. Hence, also (ii)$\Longrightarrow$(i) of the above theorems can be shown if it happens that the center of $\mathcal{N}$ is included in $e_0\mathfrak{Z}$. However this does not always happen. So we cannot, at present, remove the separability assumption of $\mathcal{H}$ in our main results.

REMARK 4.6. When $\mathcal{M}$ is a general (not necessarily semifinite) von Neumann algebra, the spectral relations considered in this paper cannot be defined from the lack of the notion of generalized s-numbers. However the unitary mixing for elements in $\mathcal{M}_*$ (moreover $\mathcal{M}^*$) is meaningful. Now let φ, $\psi\in\mathcal{M}_*^+$. In [2], φ is said to be *more chaotic* (or *more unitarily mixed*) than ψ if φ is in the norm-closure of conv $U(\psi)$ where $U(\psi)=\{u\psi u^*: u\in U(\mathcal{M})\}$ and $u\psi u^*=\psi(u^*\cdot u)$. But we observe (cf. [2, § 4.4]) that if $\mathcal{M}$ is a von Neumann algebra of type III on a separable Hilbert space, then φ is more chaotic than ψ whenever $\varphi|\mathfrak{Z}=\psi|\mathfrak{Z}$. In particular if $\mathcal{M}$ is a factor of type III_1, then φ is in the norm-closure of $U(\psi)$ for every φ, $\psi\in\mathcal{M}_*^+$ with $\varphi(1)=\psi(1)$ (see [8]). Hence we may conclude that the concept of unitary mixing is more or less simple on the type III part and rather crucial on the semifinite part.

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