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A characterization of some spreads of order q^3 that admit GL(2, q) as a collineation group

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Introbuction

Let $\mathscr{F}\cong GF(q)$, $q=p^m$, be a field of matrices in $GL(3m, p)\cup\{O_{3m}\}$. Then $\pi=\boldsymbol{F}_p^{3m}\oplus\boldsymbol{F}_p^{3m}$ becomes a $\mathscr{G}=GL(2, \mathscr{F})$-module under the natural action of $\mathscr{G}\cong GL(2, q)$ on Π:

$$\underline{x}\oplus\underline{y}\longmapsto(\underline{x}a+\underline{y}b)\oplus(\underline{x}c+\underline{y}d)$$

whenever a, b, c, $d\in\mathscr{F}$ and $ad-bc\neq 0$.

We regard this action of $\mathscr{G}$ on π as defining the Desarguesian representation of $GL(2, q)$, of order q^3, because under this representation $GL(2, q)$ leaves invariant (several) Desarguesian spreads of order q^3, on π. Thus, if $\mathscr{M}\cong GF(q^3)$ is a field of matrices containing $\mathscr{F}$, then $\mathscr{M}$ defines a $\mathscr{G}$-invariant Desarguesian spread $\Gamma_{\mathscr{M}}$ whose components, apart from $Y=O_{3m}\oplus\boldsymbol{F}_p^{3m}$, have the generic form:

$$\{(\underline{x}, \underline{x}M):\ \underline{x}\in\boldsymbol{F}_p^{3m}\}\quad\text{for}\quad M\in\mathscr{M}.$$

Also there are often many non-Desarguesian spreads on π that are invariant under the Desarguesian representation of $GL(2, q)$. The first infinite families of such spreads were discovered by Kantor [7, 8], and later more examples were given in Bartolone-Ostrom [1]. Recently [5], we described a technique for constructing such spreads that yields all the abovementioned spreads of Kantor and Bartolone-Ostrom, and, in addition, yields many new examples. Our method allows one to construct a $\mathscr{G}$-invariant spread "$\pi_{\mathscr{O}}$", whenever one has a fixed-point-free collineation $\mathscr{O}\in P\Gamma L(3, q)$-$PGL(3, q)$ of the Desarguesian projective plane $PG(2, q)$. These "$\pi_{\mathscr{O}}$", which we shall call "generalized Desarguesian spreads", seem too numerous to classify as nonconjugate $\mathscr{O}$ usually yield nonisomorphic spreads.

The object of this note is to show that the only non-Desarguesian spreads invariant under the Desarguesian action of $GL(2, q)$ (of order q^3) are the generalized Desarguesian spreads.

THEOREM A. *Let π be a Desarguesian $GL(2, q)$-module of order q^3.*

Then the only spreads on π invariant under this Desarguesian representation of $GL(2, q)$ are either Desarguesian spreads or generalized Desarguesian spreads.

REMARK. The converse, that every generalized Desarguesian spread of order q^3 (cf. result 1. 4) is invariant under a Desarguesian representation of $GL(2, q)$, has been proved in [5].

The following corollary also follows from Bartolone and Ostrom [1].

COROLLARY B. *Let p be a prime and let π admit a Desarguesian representation of $GL(2, q)$ of order p^3. Then any spread on π, invariant under this representation, is necessarily a Desarguesian spread.*

The module-theoretic characterization of the generalized Desarguesian spreads given in Theorem A, seems a good base from which to obtain geometric characterizations of the corresponding translation planes. One such characterization will be given in a sequel [6].

1. The Generalized Desarguesian Spreads.

In this section, we review our construction of the generalized Desarguesian spreads [5], and take the opportunity to introduce some notation.

Throughout the article, $\pi=\boldsymbol{F}_p^{3m}\oplus\boldsymbol{F}_p^{3m}$ is a vector space of $3m$-tuples over the prime field $GF(p)$, and $\mathscr{F}\cong GF(q)$, $q=p^m\geq p$, is a field of $3m\times 3m$ matrices over $GF(p)$. π is regarded as a $\mathscr{G}=GL(2, \mathscr{F})$-module defined by the following action:

$$(\underline{x}\oplus\underline{y})\begin{bmatrix} A & B \\ C & D \end{bmatrix}=(\underline{x}A+\underline{y}C)\oplus(\underline{x}B+\underline{y}D)$$

whenever

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\in\mathscr{G} \quad \text{and} \quad \underline{x},\ \underline{y}\in\boldsymbol{F}_p^{3m}.$$

If M is any $3m\times 3m$ matrix over $GF(p)$, we define the corresponding subspace of π by

$$\underline{\text{“}\, y=xM\text{ ”}=\{(\underline{x}, \underline{x}M):\ \underline{x}\in\boldsymbol{F}_p^{3m}\}.}$$

Thus, if M is nonsingular, “ $y=xM$ ” is a $GF(p)$-subspace of order q^3 (and rank $3m$). A routine computation allows one to determine the image of “ $y=xM$ ” under any $g\in\mathscr{G}$.

1. LEMMA. *Suppose* $g=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is any element of* $\mathscr{G}$: *thus* a, b, c, d *are block-matrices in* $\mathscr{F}$, *with* $ad-bc\neq 0$. *Then*

$$g: \text{“}\, y=xM \,\text{”} \longmapsto \text{“}\, y=x(a+Mc)^{-1}(b+Md) \,\text{”}$$

whenever $a+Mc$ *is a nonsingular matrix.*

The following partial spread of $q+1$ components will be contained in the generalized Desarguesian spreads that we shall define shortly.

2. NOTATION. $\delta_{\mathscr{F}}=\{\text{“}\, y=xf \,\text{”} : f\in\mathscr{F}\}\cup\{O\oplus \boldsymbol{F}_p^{3m}\}$.

In passing, we note that $\delta_{\mathscr{F}}$ is actually a Desarguesian partial spread (it lies in $\Gamma_{\mathscr{M}}$, defined in the Introduction). To extend $\delta_{\mathscr{F}}$ to a non-Desarguesian spread we require the following concept.

3. DEFINITION. If $T\in GL(3m, p)$, we call $(T, \mathscr{F})$ an irreducible pair on $V=\boldsymbol{F}_p^{3m}$ (which is chosen to be a column space, for convenience) if

(i) $T\in N_{GL(3m,p)}(\mathscr{F})-C_{GL(3m,p)}(\mathscr{F})$: and

(ii) T does not fix any nonzero proper $\mathscr{F}$-subspace of V, with T and $\mathscr{F}$ acting on V from the left.

It is easily seen (cf. [5]) that an irreducible pair exists on the column $3m$-tuples $V=\boldsymbol{F}_p^{3m}$ iff the Desarguesian plane $PG(2, p^m)$ admits a fixed-point-free collineation $\mathscr{O}\in P\Gamma L(3, q)-PGL(3, q)$: however, we shall not use this connection—we only mention it to indicate that large numbers of irreducible pairs exist (in fact, whenever $q=p^m>p$, cf. [5, section 6]).

We now define the generalized Desarguesian spreads on π via the following result.

4. RESULT. (Jha and Johnson [5]). Let $(T, \mathscr{F})$ be an irreducible pair on the column space $V=\boldsymbol{F}_p^{3m}$, $p^m=q$. Then

$$\pi_{T,\mathscr{F}}=\delta_{\mathscr{F}}\cup Orb_{\mathscr{G}}(\text{“}\, y=xT \,\text{”})$$

is a $\mathscr{G}$-invariant (strictly non-Desarguesian) spread on the “row” space $\pi=\boldsymbol{F}_p^{3m}\oplus\boldsymbol{F}_p^{3m}$.

Any spread isomorphic to a $\pi_{(T,\mathscr{F})}$ will be called a generalized Desarguesian spread.

2. Proof of Theorem A.

To prove Theorem A, we assume Γ to be any spread on π whose components are left invariant by the action of $\mathscr{G}$ on π. So our objective is to show that either $\Gamma=\pi_{(T,\mathscr{F})}$, where $(T, \mathscr{F})$ is an irreducible pair, or Γ is

Desarguesian.

1. LEMMA. *$\delta_{\mathscr{F}}$ is a $\mathscr{G}$-orbit and $\Gamma\supset\delta_{\mathscr{F}}$.*

PROOF. It is a routine matter (based mostly on Lemma 1.1) to verify that $\delta_{\mathscr{F}}$ is a $\mathscr{G}$-orbit. To show that $\Gamma\supset\delta_{\mathscr{F}}$ let $\{P_i: 1\leq i\leq q+1\}$ denote the set of Sylow p-subgroups of $\mathscr{G}$ and write $F_i=\mathrm{Fix}(P_i)$. If P_1 is the upper-triangular Sylow p-subgroup in $\mathscr{G}$, then $F_1=O\oplus \boldsymbol{F}_p^{3m}\in\delta_{\mathscr{F}}$. Hence by the transitivity of $\mathscr{G}$ on $\delta_{\mathscr{F}}$ we find that $\delta_{\mathscr{F}}=\{F_i: 1\leq i\leq q+1\}$, and each $|F_i|=q^3$. In particular, Sylow p-subgroups of $\mathscr{G}$ are either all elation groups of order q or are all Baer groups of same order. In the latter event, q is a square and so, since now $q\geq 4$, $\{P_i: 1\leq i\leq q+1\}$ generates the nonsolvable group $SL(2, q)$. We now have enough conditions to force Γ to be either a Hall spread, or have order 16, using [3, Theorem 1. 2] and [4, Theorem B]; but in either case we have a contradiction, since 16 is not a cube, and a Hall plane of order q^3 cannot admit $GL(2, q)$. Thus each member of $\delta_{\mathscr{F}}$ is an elation axis of a nontrivial elation group in Aut Γ. Hence $\delta_{\mathscr{F}}$ consists of $q+1$ components of Γ and the lemma is proved.

If we write $J=\boldsymbol{F}_p^{3m}$ then the lemma above implies that

$$X=J\oplus O,\quad Y=O\oplus J\quad \text{and}\quad I=\{(\underline{x}, \underline{x}): \underline{x}\in J\}$$

are all in Γ. Hence, we have the well-known fact (cf. Foulser [2]) that every component of Γ-$\{Y\}$ can be written uniquely in the form "$y=xS$", where $S\in GL(3m, p)\cup\{O_{3m}\}$, and the set of all such matrices $\mathscr{S}$ is the spread-set associated with Γ; thus

$$\mathscr{S}=\{S\in GL(3m, p): \text{“} y=xS \text{”}\in\Gamma\}\cup\{O_{3m}\}.$$

Any spread-set satisfies an appropriate generalization of the following easily-checked conditions.

2. LEMMA. *$\mathscr{S}$ is a set of q^3 matrices such that*
(i) *O_{3m} and $I_{3m}\in\mathscr{S}$; and*
(ii) *$X, Y\in\mathscr{S}\Longrightarrow X-Y$ is nonsingular or zero.*

We now list properties of the spread-set $\mathscr{S}$ that take into account the $\mathscr{G}$-invariance of Γ. These properties will sometimes be used without explicit citation.

3. LEMMA. (a) $\mathscr{S}\supset\mathscr{F}$.
(b) $M\in\mathscr{S}^*=(\mathscr{S}-\{\underline{O}\})\Longrightarrow M^{-1}\in\mathscr{S}^*$.
(c) $M\in\mathscr{S}-\mathscr{F}\Longrightarrow\alpha+\beta M\gamma\in\mathscr{S}^*\ \forall\ \alpha, \beta, \gamma\in\mathscr{F}^*$.
(d) $M\in\mathscr{S}-\mathscr{F}\Longrightarrow(\mathscr{F}\oplus M\mathscr{F})\cup(\mathscr{F}\oplus\mathscr{F}M)\subseteq\mathscr{S}$.

(N. B.: *In part* (d), *and the rest of the section, we minimize brackets by assuming* $\mathscr{F}\oplus x\mathscr{F}$ (*resp.,* $\mathscr{F}\oplus\mathscr{F}x$) *denotes* $\mathscr{F}\oplus(x\mathscr{F})$: *a similar convention holds even when the sum is not direct.*)

PROOF. (a) is the spread-set version of Lemma 1. By Lemma 1.1, we get the image of " $y=xM$ " to be " $y=xM^{-1}$ " if we apply the transformation: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus (b) holds. Part (c) is similarly obtained by computing the images of " $y=xM$ " under the upper-triangular group in $\mathscr{G}$. Part (d) is a special case of (c) if we note that $\mathscr{F}+\mathscr{F}M=\mathscr{F}\oplus\mathscr{F}M$, since otherwise $M\in\mathscr{F}$.

DEFINITION. $\mathscr{R}$ is the set of all $\mathscr{F}$-spaces of type $\mathscr{F}\oplus\mathscr{F}M$ where $M \in\mathscr{S}-\mathscr{F}$.

4. LEMMA. $|\mathscr{R}|=q+1$.

PROOF. The collection $\{(\mathscr{F}\oplus\mathscr{F}M)\backslash\mathscr{F}\colon M\in\mathscr{S}\backslash\mathscr{F}\}$ partitions the q^3-q elements of $\mathscr{S}-\mathscr{F}$ into classes of size q^2-q.

We now verify that $\mathscr{F}^*$ induces a permutation group on $\mathscr{R}$, under right multiplication by $\mathscr{F}^*$.

5. LEMMA. *If* $x\in\mathscr{F}^*$, *then*

$$\begin{aligned} \bar{x}:\ & \mathscr{R} \longrightarrow \mathscr{R} \\ & \mathscr{F}\oplus\mathscr{F}M \longrightarrow \mathscr{F}\oplus\mathscr{F}Mx \end{aligned}$$

is a bijection on $\mathscr{R}$. *Hence*

$$\begin{aligned} \mathscr{F}^* & \longrightarrow \overline{\mathscr{F}^*} \\ x & \longmapsto \bar{x} \end{aligned}$$

is a group homomorphism from $\mathscr{F}^*$ *into* symm$(\mathscr{R})$.

PROOF. By lemma 3, $M\in\mathscr{S}-\mathscr{F}\Longrightarrow \mathscr{F}\ Mx\in\mathscr{S}-\mathscr{F}$. Hence

$$\mathscr{F}\oplus\mathscr{F}M\in\mathscr{R}\Longrightarrow\mathscr{F}\oplus\mathscr{F}Mx\in\mathscr{R}.$$

Further, the image of $\mathscr{F}\oplus\mathscr{F}M$ under $\bar{x}$ is unambiguous because

$$\begin{aligned} & \mathscr{F}\oplus\mathscr{F}M=\mathscr{F}\oplus\mathscr{F}N \\ \Longrightarrow & N=\alpha+\beta M \ \exists\ \alpha\in\mathscr{F},\ \beta\in\mathscr{F}^* \\ \Longrightarrow & \mathscr{F}\oplus\mathscr{F}Nx=\mathscr{F}\oplus\mathscr{F}Mx. \end{aligned}$$

Now $x\longrightarrow\bar{x}$ is easily seen to be a group homomorphism from $\mathscr{F}^*$ into Symm$(\mathscr{R})$.

We now wish to show that $\overline{\mathscr{F}^*}$ fixes an element of $\mathscr{R}$.

6. LEMMA. *Let u be an odd prime divisor of $|\mathscr{F}^*|=q-1$, and U the Sylow u-subgroup of $\mathscr{F}^*$. Then $\exists\ M\in\mathscr{S}-\mathscr{F}$ such that $\mathscr{F}\oplus\mathscr{F}M\gamma=\mathscr{F}\oplus\mathscr{F}M\ \forall\ \gamma\in U$.*

PROOF. By lemma 4, $(|U|,|\mathscr{R}|)=1$. Hence, by lemma 5, $\bar{U}$ fixes a member of $\mathscr{R}$, and this is the required conclusion.

We can now strengthen lemma 6.

7. LEMMA. *$\exists\ T\in\mathscr{S}-\mathscr{F}$ such that $T\mathscr{F}\subseteq\mathscr{F}\oplus\mathscr{F}T$.*

PROOF. If $M\in\mathscr{S}-\mathscr{F}$ we write

$$\boldsymbol{L}_M=\{x\in\mathscr{F}:\ \mathscr{F}\oplus\mathscr{F}Mx\subseteq\mathscr{F}\oplus\mathscr{F}M\}.$$

Since $\boldsymbol{L}_M$ is additively and multiplicatively closed, and contains 0 and 1, it must be a subfield of $\mathscr{F}$. If $\mathscr{F}$ is a prime field then $\boldsymbol{L}_M=\mathscr{F}$, and we are done. Hence we assume $q=p^m>p$, and now by Zsygmondy's theorem we need to consider the following cases:

(i) $q-1$ has a primitive divisor u;

(ii) $q=64$;

(iii) $q=p^2$ and $p+1=2^m>2$.

In case (i), let U be the Sylow u-subgroup of $\mathscr{F}^*$. Hence by the last lemma,

$$\exists\ T\in\mathscr{S}-\mathscr{F}\text{ such that the field }\boldsymbol{L}_T\supseteq U.$$

But as U cannot lie in a proper subfield of $\mathscr{F}^*$, we have $\boldsymbol{L}_T=\mathscr{F}$ as required. Case (ii) is proved in the same way if we choose U to be the cyclic group of order 9 in $GF(2^6)^*$. To treat case (iii), let V be the Sylow 2-subgroup of $\mathscr{F}^*$. Since $2\|q+1$, lemmas 4 and 5 imply that an index 2-subgroup of V, say R, must fix some $\mathscr{F}\oplus\mathscr{F}T$ in $\mathscr{R}$: that is, $\mathscr{F}\oplus\mathscr{F}TR\subseteq\mathscr{F}\oplus\mathscr{F}T$. But since $|R|=2^{m+1}/2=p+1$, R cannot lie in a subfield of $\mathscr{F}$. Hence we again have $\boldsymbol{L}_T=\mathscr{F}$, since $\boldsymbol{L}_T\supseteq R$. Thus the lemma is proved.

8. LEMMA. *Suppose $M\in\mathscr{S}-\mathscr{F}$ is chosen so that $M\mathscr{F}\subseteq\mathscr{F}\oplus\mathscr{F}M$. Then*

$$W\in\mathscr{F}\oplus\mathscr{F}M\backslash\mathscr{F}\Longrightarrow W^2\notin\mathscr{F}\oplus\mathscr{F}M.$$

PROOF. It is obvious that $\mathscr{F}\oplus\mathscr{F}M=\mathscr{F}\oplus\mathscr{F}W$, whenever $W\in\mathscr{F}\oplus\mathscr{F}M\backslash\mathscr{F}$. Since by hypothesis $M\mathscr{F}\subseteq\mathscr{F}\oplus\mathscr{F}M$, we have

$$\mathscr{F}W\mathscr{F}\subseteq\mathscr{F}\oplus\mathscr{F}W. \tag{i}$$

If the lemma is false, we further have

$$W^2 \in \mathscr{F} \oplus \mathscr{F} W \ \exists \ W \in \mathscr{F} \oplus \mathscr{F} M \backslash \mathscr{F}.$$

Now we claim $\mathscr{F} \oplus \mathscr{F} W$ is multiplicatively closed. For if $x, y, u, v \in \mathscr{F}$ then

$$(x+yW)(u+vW) \equiv yWu + yWvW \mod \mathscr{F} \oplus \mathscr{F} W. \qquad \text{(ii)}$$

But by (i), $yWu \in \mathscr{F} \oplus \mathscr{F} W$, and

$$Wv = v_1 + v_2 W \ \exists \ v_1, \ v_2 \in \mathscr{F}.$$

Thus (ii) yields

$$(x+yW)(u+vW) \equiv y(v_1 + v_2 W) W \mod (\mathscr{F} \oplus \mathscr{F} W),$$

and now, since $W^2 \in \mathscr{F} \oplus \mathscr{F} W$, we have

$$(x+yW)(u+vW) \in \mathscr{F} \oplus \mathscr{F} W.$$

Thus $\mathscr{F} \oplus \mathscr{F} W$ is multiplicatively, as well as additively closed. Since the nonzero elements of $\mathscr{F} \oplus \mathscr{F} W$ are nonsingular (being in the spread set $\mathscr{S}$, because of lemma 3) we therefore conclude that

$$\mathscr{M} = \mathscr{F} \oplus \mathscr{F} W \cong GF(q^2).$$

This means that the spread Γ, "coordinatized" by the spread-set $\mathscr{S}$, contains the rational Desarguesian partial spread

$$\delta_{\mathscr{M}} = \{\text{“} y = xM \text{”}: \ M \in \mathscr{M}\} \cup \{O \oplus \boldsymbol{F}_p^{3m}\}.$$

Thus the translation plane of order q^3, associated with Γ, contains (Desarguesian) subplanes of order q^2 (whose lines include all the members of $\delta_{\mathscr{M}}$), and we contradict the Baer condition. The lemma follows.

We shall call an additive subgroup $\Sigma \subseteq GL((N, p) \cup \{O_N\}$ an <u>additive</u> spread-set if $|\Sigma| = p^N$. Thus, an additive spread-set is a spread-set Σ which happens to be an additive group of matrices. It is generally realized that all additive spread sets arise as matrix representations of the additive group of slope maps (cf. Foulser [2]) of some semifield (but we shall avoid using this fact).

9. Lemma. $\exists \ T \in \mathscr{S} - \mathscr{F}$ *and* $\sigma \in Aut(\mathscr{F}, +, \cdot)$ *such that*
(i) $fT = Tf^{\sigma} \ \forall \ f \in \mathscr{F}$; *and*
(ii) $\boldsymbol{A} = \mathscr{F} \oplus \mathscr{F} T \oplus \mathscr{F} T^2$ *is an additive spread-set.*

Proof. By lemma 7 we may choose an $M \in \mathscr{S} - \mathscr{F}$ such that

$$M\mathcal{F}\subseteq\mathcal{F}\oplus\mathcal{F}M.$$

Assign to each $f\in\mathcal{F}$ the map

$$\begin{aligned}\hat{f}:\ &\mathcal{F}\oplus\mathcal{F}M\longrightarrow\mathcal{F}\oplus\mathcal{F}M\\ &\alpha+\beta M\longrightarrow(\alpha+\beta M)f.\end{aligned}$$

Then $\hat{\mathcal{F}}^*$ is a group of $\mathcal{F}$-linear maps of the <u>left</u> vector space $\Lambda=\mathcal{F}\oplus\mathcal{F}M$, and this group leaves invariant the subspace $\mathcal{F}$. Hence $\hat{\mathcal{F}}^*$ also fixes a Maschke complement of $\mathcal{F}$ in Λ. Thus

$$\exists\ T\in\mathcal{F}\oplus\mathcal{F}M-\mathcal{F}\text{ and a map }\mathcal{T}:\mathcal{F}\longrightarrow\mathcal{F}$$

such that

$$Tf=f^{\mathcal{T}}T\ \ \forall\ f\in\mathcal{F}.$$

It is obvious now that $\mathcal{T}$ is an additive and multiplicative bijection of the field $\mathcal{F}$, and hence (i) follows if we choose $\sigma=\mathcal{T}^{-1}$. To prove (ii) it is sufficient to establish that

(*) $\quad x+yT+zT^2$ is nonsingular $\forall\ x,\ y,\ z\in\mathcal{F}$ unless $x=y=z=0$

Since $\mathcal{F}\oplus\mathcal{F}T\subset\mathcal{S}$ (lemma 3), we may further assume that $z\neq0$. Right-multiplying (*) by T^{-1}, it is now sufficient to verify that

(**) $\quad xT^{-1}+y+zT$ is nonsingular if $z\neq0$.

But by lemma 3, $-xT^{-1}$ and $y+zT$ are both in $\mathcal{S}$, and this forces their difference $xT^{-1}+y+zT$ to be either nonsingular or zero. But if $xT^{-1}+y+zT=0$ then $-zT^2=x+yT$, forcing $T^2\in\mathcal{F}\oplus\mathcal{F}T$, contrary to lemma 8. Hence (**) holds, and the required result follows.

In the following corollary, T and σ are as in the lemma above.

10. COROLLARY. *Let $V=\boldsymbol{F}_p^{3m}$ be the column-space of $3m$-tuples over $GF(p)$. Define*

$$\begin{aligned}\bar{T}:\ &V\longrightarrow V\\ &\underline{x}\longrightarrow T\underline{x}.\end{aligned}$$

Then

(i) $\bar{T}(f\underline{x})=f^{\sigma}\bar{T}(\underline{x})\ \forall\ \underline{x}\in V,\ f\in\mathcal{F}$, *and*

(ii) $\bar{T}$ *does not fix any $\mathcal{F}$-subspaces of V, other than $\underline{O}$ and V.*

PROOF. The semilinearity condition is lemma 9(i). To verify that $\bar{T}$ does not fix any proper nonzero subspace, assume, in order to get a contradiction, that for some nonzero $\underline{x}\in V$ the $\mathcal{F}$-subspace W generated by

$\{\underline{x}, T\underline{x}\}$ is T-invariant. Since the $\mathscr{F}$-rank of $W \leq 2$, and $T^2\underline{x} \in W$, we obtain

$$T^2\underline{x} = \alpha\underline{x} + \beta T\underline{x} \ \exists \ \alpha, \ \beta \in \mathscr{F}.$$

Hence $T^2 - \beta T - \alpha$ is a singular matrix, contradicting lemma 9(ii). The lemma follows.

If $\sigma \neq$ identity, then the corollary asserts that $(T, \mathscr{F})$ of lemma 9 is an irreducible pair (cf. definition 1.3). But if σ=identity, then $\boldsymbol{A}$ of lemma 9 (ii) centralizes the irreducible $\mathscr{F}$-linear map $\overline{T}$; so by Schur's lemma, $\boldsymbol{A}$ must coincide with the matrix field $\cong GF(q^3)$ centaralizing T. Thus, using lemma 9 and corollary 10 we have shown

11. LEMMA. *$\mathscr{S}$ contains an element T such that $(T, \mathscr{F})$ is an irreducible pair, on $V = \boldsymbol{F}_p^{3m}$, except when:*
T centralizes $\mathscr{F}$ and $\boldsymbol{A} = \mathscr{F} + \mathscr{F}T + \mathscr{F}T^2 \cong GF(q^3)$.

Now consider Θ, the $\mathscr{G}$-orbit of "$y = xT$". Since $\mathscr{G}$ leaves invariant the spread Γ, one of whose components is "$y = xT$", we have

$$\Theta = \mathrm{Orb}_{\mathscr{G}}(\text{“}y = xT\text{”}) \subseteq \Gamma.$$

Since $\delta_{\mathscr{F}}$ is also a $\mathscr{G}$-orbit (lemma 1) we obtain

$$\Gamma \supseteq \Theta \cup \delta_{\mathscr{F}}.$$

The RHS is a disjoint union of two $\mathscr{G}$-orbits such that $|\delta_{\mathscr{F}}| = q+1$ and $|\Theta| = q^3 - q$: to see the latter, compute the stabilizer of "$y = xT$" using lemma 1.1 (for full details, see [5, section 4]). But now $|\Gamma| = q^3 + 1 = |\Theta| + |\delta_{\mathscr{F}}|$, and so

$$\Gamma = \mathrm{Orb}_{\mathscr{G}}(\text{“}y = xT\text{”}) \cup \delta_{\mathscr{F}}.$$

Thus we have the following refinement of lemma 11.

12. LEMMA. *$\exists \ T \in \mathscr{S} - \mathscr{F}$ such that*
(i) $\Gamma = \mathrm{Orb}(\text{“}y = xT\text{”}) \cup \delta_{\mathscr{F}}$: *and*
(ii) *$(T, \mathscr{F})$ is an irreducible pair except when*

$$\boldsymbol{A} = \mathscr{F} + \mathscr{F}T + \mathscr{F}T^2 \cong GF(q^3).$$

Let us now take a closer look at case (ii):

$$\boldsymbol{A} = \mathscr{F} + \mathscr{F}T + \mathscr{F}T^2 \cong GF(q^3).$$

Now

$$\Gamma_{\boldsymbol{A}} = \{\text{“}y = ax\text{”}: \ a \in \boldsymbol{A}\} \cup O \oplus \boldsymbol{F}_p^{3m}\}$$

is a Desarguesian spread of order q^3 on π and

$$\Gamma_A \cap \Gamma \supseteq \{\text{“ } y=xT \text{ ”}\} \cup \delta_{\mathscr{F}}. \tag{1}$$

Now, by applying lemma 1.1, we can easily verify that any image of "$y=xT$", under $\mathscr{G}$, has ite generic form

$$\text{“ } y=x(a+Tc)^{-1}(b+Td) \text{ ”}. \tag{2}$$

But since $\boldsymbol{A}$ is a field, $\boldsymbol{A}^*$ is closed under inversion and multiplication, and so (2) yields

$$\mathrm{Orb}_{\mathscr{G}}(\text{“ } y=xT \text{ ”}) \subset \Gamma_A. \tag{3}$$

But since the LHS of (3) is in Γ as well, equations (1) and (3) show that

$$\Gamma_A \cap \Gamma \supseteq \mathrm{Orb}_{\mathscr{G}}(\text{“ } y=xT \text{ ”}) \cup \delta_{\mathscr{F}},$$

and since the RHS is actually the spread Γ (lemma 12(i)), we get $\Gamma=\Gamma_A$. Thus lemma 12 refines to

13. PROPOSITION. $\exists\ T \in \Gamma - \mathscr{F}$ *such that either*

(i) $(T, \mathscr{F})$ *is an irreducible pair, and* Γ *is the generalized Desarguesian spread*

$$\pi_{T,\mathscr{F}} = \mathrm{Orb}_{\mathscr{G}}(\text{“ } y=xT \text{ ”}) \cup \delta_{\mathscr{F}}$$

(*cf.* 1.4); *or*

(ii) Γ *is a Desarguesian spread.*

In effect, proposition 13 is theorem A of the introduction.

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