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Decomposition of quotients of bounded operators with respect to closability and Lebesgue-type decomposition of positive operators

Saichi IZUMINO
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1. Introduction

Let A and B be bounded linear operators on an infinite dimensional Hilbert space H with the kernel condition

(1.1) $\quad \ker A \subset \ker B.$

Then we define a quotient $[B/A]$ as the linear operator: $Ax \mapsto Bx$, $x \in H$. In [5] we showed that both the adjoint and the closure of $[B/A]$ are also represented as reasonable quotients if they exist. Let $P = P_{A^*,B^*}$ be the orthogonal projection onto the closure of the set $B^{*(-1)}(A^*H) := \{x ; B^*x \in A^*H\}$, and let $P^{\perp} = 1 - P$. Then, applying Jorgensen decomposition [6] (Ôta [10]) to $[B/A]$, we obtain the sum decomposition [5] $[B/A] = [PB/A] + [P^{\perp}B/A]$ of $[B/A]$ into the closable part $[PB/A]$ and the singular part $[P^{\perp}B/A]$. Extending this notion, we call the decomposition

$$[B/A] = [QB/A] + [Q^{\perp}B/A]$$

J-decomposition of $[B/A]$ by Q, if Q is an orthogonal projection such that $[QB/A]$ is closable and $[Q^{\perp}B/A]$ is singular.

Another decomposition is Lebesgue-type (or shortly L-) decomposition of (bounded) positive operators, which was introduced by Ando [2]; if S is a positive operator then every positive operator T is decomposed into the sum $T = U + V$ of two positive operators U and V such that U is S-absolutely continuous and V is S-singular. It was proved in [2] that a positive operator T is S-absolutely continuous if and only if $T^{1/2(-1)}(S^{1/2}H)$ is dense in H. The latter condition is, as a matter of fact, just what guarantees closability of $[T^{1/2}/S^{1/2}]$ when $\ker S \subset \ker T$ [5], [9]. This suggests close connections between J-decomposition and L-decomposition.

In this paper we first consider J-decomposition of quotients and give some equivalent conditions for uniqueness of this decomposition. Next we show that every J-decomposition of a quotient $[B/A]$ induces an L-decomposition of B^*B with respect to A^*A , and conversely that every

L-decomposition of T with respect to S, under the condition ker $S\subset$ker T, is induced from a J-decomposition of $[B/A]$ such that $A^*A=S$ and $B^*B=T$.

To avoid triviality we assume that the Hilbert space H has infinite dimension. An operator is assumed to be bounded linear, defined on H, unless specially stated otherwise.

2. J-decomposition of quotients

For given operators A and B, put

(2.1) $\qquad R=R_{A,B}=(A^*A+B^*B)^{1/2}.$

Then as a basic fact we have $RH=A^*H+B^*H$ [4, Theorem 2.2]. If we consider the equations

(2.2) $\qquad XR=A$ and $YR=B$,

then, since $A^*H\subset RH$ and $B^*H\subset RH$ we can fined operators X and Y satisfying (2.2) [4, Theorem 2.1]. Furthermore, with the restrictions ker $X\supset$ker R and ker $Y\supset$ker R each of the equations has a unique solution, so that we then denote by $X=A_\iota(=A_{B,\iota})$ and $Y=B_\iota(=B_{A,\iota})$ [5]. Following [4] we now define

(2.3) $\qquad A^*A : B^*B=A^*A_\iota B_\iota^*B,$

and call it the parallel sum of A^*A and B^*B. (If $A^*A=C^*C$ for an operator C, then we can see $A^*A_\iota=C^*C_\iota$, so that $A^*A : B^*B$ is really well-defined by (2.3).) In [5] we proved the following facts which are useful for our discussions.

LEMMA 2.1 (cf. [5, Lemma 2.3]). *Let A, B be operators on H, and let R, A_ι and B_ι are operators defined as before. Then*

(1) $\qquad A_\iota^*A_\iota+B_\iota^*B_\iota=P_R$, *the orthogonal projection onto the closure $(RH)^-$ of RH.*

(2) $\qquad A^*A : B^*B=B^*B : A^*A=A^*(1-A_\iota A_\iota^*)A=B^*(1-B_\iota B_\iota^*)B.$

(3) $\qquad A^*H\cap B^*H=(A^*A : B^*B)^{1/2}H.$

(4) $\qquad B^{*(-1)}(A^*H)=(1-B_\iota B_\iota^*)^{1/2}H.$

Denote by P_{A^*,B^*} (or $P(A^*, B^*)$) the orthogonal projection onto $\{B^{*(-1)}(A^*H)\}^-$. Then we have

LEMMA 2.2 *Let V_ι be the partial isometry obtained from the polar decomposition $A_\iota=V_\iota(A_\iota^*A_\iota)^{1/2}$ of A_ι. Then*

(1) $P_{A^*,B^*}=1-B_lB_l^*+B_lV_l^*V_lB_l^*$.
(2) $P_{A^*,B^*}B=B_lV_l^*V_lR$.

PROOF. From Lemma 2.1 (1) we see that $A_l^*A_l$ and $B_l^*B_l$ commute. Hence we have easily

(2.4) $V_l^*V_lB_l^*B_l=B_l^*B_lV_l^*V_l$.

To prove (1), let $P=P_{A^*,B^*}$ and denote by Q the right hand side of (1). Then, using Lemma 2.1 (1) again and (2.4), we can see that $Q^2=Q$, that is, Q is an orthogonal projection. Hence, since $1-B_lB_l^*\leqq Q$ (or $Q-(1-B_lB_l^*)$ is positive), we have $PH\subset QH$. For the converse inclusion, first note that $B_l^*(1-B_lB_l^*)=(P_R-B_l^*B_l)B_l^*=A_l^*A_lB_l^*$, and that $\ker A_l^*A_lB_l^*=\ker V_lB_l^*$. Hence we have

(2.5) $\ker(1-B_lB_l^*)\subset\ker V_lB_l^*$.

Hence $\ker(1-B_lB_l^*)\subset\ker Q$, which implies $PH\supset QH$. Now the identity (2) can be obtained from (1), (2.4) and Lemma 2.1 (1).

Let $[B/A]$ be a quotient of operators (with the kernel condition (1.1)). If AH is dense in H, then the adjoint $[B/A]^*$ of $[B/A]$ exists, and it is represented [5, Theorem 4.1] as

(2.6) $[B/A]^*=[V_lB_l^*/(1-B_lB_l^*)^{1/2}]$.

In [5], assuming that AH is dense in H, we defined $[B/A]$ to be closable if the domain $(1-B_lB_l^*)^{1/2}H$ of $[B/A]^*$ is dense in H. Here we, however, want to define $[B/A]$ to be closable (cf. [7, p. 165]) if

(2.7) $Ax_n\to 0$ and $Bx_n\to y$ for a sequence $\{x_n\}$ in H imply $y=0$.

Consequently, we do not assume the denseness of AH in H for closability of $[B/A]$. Denote by $[B/A]^-$ the closure of $[B/A]$ when it exists. Then we have

LEMMA 2.3 (cf. [5, Theorem 4.2], [8, Lemma 3]). *Let $[B/A]$ be a quotient. Then the following conditions are equivalent;*

(1) *$[B/A]$ is closable, (i.e., (2.7) is assumed.)*
(2) $\ker A_l\subset\ker B_l$.
(3) *$(1-B_lB_l^*)^{1/2}H$ $(=B^{*(-1)}(A^*H))$ is dense in H.*

If one of (1)−(3) holds, then $[B/A]^-=[B_l/A_l]$.

PROOF. (1)⇒(2); Let $A_lu=0$, $u\in H$. Then, since A_l is defined as a natural extension of the mapping $Rx\mapsto Ax$, $x\in H$, we can find a sequence $\{x_n\}$

such that $Rx_n \to u$ and $Ax_n \to A_l u = 0$. Hence $Bx_n = B_l Rx_n \to B_l u$, which implies $B_l u = 0$.

(2)⇒(3); Let $(1 - B_l B_l^*)u = 0$. Then we have to show that $u = 0$. By (2.5) we see that $B_l^* u \in \ker V_l = \ker A_l$. Hence $B_l B_l^* = 0$, so that $u = (1 - B_l B_l^*)u + B_l B_l^* u = 0$.

(3)⇒(1); Let $Ax_n \to 0$ and $Bx_n \to y$. Then $\{Rx_n\}$ is convergent. Put $z = \lim_{n\to\infty} Rx_n$. Then $A_l z = \lim_{n\to\infty} A_l Rx_n = \lim_{n\to\infty} An_n = 0$. Hence $(1 - B_l B_l^*)B_l z = B_l(P_R - B_l^* B_l)z = B_l A_l^* A_l z = 0$. Since $\ker(1 - B_l B_l^*) = \{0\}$, we have $B_l z = 0$. Hence $y = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} B_l Rx_n = B_l z = 0$.

For the closure $[B/A]^-$, we first note that $[B_l/A_l]$ is an extension of $[B/A]$, because $A = A_l R$ and $B = B_l R$. Since $A_l^* A_l + B_l^* B_l = P_R$ (Lemma 2.1 (1)), we see that $A_l^* H + B_l^* H$ is closed in H. Hence from [8, Theorem 1] (or by a direct computation) we can show that $[B_l/A_l]$ is closed. Now, since AH is dense in $A_l H$ we can conclude that $[B/A]^- = [B_l/A_l]$.

Among general (possibly unbounded) operators a singular operator L is defined ([6] and [10]) as one which has dense domain $D(L)$ in H and satisfies the condition $L(D(L)) \subset D(L^*)^\perp$, that is, the range of L is orthogonal to the domain of L^*. Since the domain of the adjoint of a quotient $[B/A]$ is $(1 - B_l B_l^*)^{1/2} H$, we naturally assume that a singular quotient $[B/A]$ satisfies the condition $BH \subset \{(1 - B_l B_l^*)H\}^\perp$, or equivalently

(2.8) $\qquad BH \subset \ker P_{A^*, B^*}$.

We here adopt (2.8) as the definition of $[B/A]$ to be singular, and we do not request the denseness of AH in H (cf. [5]). Now on singularity of quotients we can show the next equivalences, the proof of which is almost similar to that in [5].

LEMMA 2.4 [5, Theorem 5.5]. *Let $[B/A]$ be a quotient. Then the following conditions are equivalent*;

(1) $\qquad [B/A]$ *is singular, (i.e., (2.8) is assumed.)*
(2) $\qquad A_l B_l^* = 0$.
(3) $\qquad A^*A : B^*B = 0$.
(4) $\qquad A^*H \cap B^*H = \{0\}$.

Recall that for a quotient $[B/A]$ and an orthogonal projection Q the decomposition

(2.9) $\qquad [B/A] = [QB/A] + [Q^\perp B/A]$

is a J-decomposition by Q if $[QB/A]$ is closable and $[Q^\perp B/A]$ is singular. Easily we see that $(QB)^{*(-1)}(A^*B) = Q^{(-1)}(B^{*(-1)}(A^*H))$, and that the relation $(Q^{\perp}B)^*H\cap A^*H=\{0\}$ is equivalent to $Q^{\perp}H\cap B^{*(-1)}(A^*H)\subset\ker B^*$. Hence from Lemmas 2.3 and 2.4 we have

THEOREM 2.5. *Let $[B/A]$ be a quotient, and let Q be an orthogonal projection. Then $[B/A]=[QB/A]+[Q^{\perp}B/A]$ is a J-decomposition if and only if the following two conditions hold.*

(1) $Q^{(-1)}(B^{*(-1)}(A^*H))$ *is dense in* H.

(2) $Q^{\perp}H\cap B^{*(-1)}(A^*H)\subset\ker B^*$.

It is easy to see that the orthogonal projection $P=P_{A^*,B^*}$ satisfies the above conditions (1) and (2). Hence

(2.10) $[B/A]=[PB/A]+[P^{\perp}B/A]$

is really a J-decomposition of $[B/A]$ [5, Theorem 5.4].

COROLLARY 2.6. *Let Q be an orthogonal projection such that $[B/A]=[QB/A]+[Q^{\perp}B/A]$ is a J-decomposition. Then $Q\leqq P_{A^*,B^*}$.*

PROOF. Note that $Q^{(-1)}(B^{*(-1)}(A^*H))\subset Q^{(-1)}(PH)$ $(P=P_{A^*,B^*})$, and that $Q^{(-1)}(PH)$ is closed. Hence, by the theorem, $Q^{(-1)}(PH)=H$, so that $QH\subset PH$ or $Q\leqq P$.

On the closure $[PB/A]^-$ of the closable part $[PB/A]$ of $[B/A]$ in the decomposition (2.10), we have

PROPOSITION 2.7. $[PB/A]^-=[B_lV_l^*V_l/A_l]$.

PROOF. From Lemma 2.2 (2) we see that $[B_lV_l^*V_l/A_l]$ is an extension of $[PB/A]=[B_lV_l^*V_lR/A_lR]$. Since $A_l^*A_l+(B_lV_l^*V_l)^*(B_lV_l^*V_l)=V_l^*V_l$ is an orthogonal projection, we can see that $[B_lV_l^*V_l/A_l]$ is closed (as in the proof of Lemma 2.3). Now since AH is dense in A_lH, we have the desired identity.

A quotient $[B/A]$ is bounded as an operator on AH if and only if there exists some $\alpha>0$ such that $\|Bx\|\leqq\alpha\|Ax\|$, $x\in H$. An equivalent condition for the boundedness of $[B/A]$ is the relation $B^*H\subset A^*H$ (e.g. by [4, Theorem 2.1]). The following theorem characterizes a quotient whose closable part of the decomposition (2.10) is bounded.

THEOREM 2.8. *The following conditions are equivalent;*

(1) $[P_{A^*,B^*}B/A]$ *is bounded on* AH.

(2) A_l *has closed range.*

(3) $B^{*(-1)}(A^*H)$ *is closed in* H.

PROOF. (1)$\Rightarrow$(2); Write $P=P_{A^*,B^*}$ briefly. Since (1) is equivalent to $B^*PH\subset A^*H$, we have $B^*P=A^*X$ for some operator X. Hence by Lemma 2.2 (2) we have $RV_l^*V_lB_l^*=RA_l^*X$, or $V_l^*V_lB_l^*=A_l^*X$. Hence $V_l^*V_l=V_l^*V_l(A_l^*A_l+B_l^*B_l)V_l^*V_l=A_l^*A_l+A_l^*XX^*A_l\leqq(1+\|x\|^2)A_l^*A_l$. This implies that $V_l^*H\subset A_l^*H$, so that A_l^* and hence also A_l has closed range.

(2)$\Rightarrow$(3); Note that $B^{*(-1)}(A^*H)=B_l^{*(-1)}(A_l^*H)$, and that the inverse image $B_l^{*(-1)}(A_l^*H)$ of the closed set A_l^*H is closed.

(3)$\Rightarrow$(1); If $B^{*(-1)}(A^*H)$ is closed, then $PH=B^{*(-1)}(A^*H)$, so that $B^*PH\subset A^*H$. This implies boundedness of $[PB/A]$.

On uniqueness of the J-decomposition, we have

THEOREM 2.9. *A quotient* $[B/A]$ *has the unique J-decomposition* (2.10) *if and only if one of the conditions* (1)$-$(3) *in Theorem* 2.8 *holds.*

PROOF. Suppose that (1) of Theorem 2.8 holds, or equivalently, that $B^*PH\subset A^*H$ $(P=P_{A^*,B^*})$. Let Q be an orthogonal projection which yields a J-decomposition (2.8). Then, by Corollary 2.6 P and Q commute, so that $B^*Q^\perp PH=B^*PQ^\perp H\subset A^*H$. Since $[Q^\perp B/A]$ is singular, we have $A^*H\cap B^*Q^\perp H=\{0\}$ from Lemma 2.4. Hence $B^*Q^\perp PH=\{0\}$ or $B^*Q^\perp P=0$, which implies $QB=PB$, uniqueness of J-decomposition of $[B/A]$.

To see the converse assertion, suppose that $B^*PH\not\subset A^*H$. Then there is a vector $u\in H$ such that $B^*Pu\notin A^*H$. We can assume that $u\in PH$ *and* $\|u\|=1$. Put $Q=P(1-u\otimes u)$ $(=(1-u\otimes u)P)$, where $u\otimes u$ is an operator defined by $(u\otimes u)x=\langle x,u\rangle u$, $x\in H$. ($\langle\cdot,\cdot\rangle$ is the inner product of H.) Then clearly Q is an orthogonal projection. Now we want to show that this Q yields a J-decomposition of $[B/A]$ which is deferent from (2.10). It suffices to prove that

(i) $OB\neq PB$, and
(ii) $[QB/A]$ is closable and $[Q^\perp B/A]$ is singular.

For (i), since $B^*u\notin A^*H$, we see that $B^*u\neq 0$, so that $PB-QB=(u\otimes u)B=u\otimes B^*u\neq 0$. For (ii), first note that $[QB/A]=[QPB/A]$ has an extension $[QB_lV_l^*V_l/A_l]$ $(PB_l=B_lV_l^*V_l)$. By a simple computation we can see that $A_l^*A_l+(QB_lV_l^*V_l)^*(QB_lV_l^*V_l)=V_l^*V_l-B_l^*u\otimes B_l^*u$ is an operator with closed range. Hence $[QB_lV_l^*V_l/A_l]$ is a closed extension of $[QB/A]$. Next in order to see that $[Q^\perp B/A]$ is singular, we want to show that $A^*H\cap B^*Q^\perp H=\{0\}$. Let $v\in A^*H\cap B^*Q^\perp H$. Then $v=A^*x=B^*Q^\perp y$ for some x, $y\in H$. Hence $R(A_l^*x-B_l^*Q^\perp y)=0$, or equivalently, $A_l^*x=B_l^*Q^\perp y$. Since $B_l^*Q^\perp=B_l^*(P^\perp+u\otimes u)=(1-V_l^*V_l)B_l^*+B_l^*u\otimes u$, we have

$$A_l^* x = (1 - V_l^* V_l) B_l^* y + \langle y, u\rangle B_l^* u.$$

Multiplying this identity by $RV_l^* V_l$ from the left, we have $RA_l^* x = \langle y, u\rangle RV_l^* \times V_l B_l^* u$, that is, $A^* x = \langle y, u\rangle B^* Pu$. Hence, from the assumption $B^* Pu \notin A^* H$ we conclude that $A^* x = 0$, or $v = 0$.

3. Relations between J-decompositions and L-decompositions

We begin with the definition of L-decomposition of positive operators. Let S be a positive operator. Then a positive operator U is said to be S-absolutely continuous if there exists a sequence $\{U_n\}$ of positive operators such that $U_n \leqq U_{n+1}$, $U_n \leqq \alpha_n S_n$ for some $\alpha_n > 0$ $(n = 1, 2, \ldots)$ and $\lim_{n\to\infty} U_n = U$ (strong limit). A positive operator V is S-singular if any operator W satisfying $0 \leqq W \leqq V$, $W \leqq S$ is identical to 0. Let T be a positive operator, and let

$$(3.1) \qquad T = U + V$$

for two positive operators U and V with the conditions defined as above. Then we call (3.1) an L-decomposition of T with respect to S [2].

Recall that the parallel sum $S : T$ of two positive operators S and T is defined (see(2.3)) by $S : T = S^{1/2}(S^{1/2})_l (T^{1/2})_l^* T^{1/2}$. Easily we see that $S : T$ is bounded by S and T (e.g. by Lemma 2.1 (2)). Furthermore, it is monotone [4, Theorem 4.4], that is, $S : T_1 \leqq S : T_2$ if $0 \leqq T_1 \leqq T_2$. Using the parallel sum, Ando [2] introduced an S-absolutely continuous operator

$$[S]T = \lim_{n\to\infty} (nS) : T,$$

and proved that

$$(3.2) \qquad T = [S]T + (T - [S]T)$$

is an L-decomposition of T with respect to S. (In defining the operator $[S]T$, Ando, however, adopted a different but equivalent definition [1, Theorem 9] of the parallel sum; $\langle (S : T)x, x\rangle = \inf\{\langle Sy, y\rangle + \langle Tz, z\rangle ;\ y + z = x\}$.)

Now, as a relation combining the J-decomposition (2.10) and the L-decomposition (3.2), we have the following result which was essentially obtained by Kosaki [9]. For completeness we shall prove it.

THEOREM 3.1 (cf. [9, Theorem 6]). *Let A, B be operators, and let* $S = A^* A$ and $T = B^* B$. Then $[S]T = B^* P_{A^*, B^*} B$.

PROOF. Let $R_n = R_{nA, B}$ (cf. (2.1)), and let $X = A_n = (nA)_{B,l}$, $Y = B_n = B_{nA,l}$ be the unique solutions of the equations $XR_n = nA$, ker $X \supset$ ker R_n and

$YR_n=B$, ker $Y\supset$ker R_n, respectively (cf. (2. 2)). Then we easily have the following facts.

(1) $\|A_n\|\leqq 1$, $\|B_n\|\leqq 1$.

(2) (n^2S) : $T=B^*(1-B_nB_n^*)B$. (By Lemma 2. 1 (2).)

(3) $(1-B_nB_n^*)^{1/2}H=B^{*(-1)}(nA^*H)=B^{*(-1)}(A^*H)$. (By Lemma 4. 1 (4).)

(4) $1-B_nB_n^*\leqq P_{nA^*,B^*}=P_{A^*,B^*}$. (By Lemma 2. 2 (1).)

We want to add more two facts.

(5) $\{1-B_nB_n^*\}$ is an increasing squence.

(6) $(1-B_lB_l^*)B_nB_n^*=(1/n)B_lA_l^*A_nB_n^*$.

For (5), since $R_m^2\leqq R_n^2$ for $m\leqq n$, we have the unique operator $Z=Z_{mn}$ such that $R_m=R_nZ=Z^*R_n$, ker $Z^*\supset$ker R_n. Since $B_nR_n=B=B_mR_m=B_mZ^*R_n$, we can see that $B_n=B_mZ^*$. Hence, since $\|Z\|\leqq 1$, we have $B_nB_n^*=B_mZ^*\times ZB_m^*\leqq B_mB_m^*$, which implies (5).

For (6), we can first obtain $A_1Z_{1n}^*=(1/n)A_n$ and $B_1Z_{1n}^*=B_n$ by a similar argument to that used above (to get $B_n=B_mZ^*$). Note that $A_1=A_l$ and $B_1=B_l$. Hence, from Lemma 2. 1 (2), we have

$$(1-B_lB_l^*)B_n=(1-B_lB_l^*)B_lZ_{1n}^*=B_l(P_R-B_l^*B_l)Z_{1n}^*$$
$$=B_lA_l^*A_lZ_{1n}^*=(1/n)B_lA_l^*A_n.$$

We now get (6) immediately.

To show the desired identity $[S]T=B^*PB$, where $P=P_{A^*,B^*}$, let $Q=\lim_{n\to\infty}(1-B_nB_n^*)$. Then, by (2), what we have to do is to show $Q=P$. Letting $n\to\infty$ in (6), we obtain $(1-B_lB_l^*)(1-Q)=0$. Hence we have easily $P(1-Q)=0$, or $P=PQ$. From (4) we can also have $PQ=Q$, which completes the proof.

From the fact ker $B^*\subset PH$, we can see that $P=1$ is equivalent to $PB=B$. Between closability of quotients and absolute continuity of positive operators, we have

COROLLARY 3. 2 (cf. [2, Theorem 5], [9, Lemma 3]). *Let* $[B/A]$ *be a quotient. Then the following conditions are equivalent*;

(1) $[B/A]$ *is closable.*

(2) $P_{A^*,B^*}=1$.

(3) B^*B *is* A^*A*-absolutely continuous.*

PROOF. (1)$\Longleftrightarrow$(2); Clear by Lemma 2. 3.

(2)⇒(3) ; From (2) we have $[A^*A](B^*B)=B^*B$, which implies (3).
(3)⇒(1) ; If (3) is assumed, then there is a sequence $\{T_n\}$ of positive operators such that $T_n^2\leqq T_{n+1}^2$, $T_n^2\leqq \alpha_n A^*A$ for some $\alpha_n>0$ and $\lim_{n\to\infty} T_n^2=B^*B$. Then, since $T_nH\subset A^*H$, we see that $T_n^{(-1)}(A^*H)=H$. Hence $P_{A^*,T_n}=1$, so that $[A^*A]T_n^2=T_n^2$. Hence $[A^*A](B^*B)\geqq[A^*A]T_n^2=T_n$. Taking the limit, we have $[A^*A](B^*B)\geqq B^*B$, or equivalently, $[A^*A](B^*B)=B^*B$. From this identity, we can easily obtain $P_{A^*,B^*}B=B$, which implies (2).

For the singularity of quotients and positive operators, we have

COROLLARY 3.3 (cf. [2, Corollary 3]). *Let $[B/A]$ be a quotient. Then the following conditions are equivalent* ;

(1) *$[B/A]$ is singular.*
(2) $P_{A^*,B^*}B=0$.
(3) *B^*B is A^*A-singular.*

PROOF. The equivalence (1)⟺(2) is clear by the definition (2.8). By Theorem 3.1 the condition (2) is equivalent to the identity

$$(2')\quad [A^*A](B^*B)=0.$$

From the definitions of $[A^*A](B^*B)$ and A^*A-singularity, we can see the equivalences (2')⟺(n^2A^*A) : $B^*B=0$ $(n=1,2,\ldots)$⟺(3).

Let $[B/A]=[QB/A]+[Q^\perp B/A]$ be a J-decomposition of $[B/A]$ by an orthogonal projection Q. Then by Corollaries 3.2 and 3.3 we see that $B^*B=B^*QB+B^*Q^\perp B$ is an L-decomposition of B^*B with respect to A^*A. Hence every J-decomposition of $[B/A]$ induces an L-decomposition of B^*B with respect to A^*A. As the converse to this fact we have

THEOREM 3.4. *Let S and T be positive operators with* ker $S\subset$ker T, *and let $T=U+V$ be an L-decomposition of T such that U and V are S-absolutely continuous and S-singular positive opeators, respectively. Then there exist an operator B and an orthogonal projection Q such that $U=B^*QB$ and $V=B^*Q^\perp B$. Hence, if A is an operator with $A^*A=S$, then $[B/A]=[QB/A]+[Q^\perp B/A]$ is a J-decomposition of $[B/A]$ by Q, which induces the given L-decomposition of T.*

PROOF. Since the dimension of H is infinite, we can find mutually orthogonal closed linear subspaces M and N in H such that dim $M=$dim $(UH)^-$ and dim $N=$dim $(VH)^-$. Then there exist partial isometries X and Y such that

(3.3) $XX^*=P_U,\ YY^*=P_V,\ XY^*=0.$

Here P_U and P_V are the orthogonal projections onto $(UH)^-$ and $(VH)^-$, respectively. Put $B=X^*U^{1/2}+Y^*V^{1/2}$ and $Q=X^*X$. Then we can obtain all that we desire.

THEOREM 3.5. *If we add the assumption* $U^{1/2}H\cap V^{1/2}H=\{0\}$ *to Theorem* 3.4, *then we have a J-decomposition of* $[T^{1/2}/S^{1/2}]$ *by some orthogonal projection* Q *which induces the given L-decomposition* $T=U+V$.

PROOF. Let X and Y be, respectively, the unique solutions of the equations $XT^{1/2}=U^{1/2}$ and $YT^{1/2}=V^{1/2}$ such that ker $X\subset$ker T and ker $Y\subset$ker T. Then we can see that $X^*X+Y^*Y=P_T$, and that $T^{1/2}=X^*U^{1/2}+Y^*V^{1/2}$ or $U^{1/2}=XT^{1/2}=XX^*U^{1/2}+XY^*V^{1/2}$. Hence $(P_U-XX^*)U^{1/2}=XY^*V^{1/2}$. Taking the adjoints, we have $U^{1/2}(P_U-XX^*)=V^{1/2}YX^*$. Hence by the assumption $U^{1/2}H\cap V^{1/2}H=\{0\}$, we have $P_U-XX^*=YX^*=0$. Similarly we can obtain $P_V-YY^*=0$. Hence we have (3.3) for those X and Y. Now, letting $B=T^{1/2}(A=S^{1/2}$ and $Q=X^*X)$, we obtain the desired J-decomposition of $[T^{1/2}/S^{1/2}]$.

On uniqueness of L- and J-decompositions we have

THEOREM 3.6. *Let* S *and* T *be positive operators with* ker $S\subset$ker T. *Then* T *has a unique L-decomposition with respect to* S *if and only if* $[T^{1/2}/S^{1/2}]$ *has a unique J-decomposition.*

PROOF. Suppose that T has a unique L-decomposition with respect to S, and let $[T^{1/2}/S^{1/2}]=[QT^{1/2}/S^{1/2}]+[Q^\perp T^{1/2}/S^{1/2}]$ be a J-decomposition of $[T^{1/2}/S^{1/2}]$. Then $T^{1/2}QT^{1/2}=T^{1/2}PT^{1/2}$, where $P=P(S^{1/2},\ T^{1/2})$. Hence by Corollary 2.6, $QT^{1/2}=PT^{1/2}$, which implies that $T^{1/2}$ has a unique J-decomposition. Conversely, suppose that $[T^{1/2}/S^{1/2}]$ has a unique J-decomposition, and let $T=U+V$ be an L-decomposition of T such that U is S-absolutely continuous and V is S-singular. Then by the monotone property of the operation $[S]$, we have $U=[S]U\leqq[S]T=T^{1/2}PT^{1/2}$, so that $U^{1/2}H\subset T^{1/2}PH$. By Theorem 2.8 (1) and Theorem 2.9, we see that $T^{1/2}PH\subset S^{1/2}H$. Hence we have $U^{1/2}H\subset S^{1/2}H$. On the other hand, since $[V^{1/2}/S^{1/2}]$ is singular we have $V^{1/2}H\cap S^{1/2}H=\{0\}$. Hence $U^{1/2}H\cap V^{1/2}H=\{0\}$. Now by Theorem 3.5 we can find an orthogonal projection Q such that $[T^{1/2}/S^{1/2}]=[QT^{1/2}/S^{1/2}]+[Q^\perp T^{1/2}/S^{1/2}]$ is a J-decomposition which induces the L-decomposition $T=U+V$. Hence, from uniqueness of the J-decomposition we obtain $QT^{1/2}=PT^{1/2}$, so that $U=T^{1/2}QT^{1/2}=T^{1/2}PT^{1/2}$ and $V=T^{1/2}P^\perp T^{1/2}$. This implies uniqueness of the L-decomposition of T.

COROLLARY 3.7 (cf. [2, Theorem 6]). *Let S and T be positive operators with* $\ker S \subset \ker T$. *Then T has a unique L-decomposition with respect to S if and only if $[S]T \leqq aS$ for some $a>0$.*

PROOF. By Theorems 2.7, 2.8 and 3.6 we see that T has a unique L-decomposition with respect to S if and only if $[PT^{1/2}/S^{1/2}]$ is bounded on $S^{1/2}H$. The latter condition is equivalent to $T^{1/2}PH \subset S^{1/2}H$. Since $[S]T = T^{1/2}PT^{1/2}$, we now obtain $[S]T \leqq aS$ for some $a>0$ as an equivalent condition for uniqueness of the L-decomposition of T.

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Faculty of Education
Toyama University