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Some remarks on the Dirichlet problem for semi-linear elliptic equations with the Ambrosetti-Prodi conditions.

J. CHABROWSKI

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1. Introduction.

In this paper we investigate the solvability of the Dirichlet problem for semi-linear equation

$$(1_s)\qquad Lu=-\sum_{i,j=1}^{n}D_i(a_{ij}(x)D_ju)=f(u)+s\theta(x)+h(x) \ \ in \ Q,$$

$$(2_t)\qquad u(x)=t\phi(x) \ \ on \ \partial Q,$$

in a bounded domain $Q\subset \boldsymbol{R}^n$, with the boundary ∂Q of class C^2, where s and t are real parameters, θ is the first eigenfunction of L and $\theta\perp h$.

In the case, where $t=0$ anf f satisfies the Ambrosetti-Prodi conditions

$$(3)\qquad \lim_{t\to-\infty}\frac{f(t)}{t}<\lambda_1<\lim_{t\to\infty}\frac{f(t)}{t},$$

the problem (1_s), (2_0) has an extensive literature (see [1], [2], [3], [8], [10], [12], [13] and [14]). Here λ_1 denotes the first eigenvalue of L. In these papers, under suitable regularity assumptions on a_{ij} $(i, j=1, \dots, n)$ f and h, the following result was established. There exists a constant s_0 such that the problem (1_s), (2_0) has 2, 1 or 0 solutions depending on whether s is less than, equal to or greater than s_0.

The purpose of this article is to investigate the dependence of the existence of solutions of (1_s), (2_t) on a parameter t.

The main result can be summarized as follows. Suppose that ϕ is sufficiently smooth, $\phi\geq 0$ and $\phi\not\equiv 0$ on ∂Q. Then there exists a number $s_0=s_0(h, \phi, f)$ such that for every $s\leq s_0$ there exists $t^*(s)$ such that for $t<t^*(s)$ the problem (1_s), (2_t) has at least one solution and no solution for $t>t^*(s)$.

2. Preliminaries.

Throughout this paper we make the following assumptions:

(A) There exists a constant $\gamma>0$ such that

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$$

for all $\xi \in \mathbf{R}^n$ and $x \in Q$, moreover $a_{ij} \in C^1(\bar{Q})$ and $a_{ij} = a_{ji}$, $i, j = 1, \dots, n$.

(B) The nonlinearity $f: \mathbf{R} \to \mathbf{R}$ is Lipschitz and satisfies the Ambrosetti-Prodi conditions (3).

(C) The boundary data $\phi \in L^\infty(\partial Q)$, $h \in L^\infty(Q)$ and $h \perp \theta$.

It is well known that $\theta(x)$ can be taken positive on Q. We always assume that θ is normalized, that is, $\int_Q \theta(x)^2 dx = 1$.

A function u is said to be a weak solution of the equation (1_s), if $u \in W^{1,2}_{loc}(Q)$ and u satisfies

$$(4) \qquad \int_Q \sum_{i,j=1}^{n} a_{ij}(x) D_i u D_j v \, dx = \int_Q [f(u) + s\theta(x) + h(x)] v(x) \, dx$$

for every $v \in W^{1,2}(Q)$ with compact support in Q.

Since not every function ϕ in $L^\infty(\partial Q)$ is a trace of an element from $W^{1,2}(Q)$ the boundary condition (2) requires a proper formulation. We adopt here the L^2-approach developed in papers [4], [5], [6] and [16]. To formulate the meaning of the boundary condition (2) we need some terminology and definitions.

It follows from the regularity of the boundary ∂Q that there is a number $\delta_0 > 0$ such that for $\delta \in (0, \delta_0]$ the domain $Q_\delta = Q \cap \{x ; \min_{y \in \partial Q} |x - y| > \delta\}$, with the boundary ∂Q_δ, possesses the following property: to each $x_0 \in \partial Q$ there is a unique point $x_\delta(x_0) \in \partial Q_\delta$ such that $x_\delta(x) = x_0 - \delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The above relation gives a one-to-one mapping, of class C^1, of ∂Q onto ∂Q_δ.

According to Lemma 14. 16 in [11] (p. 355), the distance $r(x) = \text{dist}(x, \partial Q)$, $x \in \bar{Q}$, belongs to $C^2(\bar{Q} - Q_{\delta_0})$ if δ_0 is sufficiently small. Denote by $\rho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties: $\rho(x) = r(x)$ for $x \in \bar{Q} - Q_{\delta_0}$, $\rho \in C^2(\bar{Q})$, $\rho(x) \geq \frac{3\delta_0}{4}$ for $x \in Q_{\delta_0}$, $\gamma_1^{-1} r(x) \leq \rho(x) \leq \gamma_1 r(x)$ for $x \in C^2(\bar{Q})$ for some positive constant γ_1, $\partial Q_\delta = \{x ; \rho(x) = \delta\}$ for $\delta \in (0, \delta_0]$ and finally $\partial Q = \{x ; \rho(x) = 0\}$.

Guided by the results of [4], [5], [6] and [16] we adopt the following approach to the Dirichlet problem (1_s), (2_t).

Let $\phi \in L^\infty(\partial Q)$. A weak solution u in $W^{1,2}_{loc}(Q)$ of (1) is a solution of the Dirichlet problem with the boundary condition (2_t) if

$$\lim_{\delta\to 0}\int_{\partial Q}[u(x_\delta(x))-t\phi(x)]^2dS_x=0$$

It follows from Theorem 5 in [5] (see also Theorem 1 in [4]), that if the problem (1_s), (2_t) admits a solution u in $W^{1,2}_{loc}(Q)$, then $u\in\tilde{W}^{1,2}(Q)$, where $\tilde{W}^{1,2}(Q)$ is a weighted Sobolev space defined by

$$\tilde{W}^{1,2}(Q)=\{u\,;\,u\in W^{1,2}_{loc}(Q)\ \text{and}$$
$$\int_Q|Du(x)|^2r(x)\,dx+\int_Q u(x)^2dx<\infty\}$$

and equipped with the norm

$$\|u\|^2_{\tilde{W}^{1,2}}=\int_Q|Du(x)|^2r(x)\,dx+\int_Q u(x)^2dx.$$

3. Main result.

We commence with the following lemma, which shows that a solution of (1_s), (2_t) for fixed s does not exist for t sufficiently large.

Let us denote by $K\phi$ a unique solution in $\tilde{W}^{1,2}(Q)\cap L^\infty(Q)$ of the problem

(5) $\quad Lu=0$ in Q.

(6) $\quad u(x)=\phi(x)$ on ∂Q.

The existence of $K\phi$ follows from Theorem 6 in [5] (see also Lemma 2 in [6]).

LEMMA 1. *If* $\int_Q K\phi(x)\theta(x)\,dx>0$ $\left(\int_Q K\phi(x)\theta(x)\,dx<0\right)$ *then for every* $s\in\boldsymbol{R}$ *there exists a constant* $t_0=t_0(s)$ *such that the problem* (1_s), (2_t) *has no solution in* $\tilde{W}^{1,2}(Q)$ *for* $t>t_0$ $(t<t_0)$.

PROOF. It follows from (3) that there exists a constant b such that

(7) $\quad \lambda_1u-f(u)\le b$

for all $u\in\boldsymbol{R}$. If u is a solution of (1_s), (2_t), then the function $v=u-tK\phi$ is a solution in $\overset{\circ}{W}{}^{1,2}(Q)$ to the problem

$$Lv=f(v+tK\phi)+s\theta(x)+h(x)\ \text{in}\ Q,$$
$$v(x)=0\ \text{on}\ \partial Q.$$

We only consider the case $\int_Q K\phi(x)\theta(x)\,dx>0$. It is clear that

$$0=\int_Q\sum_{i,j=1}^n a_{ij}(x)D_ivD_j\theta dx-\lambda_1\int_Q v\theta dx=\int_Q f(v+tK\phi)\theta dx$$

$$+s-\lambda_1\int_Q v\ \theta\ dx.$$

The estimate (7) yields that

$$(8)\qquad \lambda_1 t\int_Q K\phi\cdot\theta dx=\lambda_1\int_Q(v+tK\phi)\cdot\theta dx-\int_Q f(v+tK\phi)\theta dx-s \\ \leq b\int_Q\theta dx-s.$$

We obtain the assertion of lemma if we set

$$t_0(s)=\frac{b\int_Q\theta(x)dx-s}{\int_Q K\phi\cdot\theta dx}.$$

To proceed further let us denote by $L^\infty_{m,M}(\partial Q)$ $(0<m<M<\infty)$ the set of all functions ϕ in $L^\infty(\partial Q)$ such that $m\leq\phi(x)\leq M$ a.e. on ∂Q.

We also need a slightly modified definition of a super-and subsolution of (1_t), (2_s). We recall that if $\phi\in H^{1/2}(\partial Q)$ *then a function* U in $W^{1,2}(Q)$ is a supersolution of the problem (1_s), (2_t) if

$$\int_Q\sum_{i,j=1}^n a_{ij}(x)D_iUD_jvdx\geq\int_Q[f(U)v+s\theta(x)+h(x)]vdx$$

for every non-negative v in $\overset{\circ}{W}{}^{1,2}(Q)$ and $U(x)\geq t\phi(x)$ on ∂Q in the sense of $H^{1/2}(\partial Q)$. We define a sub-solution of the problem (1_s), (2_t) by reversing the inequality signs in this definition.

If $\phi\in L^\infty(\partial Q)$, then in general $\phi\notin H^{1/2}(\partial Q)$. Therefore we introduce the following modification of this definition.

Let $\phi\in L^\infty(\partial Q)$. A function $U\in W^{1,2}(Q)$ is a supersolution of the problem (1_s), (2_t) if there exists a sequence of functions $\{\phi_m\}$ in $C^1(\partial Q)$ such that $\lim_{m\to\infty}\int_{\partial Q}[\phi(x)-\phi_m(x)]^2dS_x=0$ and for every m U is a supersolution of the problem (1_s), (2_t) with $\phi=\phi_m$. In an obvious way we define a subsolution.

Finally we observe that the condition (3) implies the existence of constants $0<\underline{\mu}<\lambda_1<\overline{\mu}$ and $C>0$ such that

$$(9)\qquad f(u)\geq\underline{\mu}\ u-C$$

and

$$(10)\qquad f(u)\geq\overline{\mu}\ u-C$$

for all $u\in\boldsymbol{R}$.

We are now in a position to establish the following result.

THEOREM 1. *There exists* $s_0 \in \boldsymbol{R}$ *such that for each* $s \leq s_0$ *there exists* $t^*(s)$ *such that for each* $t \leq t^*(s)$ *the problem* $(1_s),(2_t)$ *admits at least one solution in* $\widetilde{W}^{1,2}(Q) \cap L^\infty(Q)$ *for each* $\phi \in L^\infty_{m.M}(\partial Q)$. *If* $t > t^*(s)$ *then there exist functions* $\phi \in L^\infty_{m,M}(\partial Q)$ *for which the problem* (1_s), (2_t) *has no solution.*

PROOF.
Let $N > 0$ and set

$$k = \sup \{\mathrm{f(u)} + \mathrm{h(x)}\,;\ x \in Q,\ 0 \leq u \leq N\}.$$

Let Q_2 and Q_1 be open subsets of Q such that $Q_2 \subset \overline{Q}_2 \subset Q_1 \subset \overline{Q}_1 \subset Q$ with $\delta =$ meas $(Q - Q_2)$ to be determined. By H we denote a continuous function on Q such that $0 \leq H(x) \leq |k|$ on $\overline{Q}$, $H(x) = |k|$ on $\overline{Q} - Q_1$ and $H(x) = 0$ on Q_2. The Dirichlet problem

$$Lu = H(x) \text{ in } Q,$$
$$u(x) = t \cdot M \text{ on } \partial Q,$$

admits a unique solution $U \in W^{1,2}(Q) \cap C(\overline{Q})$. If $t \geq 0$, then by the maximum principle and L^p-estimates for elliptic equations we have

$$0 \leq U(x) \leq Mt + C_1 |k| \delta^{1/p}$$

on Q for some $C_1 > 0$. We now choose $t_0 > 0$ and $\delta > 0$ such that

$$Mt + C_1 k \delta^{1/2} \leq N \tag{11}$$

for $0 \leq t \leq t_0$. It is clear that there exists $s_0 < 0$ such that

$$|k| + s\theta(x) \leq H(x) \text{ on } Q$$

for $s \leq s_0$. Consequently

$$LU = H(x) \geq |k| + s\theta(x) \geq f(U) + s\theta(x) + h(x) \text{ on } Q.$$

It is easy to see that U is a supersolution (1_s), (2_t) $(s \leq s_0, 0 \leq t \leq t_0)$ for each $\phi \in L^\infty_{m,M}(\partial Q)$. To find a subsolution we consider the Dirichlet problem

$$Lu = \underline{\mu} u - C + s\theta(x) + h(x) \text{ in } Q,$$
$$u(x) = 0 \text{ on } \partial Q,$$

where $\underline{\mu}$ and C are constants from the inequality (9). We may always assume that $C > s\theta(x) + h(x)$ on Q for all $s \leq s_0$. Since $\underline{\mu} < \lambda_1$ the maximum principle yields that the solution V of this problem is negative on Q.

We now show that the problem (1_s), (2_t) has a solution in $\tilde{W}^{1,2}(Q)\cap L^\infty(Q)$ for each $\phi\in L^\infty_{m,M}(\partial Q)$ and all $s\leq s_0$, $0\leq t\leq t_0$. It $t=0$, then the existence of a solution follows from [9] and it belongs to $\mathring{W}^{1,2}(Q)\cap L^\infty(Q)$. Therefore we may assume that $t>0$. If $\phi\in L^\infty_{m,M}(\partial Q)$ then we can find a sequence $\{\phi_k\}$ in $C^1(\partial Q)$ such that $\lim_{k\to\infty}\int_{\partial Q}[\phi_k(x)-\phi(x)]^2 dS_x=0$ and $m\leq\phi_k(x)\leq M$ on ∂Q for each k. Since U and V are a super-and subsolution of (1_s), (2_t) with the boundary condition $u(x)=t\phi_k(x)$ on ∂Q for each k, it follows from [9] that the problem (1_s), (2_t) has a solution $u_k\in W^{1,2}(Q)$ satisfying the boundary condition $u_k(x)=t\phi_k(x)$ $(k=1, 2, \ldots)$. It is clear that the sequence $\{u_k\}$ is bounded in $L^\infty(Q)$. We now show that the sequence $\{u_k\}$ is bounded in $\tilde{W}^{1,2}(Q)$. To achieve this we take as a test function in (4)

$$v(x)=\begin{cases} u_k(x)(\rho(x)-\delta) & \text{on } Q_\delta, \\ 0 & \text{on } Q-Q_\delta, \end{cases}$$

and integrating by parts and letting δ tend to 0 we get

$$\int_Q\sum_{i,j=1}^n a_{ij}\ D_iu_kD_ju_k\mu dx=\int_{\partial Q}\sum_{i,j=1}^n a_{ij}\ D_i\rho D_j\rho\phi_k^2 dS_x$$
$$+\int_Q\sum_{i,j=1}^n D_i(a_{ij}D_j\rho)u_k^2dx+\int_Q f(u_k)u_k\rho dx+\int_Q(s\theta+k)u_k\rho dx.$$

Using the ellipticity condition we easily deduce from this inequality that

$$\int_Q|Du_k(x)|^2\rho(x)dx\leq C_1\Big(\int_{\partial Q}\phi_k(x)^2ds_x+\int_Q u_k(x)^2dx+1\Big)$$

for some constant $C_1>0$ independent of u_k. Since $\{\phi_k\}$ is bounded in $L^\infty(\partial Q)$, the sequence $\{u_k\}$ is bounded in $\tilde{W}^{1,2}(Q)$. Consequently, we may assume that u_k converges weakly in $\tilde{W}^{1,2}(Q)$ to a function $u\in\tilde{W}^{1,2}(Q)$. By virtue of Theorem 14. 12 in [15] we may assume that u_k converges to u in $L^2(Q)$. It is clear that u is a weak solution of (1_s) in $\tilde{W}^{1,2}(Q)\cap L^\infty(Q)$. By Theorem 5 in [4] it has a trace $\zeta\in L^\infty(\partial Q)$. Repeating a standard argument one can show that $\zeta(x)=t\phi(x)$ a. e. on ∂Q. Suppose now that for fixed $s\leq s_0$ the problem (1_s), (2_t) is solvable for some $t=t_1$.

We now show that the problem (1_s),(2_t) is solvable for all $t<t_1$ and all $\phi\in L^\infty_{m,M}(\partial Q)$. We only consider the case $t_1<0$. Since a constant function $\phi=m$ belongs to $L^\infty_{m,M}(\partial Q)$ there exists a solution $\overline{U}\in W^{1,2}(Q)$ of the problem

$$Lu=f(u)+s\theta(x)+h(x) \text{ in } Q,$$
$$u(x)=t_1\cdot M \text{ on } \partial Q,$$

and $\overline{U}$ is a supersolution of (1_s), (2_t) with $\phi=t\cdot m$, $t<t_1$. For fixed $t<t_1$, let $\overline{V}$ be a solution to the problem

$$Lu=\underline{\mu}u-C+s\theta(x)+h(x) \text{ in } Q,$$
$$u(x)=t\cdot M \text{ on } \partial Q,$$

where C and $\underline{\mu}$ are the constants from the estimate (9). By virtue of this estimate we have

$$L(\overline{U}-\overline{V})\geq\underline{\mu}(\overline{U}-\overline{V})+C \text{ in } Q$$
$$\overline{U}(x)-\overline{V}(x)=t(m-M) \text{ on } \partial Q$$

and consequently from the maximum principle we deduce that $\overline{U}(x)>\overline{V}(x)$ on Q. It is clear that $\overline{U}$ and $\overline{V}$ are a super-and subsolution of (1_s), (2_t) for each $\phi\in L^\infty_{m,M}(\partial Q)$. Repeating the argument from the previous part of the proof we can show that the problem (1_s), (2_t) is solvable in $\tilde{W}^{1,2}(Q)$ for each $\phi\in L^\infty_{m,M}(\partial Q)$. We now define for $s\leq s_0$

$$t^*(s)=\sup\{t\text{: the problem } (1_s),\ (2_t) \text{ is solvable for all } \phi\in L^\infty_{m,M}(\partial Q)\}.$$

It follows from Lemma 1 that

$$t^*(s)\leq\frac{b\cdot\int_Q\theta(x)\,dx-s}{\int_Q K\phi(x)\theta(x)\,dx}\leq\frac{b}{m}-\frac{s}{m\int_Q\theta(x)\,dx}<\infty.$$

It is evident that for fixed $s\leq s_0$ the problem (1_s), (2_t) is solvable for all $t<t^*(s)$ and all $\phi\in L^\infty_{m,M}(\partial Q)$. It also follows from the definition of $t^*(s)$ that for each $t>t^*(s)$ there must exist $\phi\in L^\infty_{m,M}(\partial Q)$ such that the problem (1_s), (2_t) is not solvable in $\tilde{W}^{1,2}(Q)$. To complete the proof we show that the problem (1_s), $(2_{t^*(s)})$ is solvable for each $\phi\in L^\infty_{m,M}(\partial Q)$. To show this we consider for a given $\phi\in L^\infty_{m,M}(\partial Q)$ the problem (1_s), (2_{t_k}) with $t_k<t^*(s)$ and $\lim_{k\to\infty} t_k=t^*(s)$. For every k there exists at least one solution u_k in $\tilde{W}^{1,2}(Q)$. First we observe that the sequence u_k is bounded below on Q. Indeed, let w_k be a solution of the problem

$$Lu=\underline{\mu}\,\bar{u}-C+s\theta(x)+h(x) \text{ in } Q,$$
$$u(x)=t_k\phi(x) \text{ on } \partial Q,$$

where C and $\underline{\mu}$ are constants from the estimate (9). It is obvious that

$$L(u_k-w_k)=f(u_k)-\underline{\mu}w_k+C\geq\underline{\mu}(u_k-w_k)+C \text{ in } Q,$$

and

$$u_k(x)-w_k(x)=0 \text{ on } \partial Q.$$

Since $u_k-w_k\in \overset{\circ}{W}{}^{1,2}(Q)$ the maximum principle implies that $u_k(x)\geq w_k(x)$ on Q. The maximum principle also implies that the sequence $\{w_k\}$ is bounded in $L^\infty(Q)$ and consequently the sequence $\{u_k\}$ is bounded below. We now show that $\{u_k\}$ is bounded in $\tilde{W}^{1,2}(Q)$. We argue by contradiction. If the sequence $\{u_k\}$ is unbounded in $\tilde{W}^{1,2}(Q)$, we may assume that $\lim_{k\to\infty}\|u_k\|_{\tilde{W}^{1,2}(Q)}=\infty$. We set $z_k(x)=u_k(x)\|u_k\|_{\tilde{W}^{1,2}}^{-1}$. Since $\|z_k\|_{\tilde{W}^{1,2}}=1$ for each k, we may also assume that z_k converges to z in $L^2(Q)$. Since u_k is bounded below on Q, $z(x)\geq 0$ on Q. It is clear that z is a solution in $\tilde{W}^{1,2}(Q)$ of the equation

$$Lz=\bar{\mu}z \text{ in } Q,$$

where $\bar{\mu}$ is a constant from the estimate (10). Repeating the argument from [7] one can show that the trace of z on ∂Q is 0 and consequently $z\in \overset{\circ}{W}{}^{1,2}(Q)$. Using as a test function

$$v(x)=\begin{cases} z_k(x)(\rho(x)-\delta) & \text{on } Q_\delta \\ 0 & \text{on } Q-Q_\delta \end{cases}$$

we can show that z_k converges to z in $\tilde{W}^{1,2}(Q)$ (see [7]). Since $\|z\|_{\tilde{W}^{1,2}}=1$, $z\geq 0$ on Q and $z\in \overset{\circ}{W}{}^{1,2}(Q)$, we obtain a contradiction with the fact that $\lambda_1<\bar{\mu}$.

4. Smooth boundary data and final remarks.

Theorem 1 becomes more transparent if $\phi\in H^{1/2}(\partial Q)\cap L^\infty(\partial Q)$, $\phi\geq 0$ and $\phi\not\equiv 0$ on ∂Q. Inspection of the proof of this theorem shows that in order to construct a super-and subsolution we can replace the boundary condition with a constant function by $u(x)=t\phi(x)$ on ∂Q at the appropriate steps of the proof. Moreover, the number $t^*(s)$ can be estimated by

$$t^*(s)\leq\frac{b\int_Q\theta(x)\,dx-s}{\int_Q K\phi(x)\theta(x)\,dx}.$$

Consequently this observation leads to the following theorem

THEOREM 2. *Let $\phi\in L^\infty(\partial Q)\cap H^{1/2}(\partial Q)$, $\phi\geq 0$ and $\phi\not\equiv 0$ on ∂Q. Then there exists a number s_0 such that for each $s\leq s_0$ there exists a constant $t^*=t^*(s)$ such that the problem (1_s), (2_t) has at least one solution in $W^{1,2}(Q)$ for $t\leq t^*(s)$ and no solution for $t>t^*(s)$.*

In the case where ϕ varies in sign we can establish a local result.

THEOREM 3. *Let $\phi \in L^{\infty}(\partial Q) \cap H^{1/2}(\partial Q)$. Then there exist constants s_* and t_0 such that the problem (1_s), (2_t) has at least one solution in $W^{1,2}(Q)$ for $s \leq s_*$ and $|t| \leq t_0$ and no solution for $s > s_*$ and $|t| \leq t_0$.*

PROOF.
Let

$$k = \max\{f(u) + h(x)\,;\ |u| \leq N,\ x \in Q\}$$

and let H a positive function defined in the proof of Theorem 1 with t_0 and δ satisfying the inequality

$$|t| \sup_{x \in \partial Q} |\phi(x)| + C\delta^{1/p} \leq N$$

for $|t| \leq t_0$. A solution U to the problem

$$Lu = H(x) \text{ in } Q,$$
$$u(x) = t\phi(x),$$

is a supersolution of (1_s), (2_t) with $s \leq s_0$ and $|t| \leq t_0$. In an obvious way we define a subsolution V such that $V \leq U$ on Q. Consequently, the existence of a solution follows from [9]. It is now a routine to show that if the problem (1_s), (2_t) is solvable for some s_1 and $|t| \leq t_0$, then it is solvable for all $s \leq s_1$ and $|t| \leq t_0$. To complete the proof we set

$$s_* = \sup\,\{s\,;\ \text{the problem } (1_s),\ (2_t) \text{ is solvable for } |t| \leq t_0\}.$$

In the next theorem, we show that for a given $t \in \boldsymbol{R}$ and $\phi \in H^{1/2}(Q) \cap L^{\infty}(\partial Q)$ there exists s such that the problem (1_s), (2_t) has a solution.

THEOREM 4. *Let $\phi \in L^{\infty}(\partial Q) \cap H^{1/2}(\partial Q)$. Then for every t there exists s_* such that the problem (1_s), (2_t) has at least one solution for $s \leq s_*$ and no solution for $s > s_*$.*

PROOF.
We modify the construction of a super-and subsolution U and V from the proof of Theorem 1.
Let

$$N > |t| \sup_{\partial Q} |\phi(x)|$$

and set

$$k = \max\,\{f(u) + h(x)\,;\ |u| \leq N,\ x \in Q\}.$$

As in the proof of Theorem 1 we define the function $H(x)$ with δ satisfying the inequality

$$|t| \sup_{\partial Q} |\phi(x)| + C_1\delta^{1/p} \leq N,$$

where C_1 is a constant from the inequality (11). There exists $s_0 < 0$ such that $|k| + s\theta(x) \leq H(x)$ for $x \in Q$ and $s \leq s_0$ and a supersolution of (1_s), (2_t) is defined as a solution of the problem

$$Lu = H(x) \text{ in } Q,$$
$$u(x) = t\phi(x) \text{ on } \partial Q.$$

The corresponding subsolution for a fixed $s \leq s_0$ is defined as a solution to the problem

$$Lu = \mu u - C + s\theta(x) + h(x) \text{ in } Q,$$
$$u(x) = \min(-|t| \sup_{\partial Q} |\phi(x)|, \min_{Q} U(x)) \text{ on } \partial Q$$

and the remaining part of the proof is similar to the proof of Theorem 1.

We point out here that this theorem continues to hold for $\phi \in L^\infty(\partial Q)$.

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The University of Queensland,
Department of Mathematics,