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An existence theorem of foliations with singularities A_k, D_k and E_k

Dedicated to Professor Masahisa Adachi on his 60th birthday

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§ 1. Introduction

In this paper a smooth (C^∞) singular foliation $\mathscr{H}$ of codimension q on a smooth manifold N of dimension n means an equivalence class of an open covering $\{V_s\}_{s\in I}$ of N and a family of smooth maps $\phi_s: V_s \to \boldsymbol{R}^q$ and C^∞-diffeomorphisms $h_{st}(x)$ for $s, t\in I$ and each $x\in V_s\cap V_t$ satisfying cocycle conditions (c. f. [9]). Recall the singularities A_k, D_k, and E_k of smooth functions in [4] and denote one of them by X_k for simplicity. It is known that there exist the submanifolds ΣX_k in $J^\infty(N, \boldsymbol{R}^q)$ such that a smooth map germ $\phi:(N, x)\to(\boldsymbol{R}^q, y)$ is C^∞ equivalent to a C^∞ stable unfolding of a smooth function germ with singularity of type X_k at the corresponding point if and only if the infinite jet map $j\phi: N\to J^\infty(N, \boldsymbol{R}^q)$ is transverse to ΣX_k and $j\phi(x)\in\Sigma X_k$. ΣA_k is the well known Boardman manifold $\Sigma^{n-q+1,1,\cdots,1,0}$ in [5] and see the difinition of ΣD_k and ΣE_k in [3]. So we say in this paper that a point x of N is a singular point of type X_k of $\mathscr{H}$ if $x\in V_s$ for some $s\in I$ and $j\phi_s(x)$ belongs to ΣX_k in $J^\infty(V_s, \boldsymbol{R}^q)$.

The purpose of this paper is to reduce an existence problem of a smooth singular foliation having a class of given singularities of type A_k, D_k and E_k to a homotopy-theoretic one. The result will be stated in a formulation motivated by [7] and [11].

Let P be another smooth manifold of dimension p with smooth (non-singular) foliation $\mathscr{F}$ of codimension q represented by a covering $\{U_i\}_{i\in J}$ of P and a family of smooth maps $\psi_i: U_i\to\boldsymbol{R}^q$. We define the submanifold $\Sigma X_k(\mathscr{F})$ in $J^\infty(N, P)$ as follows. Let $j(\psi_i): J^\infty(N, U_i)\to J^\infty(N, R^q)$ be the induced submersion of ψ_i mapping a jet $z=jf(x)$ onto $j(\psi_i\circ f)(x)$ and $j(u_i): J^\infty(N, U_i)\to J^\infty(N, P)$ be the induced jet map of the inclusion u_i of U_i into P. Then we set $\Sigma X_k(\mathscr{F})$ is the union of all submanifolds $j(u_i)(j(\psi_i)^{-1}(\Sigma X_k))$ for all $i\in J$. It does not depend on the choice of $\{U_i, \psi_i\}$. Let $\Omega(\mathscr{F})$ be any open subbundle of $J^\infty(N, P)$ consisting of a number of (possibly infinite) submanifolds $\Sigma X_k(\mathscr{F})$ and of all jets transverse to $\mathscr{F}$. Note that the adjacency relations of singularities A_k, D_k and E_k in [4] show when $\Omega(\mathscr{F})$ is open. Let π_N and π_P be the canonical projection of $J^\infty(N, P)$ onto N and P respectively. We call a homotopy s_t of sections of the fibre bundle $\Omega(\mathscr{F})$ over N an Ω-homotopy and so s_0 is Ω-homotopic to s_1. When $\pi_p\circ s$ is a proper map, we say that s is π_p-proper.

THEOREM 1.1. *Let $n\geqq q\geqq 2$ and consider the open set $\Omega(\mathscr{F})$ containing $\Sigma^{n-q+1,0}(\mathscr{F})$ at least. Let a continuous (resp. π_p-proper) section s of $\Omega(\mathscr{F})$ over N have a smooth map g defined on a neighbourhood of a given closed subset K in N where $jg=s$ such that jg is transverse to every submanifold $\Sigma X_k(\mathscr{F})$ in $\Omega(\mathscr{F})$ Then there exists a smooth map $f: N\to P$ such that $jf(N)\subset\Omega(\mathscr{F})$, jf is Ω-homotopic to s relative to a neighbourhood of K and that f is a (resp. fine) C^0 approximation of $\pi_P\circ s$.*

For the above singular foliation $\mathscr{H}$ on N consider the space $\bigcup_s(V_s\times \boldsymbol{R}^q)/\sim$ obtained by patching $V_s\times\boldsymbol{R}^q$ and $V_t\times\boldsymbol{R}^q$ for every pair (s, t) under the equivalence relation $(x, y)\sim(x, h_{st}(x)y)$ for $x\in V_s\cap V_t$ and $y\in\mathrm{Domain}(h_{st})$ and a smooth map $s_{\mathscr{H}}$ of N into this space mapping x of Vs into $(x, \phi_s(x))$. Let E be a sufficiently small neighbourhood of the image of $s_{\mathscr{H}}$ and $\mathscr{F}_{\mathscr{H}}$ be a smooth foliation on E difined by the projections $V_s\times\boldsymbol{R}^q$ onto $\boldsymbol{R}^q$. Then we can consider $\Omega(\mathscr{F}_{\mathscr{H}})$ in $J^\infty(N, E)$. As an application of Theorem 1.1 we have the following corollary motivated by [11, Theorem 1.1].

COROLLARY 1.2. Let $n\geqq q\geqq 2$ and $\Omega(\mathscr{F}_{\mathscr{H}})$ contain $\Sigma^{n-q+1,0}(\mathscr{F}_{\mathscr{H}})$ at least. If $js_{\mathscr{H}}: N\to J^\infty(N, E)$ is homotopic to a section $\Omega(\mathscr{F}_{\mathscr{H}})$ over N, then there exists a smooth singular foliation of codimension q concordant to $\mathscr{H}$ which has only singularities of type X_k such that $\Sigma X_k(\mathscr{F}_{\mathscr{H}})\subset\Omega(\mathscr{F}_{\mathscr{H}})$.

The special case of Corollary 1.2 where $\Omega(\mathscr{F}_{\mathscr{H}})$ consists of only $\Sigma^{n-q+1,0}(\mathscr{F}_{\mathscr{H}})$ and all jets transverse to $\mathscr{F}_{\mathscr{H}}$ is essentially [10, Theorem 1, 4. 1] (see also [6]). The version of a trivial foliation of Theorem 1.1 is [1 and 3, Theorem 0.1] except for the approximation property.

In order to eliminate the singularities of the highest order of $\mathscr{H}$ on N induced from $\mathscr{F}$ by deforming f, it is sufficient by Theorem 1.1 that its primary obstruction of jf vanishes. For example if f is an immersion, dim $P=n+1$ and if $\mathscr{F}$ is of codimension n, then we can obtain the precise formula of the primary obstruction for $\Sigma A_n(\mathscr{F})$ written by Stiefel-Whitney classes (c. f. [1, (1.2)]). This is also the Thom polynomial of $\Sigma A_n(\mathscr{F})$ (c. f. [8]).

§ 2. Maps with singularities A_k, D_k and E_k

In this section $\mathscr{F}$ is a trivial foliation on P, that is, every leaf consists of a single point (codim$\mathscr{F}$=dim P) and Ω means $\Omega(\mathscr{F})$. To prove this reduced case of Theorem 1.1 we shall recall the following result which is a special case of [3, Theorem 0. 1].

THEOREM 2.1. Let $\mathscr{F}$ be trivial and $P=\boldsymbol{R}^p(n\geqq p\geqq 2)$. Let a continuous section s of Ω over N have a smooth map g defined on a neighbourhood of a given closed subset K in N where $jg=s$ such that jg is transeverse to every ΣX_k in Ω. Then there exists a smooth map $f:N\to\boldsymbol{R}^p$ such that $jf(N)\subset\Omega$ and that jf is Ω-homotopic to s relative to a neighbourhood of K.

Let d be a metric on P and $\varepsilon(x)$ be any positive continuous function on N (resp. a positive constant when s is not π_P-proper). Let $\bar{s}=\pi_P\circ s$. To induce Theorem 1.1 from Theorem 2.1 we need two locally finite coverings $\{U_V\}_{V\in J}$ and $\{U'_V\}_{V\in J}$ of P defined as follows. Let $P^j(j=1,2,\cdots)$ be compact submanifolds of dimension p such that $P^1\subset P^2\subset\cdots\subset P^j\subset\cdots\subset P$ and $P=\cup_{j=1}^{\infty}P^j$. Let $\varepsilon_j=\min\{\varepsilon(x)/2|x\in(\bar{s})^{-1}(P^j)\}$. Then we can triangulate $P^j\setminus\mathrm{Int}(P^{j-1})$ so that the diameter of every simplex is less than ε_j and that the triangulation of ∂P^j, say K coming from $P^{j+1}\setminus\mathrm{Int}(P^j)$ is a subdivision of that of ∂P^j coming from $P^j\setminus\mathrm{Int}(P^{j-1})$. For every p-simplex σ of P^j having a face σ' in ∂P^j subdivided in K, we subdivide σ by joining every vertex of σ outside of $\overline{\sigma'}$ and σ'. This procedure induces a new triangulation of P^j compatible with K and does not change the triangulation of ∂P^{j-1}. Furthermore the diameter of every simplex of P^j is less than ε_j. For any vertex v of this triangulation of P we consider the open star neighbourhood $v*Lk(v)$, that is, the union of all segments $(1-t)v+ty$ for $y\in Lk(v)$ and $0\leqq t<1$. Then we set

$$U_v=\{(1-t)v+ty \mid 0\leqq t<2/3 \text{ and } y\in Lk(v)\}$$
$$U'_v=\{(1-t)v+ty \mid 0\leqq t<3/4 \text{ and } y\in Lk(v)\}.$$

It follows that the diameter of U'_v is less than $\inf\{\varepsilon(x)\mid x\in(\bar{s})^{-1}(U'_v)\}$.

For any $y\in P$ we define $c(y)$ as the number of v's such that $y\in U_v$ and $P_j=\{y|c(y)\geqq j\}$. P_j is clearly an open set with $P=P_1\supset P_2\supset\cdots\supset P_j\supset\cdots$. The procedure of proof of Theorem 1.1 for a trivial foliation $\mathscr{F}$ is the downward induction arguments on j starting from constructing a required smooth map defined near $(\bar{s})^{-1}(\bar{P}_{p+1})$ and extending it to one on $(\bar{s})^{-1}(\bar{P}_p)$ by Theorem 2.1. For this we shall prove the following.

PROPOSITION 2.2. *Let s be a section given in Theorem* 1.1 *under* *the additional assumption that $\mathscr{F}$ is trivial. For any compact subset C of N and its neighbourhood $U(C)$ there exists an Ω-homotopy $h_{j,t}(j\geqq 1)$ of s relative to a neighbourhood of K such that*

(i) $h_{j,o}=s$, $h_{j,t}|N\backslash U(C)=s|N\backslash U(C)$,
(ii) *there is a smooth map f_j defined on a neighbourhood $((\bar{s})^{-1}(\overline{P_j})\cap C)\cup K$ where $j(f_j)=h_{j,1}$*
(iii) *If $\bar{s}(x)\in U_{v_1}\cap\cdots\cap U_{v_i}$ with $i\leqq j$, then $\pi_P\circ h_{j,t}(x)\in U'_{v_1}\cap\cdots\cap U'_{v_i}$ for any i and t.*

PROOF. The proof is the downward induction on j. The assertion is clearly true for $j>p+1$ by setting $h_{j,t}(x)=s(x)$, when P_j is empty. So we shall induce the assertion of the proposition for j from that for $j+1$.

Take a neighbourhour U' of C in $U(C)$ with compact closure $\overline{U'}$. We decompose $P_j\backslash\overline{P_{j+1}}$ into the connected components, say $\{W_a\}$. Then we can choose finite $W_1,\cdots,W_w$ satisfying

$$(\bar{s})^{-1}(P_j\backslash\overline{P_{j+1}})\cap\overline{U'}=(\bar{s})^{-1}(W_1\cup\cdots\cup W_w)\cap\overline{U'}.$$

This follows from the fact that the number of v's with $(\bar{s})^{-1}(U_v)\cap\overline{U'}\neq\phi$ is finite. For $\{W_u\}(1\leqq u\leqq w)$ we shall prove the assertion that there exists an Ω-homotopy $k_{u,t}:N\to\Omega$ relative to a neighbourhood of $(N\backslash U(C))\cup K$ for $0\leqq u\leqq w$ such that

(0) $k_{0,t}=h_{j+1,t}$,
(1) $k_{u,0}=s$,
(2) there exists a smooth map $f_{j,u}$ defined on a neighbourhood of $((\bar{s})^{-1}(\overline{W_1}\cup\cdots\cup\overline{W_u}\cup\overline{P_{j+1}})\cap C)\cup K$,
(3) If $\bar{s}(x)\in U_{v_1}\cap\cdots\cap U_{v_i}$ with $i\leqq j$, then $\pi_P\circ k_{u,t}(x)\in U'_{v_1}\cap\cdots\cap U'_{v_i}$ for any t.

By the induction assumption for $j+1$ and (0), it is clear that (1), (2) and (3) holds by $f_{j,0}=f_{j+1}$ for $u=0$. Assume that the assertion above for $u-1$ is true. Then by (2) we can take a small neghbourhood T of $((\bar{s})^{-1}(\overline{W_1}\cup\cdots\cup\overline{W_{u-1}}\cup\overline{P_{j+1}}))\cap C\cup K$ such that $f_{j,u-1}$ is defined on a neighbourhood $\overline{T}$. By definition of W_u there exist vertexes $a_1,\cdots,a_j$ such that $W_u\subset U_{a_1}\cap\cdots\cap U_{a_j}$. Then we have $(\bar{s})^{-1}(\overline{W_u})\cap C\subset(\bar{s})^{-1}(U'_{a_1}\cap\cdots\cap U'_{a_j})\cap U'$. Here we take three neighbourhoods $Y_1\supset Y_2\supset Y_3$ of $(\bar{s})^{-1}(\overline{W_u})\cap C$ such that

(a) $Y_1\subset(\bar{s})^{-1}(U'_{a_1}\cap\cdots\cap U'_{a_j})\cap U'$,
(b) for any vertex v distinct from $a_1,\cdots,a_j$, $\overline{Y_1}\cap U_v\subset T$,
(c) $\overline{Y_1}$, $\overline{Y_2}$, and $\overline{Y_3}$ are submanifolds with boundaries and
(d) $\overline{Y_2}\subset Y_1$ and $\overline{Y_3}\subset Y_2$.

Now we can apply Theorem 2.1 to a section $k_{u-1,1}|Y_2: Y_2 \to \Omega|Y_2$ and a smooth map $f_{j,u-1}$ restricted on a neighbourhood of $\bar{T} \cap Y_2$ into $U'_{a_1} \cap \cdots \cap U'_{a_j}$. Then we obtain an Ω-homotopy $k'_t: Y_2 \to \Omega|Y_2$ relative to $\bar{T} \cap Y_2$ satisfying

(i) $k'_0 = k_{u-1,1}|Y_2$,

(ii) there exists a smooth map g defined on Y_2 such that $jg = k'_1$.

This yields an Ω-homotopy $k_t: Y_1 \to \Omega|Y_1$ relative to $\bar{T} \cap Y_1$ such that $\pi_P \circ k_t(Y_1) \subset U'_{a_1} \cap \cdots \cap U'_{a_j}$ and that $k_t|\bar{Y}_3 = k'_t|\bar{Y}_3$, $k_t|(Y_1 \setminus Y_2) = k_{u-1,1}|(Y_1 \setminus Y_2)$ and $k_0|Y_1 = k_{u-1,1}|Y_1$ by the homotopy extension property. Lastly we define $k_{u,t}(x)$ as follows.

$$k_{u,t}(x) = \begin{cases} k_{u-1,2t}(x) & (x \notin Y_1,\ 0 \leqq t \leqq 1/2) \\ k_{u-1,1}(x) & (x \notin Y_1,\ t > 1/2) \\ k_{u-1,2t}(x) & (x \in Y_1,\ 0 \leqq t \leqq 1/2) \\ k_{2t-1}(x) & (x \in Y_1,\ t > 1/2) \end{cases}$$

By difinition (1) is clear for $k_{u,t}$. We set $f_{j,u}$ by

$$f_{j,u}(x) = \begin{cases} f_{j,u-1}(x) & (x \notin Y_2) \\ g(x) & (x \in Y_2). \end{cases}$$

By the construction of k'_t, we have $g|(Y_2 \setminus \overline{Y_3}) = f_{j,u-1}|(Y_2 \setminus \overline{Y_3})$. Hence $f_{j,u}$ is well defined and satisfies (2). If $x \notin Y_1$, then $k_{u,t}(x)$ satisfies (3) since $k_{u-1,t}(x)$ does. If $x \in Y_1$ and $x \in Y_1 \cap U_v$ for some v distinct from $a_1, \cdots, a_j$, then $x \in T$ where $k_t(x)$ is fixed. Therefore $k_{u,t}(x)$ satisfies (3) again. Otherwise $\pi_P \circ k_{u,t}(x) \in U'_{a_1} \cap \cdots \cap U'_{a_j}$ shows (3) since $x \notin U_v$ for any v distinct from $a_1, \cdots, a_j$.

Now we finish the proof by setting $h_{j,t} = k_{w,t}$. Since $(\bar{s})^{-1}(\overline{P_{j+1}} \cup \overline{W_1} \cup \cdots \cup \overline{W_w}) \cap C = (\bar{s})^{-1}(\overline{P_j}) \cap C$, the properties (i), (ii) and (iii) of the proposition holds for $h_{j,t}$ and $f_j = f_{j,w}$. Q.E.D.

REMARK 2.3. It follows from Proposition 2,2 for $j=1$ that f is defined on a neighbourhood of $C \cup K$ and that $\pi_P \circ h_{1,t}$ is $\varepsilon(x)$-approximation of $\bar{s}$.

§ 3. Proof of Theorem 1.1

First we shall prove the following preliminary version of Theorem 1.1.

THEOREM 3.1. *Let s be a section given in Theorem* 1.1 *under the same assumption and $\varepsilon(x)$, any positive function on N. For any compact subset C of N and its neighbourhood $U(C)$ there exists an Ω-homotopy h_t relative to a neighbourhood of $(N \setminus U(C)) \cup K$ such that*

(i) $h_0 = s$,

(ii) *there exists a smooth map f defined on a neighbourhood of* $C \cup K$ *where* $jf = h_1$,

(iii) $d(\pi_P \circ h_t(x), \bar{s}(x)) < \varepsilon(x)$ *for any* $x \in N$.

PROOF. We take two special countable coordinates of $\mathscr{F}$, $\{U_j, \phi'_j \times \phi_j'\}$ and $\{U_j', \phi_j \times \phi_j'\}$ $(j=1, 2, \cdots)$ such that $\overline{U_j} \subset U_j'$, $\overline{U_j'}$ is compact and that $\phi_j \times \phi_j' | U_j$ is a restriction of the diffeomorphism $\phi_j \times \phi_j' | U_j' \to \boldsymbol{R}^q \times \boldsymbol{R}^{p-q}$ for ever j. For every $\phi_j \times \phi_j'$ we can take suitable metrices d_j on $\boldsymbol{R}^q$ and d_j' on $\boldsymbol{R}^{p-q}$ coming from the product structure of U_j' such that

$$d(x, y) \leqq d_j(\phi_j(x), \phi_j(y)) + d'_j(\phi'_j(x), \phi'_j(y))$$

for any $x, y \in U_j'$. We define a series of compact subsets N_i of N by $N_i = (\bar{s})^{-1}(\bigcup_{j=1}^{i} \overline{U_j})$ $(N_0 = \phi)$. Then for a sufficiently large number i_0 we have $N_{i_0} \supset C$. We define positive number ε' by

$$\varepsilon' = \min\{\text{distance}(\overline{U_j}, P \backslash U_j') \mid 1 \leqq j \leqq i_0\}$$

For $j \geqq 0$ we can construct a series of continuous sections $s_j : N \to \Omega(\mathscr{F})$ and Ω-homotopies $h_{j-1,t}$ of s_{j-1} up to s_j relative to a neighbourhood of $(N_{j-1} \cap C) \cup (N \backslash U(C)) \cup K$ such that there exist smooth maps f_j defined on a neighbourhood of $(N_j \cap C) \cap K$ where $jf_j = s_j$ and that for any t

$$(*) \quad d(\pi_P \circ s_{j-1}(x), \pi_P \circ h_{j-1,t}(x)) < \min(\varepsilon', \varepsilon'(x))/i_0.$$

In fact, for $j=0$ the assertion is trivial by setting $h_{-1,t} = s_0$. Assume the assertion for $j-1$. Let L_j denote $(\bar{s})^{-1}(U'_j)$. In order to construct $h_{j-1,t}$ we take a neighbourhood O of $(\bar{s})^{-1}(\overline{U_j}) \cap C$ in L_j with $\bar{O} \subset L_j$. By applying Proposition 2.2 and Remark 2.3 to the section $(j\phi_j \circ s_{j-1}) | L_j$, the compact set $(\bar{s})^{-1}(\overline{U_j}) \cap C$, its open neighbourhood O, a smooth map f_{j-1} restricted to a neighbourhood of $((N_{j-1} \cap C) \cup K) \cap L_j$ and $\min(\varepsilon', \varepsilon(x))/i_0$ it follows that there exists an Ω-homotopy of $(j\phi_j \circ s_{j-1}) | L_j$

$$k_{j-1,t} : L_j \longrightarrow \Omega | L_j \subset J^\infty(L_j, \boldsymbol{R}^q)$$

relative to $(L_j \backslash 0) \cup ((N_{j-1} \cap C) \cup K) \cap L_j$ such that there exists a smooth map f_j' defined on a neighbourhood of $((N_j \cap C) \cup K) \cap L_j$ into $\boldsymbol{R}^q$ where $jf_j' = k_{j-1,1}$ and that

$$d_j(\pi_{\boldsymbol{R}^q} \circ j\phi_j \circ s_{j-1}(x), \ \pi_{\boldsymbol{R}^q} \circ k_{j-1,t}(x)) < \min(\varepsilon', \varepsilon(x)/2i_0.$$

We can construct a homotopy $k_t(x)$ of continuous sections of $J^\infty(L_j, \boldsymbol{R}^{p-q})$ over L_j relative to $(L_j \backslash O) \cup (N_{j-1} \cap C) \cup K) \cap L_j$ such that

(a) $k_0(x) = j\phi'_j \circ s_{j-1}(x)$

(b) there is a smooth map $f''_j(x)$ defined near $((N_j \cap C) \cup K) \cap L_j$ into $\boldsymbol{R}^{p-q}$ where $jf''=k_1$ and

(c) $d'_j(k_0(x), k_t(x)) < \min(\varepsilon', \varepsilon(x))/2i_0$

since its fibre is an Eucledian space (or by the similar arguments as Proposition 2.2).

We lift $k_{j-1,t}$ to a section $\bar{k}_{j-1,t}$ of $\Omega(\mathscr{F})|L_j$ as

$$\bar{k}_{j-1,t}(x) = (j\phi_j \times \phi'_j))^{-1}(k_{j-1,t}(x), k_t(x)).$$

Since $\bar{k}_{j-1,t}$ and s_{j-1} coincide on $L_j \backslash O$, we can define $h_{j-1,t}$ and f_j by

$$h_{j-1,t}(x) = \begin{cases} s_{j-1}(x) & ,x \notin L_j \\ \bar{k}_{j-1,t}(x) & ,x \in L_j, \end{cases}$$

$$f_j(x) = \begin{cases} f_{j-1}(x) & \text{near } ((N_{j-1} \cap C) \cup K) \cap (N \backslash O) \\ (f'_j(x), f''_j(x)) & \text{near } ((N_j \cap C) \cup K) \cap L_j. \end{cases}$$

Then it follows from $d \leqq d_j + d'_j$ that the required inequality (*) holds. We note that if $\bar{s}(x) \in \overline{U_j}$, then $\pi_P \circ h_{j-1,t}(x) \in U'_j$ for any j by (*). Thus we obtain an Ω-homotopy h_t patching $h_{j,t}$ from $j=0$ to i_0-1 and a smooth map f as f_{i_0} as required. Q. E. D.

PROOF OF THEOREM 1.1. The case where N is compact in Theorem 1.1 is a direct consequence of Theorem 3.1. Therefore we let N be noncompactand $\bar{s}$, π_p-proper. Then we have a series of compact submanifolds $N_1 \subset N_2 \subset \cdots \subset N_j \subset \cdots$ with $N = \bigcup_{j=1}^{\infty} N_j$. It follows from Theorem 3.1 that we can construct a series of sections $s_j : N \to \Omega(\mathscr{F})$ and Ω-homotopies $h_{j,t}$ of s_j up to s_{j+1} relative to a neighbourhood of $N_j \cup K$ ($N_0 = \phi$ and $s_0 = s$) such that there exist smooth maps f_j defined on a neighbourhood of $N_j \cup K$ where $jf_j = s_j$ and that

$$d(\pi_P \circ s_j(x), \pi_P \circ h_{j,t}(x)) < \varepsilon(x)/2^{j+1}.$$

Now we difine an Ω-homotopy h_t of s and a smooth map of Theorem 1.1 by patching $h_{j,t}$ and $f_j (j=0, 1, 2, \cdots)$. That is, for t with $1-(1/2^j) \leqq t \leqq 1-(1/2^{j+1})$, set

$$h_t(x) = h_{j, 2^{j+1}t+2-2^{j+1}}(x) \qquad (j=0, 1, 2, \cdots)$$

and for $t=1$ and $x \in N_j$, set $h_1 = s_j(x)$. So f is defined to coincide with f_j on a neighbourhood of $N_j \cup K$. By definition it is easy to see

$$d(\bar{s}(x), \pi_P \circ h_t(x)) < \varepsilon(x).$$

This completes the proof. Q. E. D.

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References

[1] Y. ANDO, On the higher Thom polynomials of Morin singularities, Publ. R. I. M. S., Kyoto Univ., 23(1987), 195-207.

[2] ———, On local structures of the singularities A_k, D_k and E_k of smooth maps, to appear in Trans. Amer. Math. Soc..

[3] ———, Homotopy principles of smooth maps having only singularities A_k, D_k and E_k, preprint.

[4] V. I. ARNOLD, Normal forms for functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangean singularities, Func. Anal. Appl., 6(1973), 245-272.

[5] J, M, BOARDMAN, Singularities of differentiable maps, I. H. E. S. Publ., Math., 33 (1967), 21-57.

[6] Y. M. ELIASHBERG, Surgery of singularities of smooth mappings, Math. USSR Izv., 6 (1972), 1302-1326.

[7] M. L. GROMOV, Stable mappings of foliations into manifolds, Math. USSR IZV., 3 (1969), 671-694.

[8] A. HAEFLIGER and A. KOSHINSKI, Un théorème de Thom sur les singularités des applications différentiables, Séminaire H. Cartan E. N. S., 8(1956/57).

[9] A. HAEFLIGER, Homotopy and integrability, Manifolds-Amsterdam 1970, Lecture Notes in Math., 209(1971), 133-163.

[10] N. M. MISHACHEV and Y. M. ELIASHBERG, Surgery of singularities of foliations, Func. Anal. Appl., 11(1977), 43-53.

[11] W. THURSTON, The theory of foliations of codimension greater than one, Comm. Math. Helv., 49(1974), 214-231.

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