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A unit group in a character ring of an alternating group

Dedicated to Professor Kazuhiko Hirata on his 60th birthday

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1. Introduction

Throughout this paper, G denotes always a finite group, Z a ring of rational integers, Q a rational field and C a complex field. Let $\{x_1$(a principal character),$\cdots,x_h\}$ be the set of all irreducible C-characters of G. We denote this set by $\mathrm{Irr}(G)$. Let us set

$$R(G)=\{\sum_{i=1}^{h} a_i x_i | a_i \in Z\}$$

That is, $R(G)$ is the set of generalized characters of G. It is well known that $R(G)$ forms a commutative ring with an identity element x_1. We call $R(G)$ a character ring of G.

Let ζ be a primitive $|G|$-th root of unity and let $K=Q(\zeta)$ be the smallest subfield of C containing Q and ζ. We denote by A the ring of algebraic integers in K. In the paper of [9], we have proved the following theorem and corollary.

THEOREM 1.1. *Any unit of finite order in* $A\otimes_z R(G)$ *has the form* $\varepsilon\chi$ *for some linear character* χ *of* G *and some unit* ε *in* A.

COROLLARY 1.2. *Any unit of finite order in* $R(G)$ *has the form* $\pm\chi$ *for some linear character* χ *of* G.

We denote by $U(R(G))$ a unit group of $R(G)$. In section 2, we shall prove that $U(R(G))$ is finitely generated. Hence a factor group $U(R(G))/U_f(R(G))$ is a free abelian group of finite rank, where $U_f(R(G))$ is the group which consists of units of finite order in $R(G)$.

In this paper, we intend to compute the rank of $U(R(A_n))/U_f(R(A_n))$, where A_n is an alternating group on n symbols.

2. Preliminaries

We first show that $U(R(G))$ is finitely generated.

THEOREM 2.1. *For a finite group* G, $U(R(G))$ *is finitely generated.*

PROOF. Let ζ be a primitive $|G|$-th root of unity, and let $K=Q(\zeta)$ be the smallest subfield of C containing Q and ζ. Let us denote by A the ring of algebraic integers in K. Let $\mathfrak{C}_1,\cdots,\mathfrak{C}_h$ be a full set of conjugacy classes in G and let $c_1=1,\cdots,c_h$ be the representatives of $\mathfrak{C}_1,\cdots,\mathfrak{C}_h$ respectively. Let u be an element of $U(R(G))$.

Then there exists $u'\in R(G)$ such that

$$uu'=\chi_1 \text{ (a principal character).}$$

Hence $u(c_i)\cdot u'(c_i)=1$ $(i=1,\cdots,h)$. If χ is an irreducible C-character of G, then $\chi(c_i)\in A$ $(i=1,\cdots,h)$. Therefore $u(c_i)\in A$, $u'(c_i)\in A$ $(i=1,\cdots,h)$. That is, $u(c_i)$ and $u'(c_i)$ are units in A $(i=1,\cdots,h)$. We denote by $U(A)$ a unit group of A.

Now we define a mapping φ from $U(R(G))$ to a direct product of h copies of $U(A)$;

$$\varphi: U(R(G))\ni u \longrightarrow (u(c_1),\cdots,u(c_h))\in U(A)\times\cdots\times U(A) \quad (h \text{ copies})$$

Then it is clear that φ is a homomorphism and injective. Since A is the ring of algebraic integers in K, $U(A)$ is finitely generated by Dirichlet's Theorem. Therefore $U(A)\times\cdots\times U(A)$ is an abelian group which is finitely generated. As $U(R(G))$ is isomorphic to a subgroup of $U(A)\times\cdots\times U(A)$, $U(R(G))$ is finitely generated. The theorem is proved.

Q. E. D.

There are three irreducible C-characters of A_3 (an alternating group on three symbols). We denote them by χ_1, χ_2, χ_3. Each χ_i is a linear character and $\chi_i(x)\in Q(\sqrt{-3})$ for $x\in A_3$. Hence for any $\psi\in R(A_3)$, $\psi(x)\in Q(\sqrt{-3})$ for $x\in A_3$. Since $U(Q(\sqrt{-3}))=\{\pm 1, \pm\rho, \pm\rho^2\}$ where $\rho=(-1+\sqrt{-3})/2$, by the proof of Teorem 2. 1, we can see that any unit in $R(A_3)$ is of finite order. Therefore we have $U(R(A_3))=U_f(R(A_3))=\{\pm\chi_1, \pm\chi_2, \pm\chi_3\}$, by Corollary 1.2.

A_4 has four irreducible C-characters χ_1, χ_2, χ_3, χ_4 such that $\chi_1(1)=\chi_2(1)=\chi_3(1)=1$ and $\chi_4(1)=3$. For any $x\in A_4$, $\chi_i(x)\in Q(\sqrt{-3})$ $(i=1,2,3,4)$. Analogously we have $U(R(A_4))=U_f(R(A_4))=\{\pm\chi_1, \pm\chi_2, \pm\chi_3\}$.

For a natural number $n\geqq 5$, A_n is a simple group. And so $A_n=D(A_n)$ (a commutator subgroup of A_n). Hence A_n has only one linear character χ_1 (i. e. a principal character). By Corollary 1. 2, we have $U_f(R(A_n))=\{\pm\chi_1\}$.

From now on, we may assume $n\geqq 5$, when we consider about $U(R(A_n))$, and we use a notation " $U(R(A_n))/\{\pm 1\}$ " in place of " $U(R(A_n))/U_f(R(A_n))$" for simplicity, by identifying $\{\pm 1\}$ with $\{\pm\chi_1\}$.

Now we state the irreducible C-characters of an alternating group A_n. The irreducible characters of the symmetric groups which are not self-associated, are also irreducible characters of the alternating groups.

Every self-associated character of the symmetric group S_n is the sum of two irreducible characters of the alternating group A_n. These two irreducible characters of A_n take exactly half the values of the character of S_n, except for the conjugacy class for which the value of the character of S_n is ± 1. This conjugacy class splits into two for A_n, and it is for these conjugacy classes alone that the two irreducible characters of A_n differ, the characteristic values in the two conjugacy classes being interchanged for the second character.

Again we repeat these circumstances explicitly. (See p 222 of [1]) Let $[m_1,\cdots,m_r]$, $m_1+\cdots+m_r=n$ be a self-associated frame. In the following way, we can assign to $[m_1,\cdots,m_r]$ a conjugacy class of S_n with cycles of odd lengths $q_1>q_2\cdots>q_k$, $q_1+q_2+\cdots+q_k=n$; let q_1 be the length of the "hook" consisting of the first row and the first column; $q_1=2m_1-1$. If this hook is deleted, another self-associated frame remains, from which we determine q_2 in the same way; $q_2=2(m_2-1)-1=2m_2-3$. We continue thus until there is nothing left.

Here we use the following notation;
$(q_1, q_2,\cdots,q_k)$=a conjugacy class of S_n with cycles of lengths $q_1>q_2>\cdots>q_k$, $q_1+q_2+\cdots+q_k=$n.

Then the following two theorems, which play a fundamental role, are well known (See p 222-223 of [1]).

THEOREM 2.2. *The character of a self-associated representation of S_n which corresponds to a self-associated frame $[m_1,\cdots,m_r]$, $m_1+\cdots+m_r=n$ is*

$$(-1)^{\frac{1}{2}(n-k)}=(-1)^{\frac{1}{2}(p-1)}$$

in the conjugacy class $(q_1, q_2,\cdots,q_k)$ which is assigned to $[m_1,\cdots,m_r]$ where $p=q_1q_2\cdots q_k$; in all other conjugacy classes it is an even number.

THEOREM 2.3. (Frobenius's theorem) *Let χ be a self-associated character of S_n which corresponds to a self-associated frame $[m_1,\cdots,m_r]$, $m_1+\cdots+m_r=n$. Then we have*

(i) *If we consider χ as a character of A_n, χ is the sum of two irreducible characters χ_1, χ_2 of A_n; $\chi=\chi_1+\chi_2$*

(ii) *If $(q_1, q_2,\cdots,q_k)$ is a conjugacy class which is assigned to $[m_1,\cdots, m_r]$, then $(q_1, q_2,\cdots,q_k)$ splits into two conjugacy classes $\mathfrak{C}'$, $\mathfrak{C}''$ of A_n. The values of χ_1 and χ_2 are*

$$\frac{\lambda\pm\sqrt{p\lambda}}{2}$$

in the two classes $\mathfrak{C}'$, $\mathfrak{C}''$, *where* $\lambda=(-1)^{\frac{1}{2}(n-k)}=(-1)^{\frac{1}{2}(p-1)}$ *and* $p=q_1q_2\cdots q_k$. *The values of* χ_1 *and* χ_2 *are equal in all other conjugacy classes of* A_n ; $\chi_1=\chi_2=\frac{1}{2}\chi$.

DEFINITION 2.4. For a natural number n, we define a nonnegative rational integer $c(n)$ as follows;
$c(n)$=the number of self-associated frames $[m_1,\cdots,m_r]$, $m_1+\cdots+m_r=n$ such that

(i) p is not the square of a number. (i.e. $\sqrt{p}\notin Q$)
(ii) $p\equiv 1$ (mod. 4).

Where we assign to $[m_1,\cdots,m_r]$ a conjugacy class $(q_1, q_2,\cdots,q_k)$ and $p=q_1q_2\cdots q_k$.

EXAMPLE. We compute $c(15)$. There are three self-associated frames; $[8,1,\cdots,1]$, $[5,4,3,2,1]$, $[4,4,4,3]$. We can assign to $[8,1,\cdots,1]$, $[5,4,3,2,1]$, $[4,4,4,3]$ conjugacy classes of S_{15} (15), (9, 5, 1), (7, 5, 3) respectively. And conjugacy classes (15), (9, 5, 1), (7, 5, 3) determine odd numbers 15, $9\times5\times1=45$, $7\times5\times3=105$ respectively. $15\not\equiv1$ (mod. 4), $45\equiv1$ (mod. 4), $105\equiv1$ (mod. 4). Therefore we have $c(15)=2$.

In this paper our intention is to show that the rank of $U(R(A_n))/\{\pm1\}$ is equal to $c(n)$. (See Theorem 4.2.)

3. Construction of unit elements

In this section we construct a unit element of $R(A_n)$ which is not of finite order.

Let $[m_1,\cdots,m_r]$, $m_1+\cdots+m_r=n$ be a self-associated frame and let $(q_1, q_2,\cdots,q_k)$ be a conjugacy class of S_n which is assigned to $[m_1,\cdots,m_r]$. We set $p=q_1q_2\cdots q_k$. In addition we assume that $p\equiv1$ (mod. 4) and p is not the square of a number. Hence $Q(\sqrt{p})$ is the real quadratic field. Here we state several lemmata in the above situation.

LEMMA 3.1. *A conjugacy class* $(q_1, q_2,\cdots,q_k)$ *of* S_n *consists of* $|S_n|/p$ *elements.*

PROOF. Since $(q_1, q_2,\cdots, q_k)$ is a conjugacy class with cycles of lengths $q_1>q_2>\cdots>q_k$, $q_1+q_2+\cdots\ q_k=n$, then it consists of

$$\frac{n!}{q_1q_2\cdots q_k}=\frac{|S_n|}{p}$$

elements (See p 31 of [1]). The lemma is proved. Q. E. D.

LEMMA 3. 2. *We set* $p=\ell^2 p_0$, (p_0 *: square-free*). *Then we have*

(i) $p_0\equiv 1$ (*mod.* 4)

(ii) *If* $\frac{1}{2}(t+u\sqrt{p})$, $t, u\in Z$ *is an algebraic integer in* $Q(\sqrt{p_0})$, *then* $t\equiv u$ (*mod.* 2)

(iii) *If* ε_0 *is a fundamental unit of* $Q(\sqrt{p_0})$, *then the units of* $Q(\sqrt{p_0})$ *which take the form of* $\frac{1}{2}(t+u\sqrt{p})$, $t, u\in Z$, *are given by* $\pm E_0^n$ ($n=0, \pm 1, \pm 2, \cdots$), *where* $E_0=\varepsilon_0^e$ *for some natural number e.*

PROOF. It is clear that (i) and (ii) hold. For (iii), for example, see p 319 of [8]. Q. E. D.

LEMMA 3. 3. *There exists a unit of* $Q(\sqrt{p})$ *which takes the form of*

$$\frac{1}{2}(a+b\sqrt{p})+1,\ a, b\in Z,\ p\mid a\ (i.\ e.\ a\ divides\ by\ p)$$
$$b\neq 0$$

and of which the norm over Q is equal to 1.

PROOF. By Lemma 3. 2, there exists a unit $\eta=\frac{1}{2}(t+u\sqrt{p})$, $t, u\in Z$ such that $N\eta=1$ where $N\eta$ denotes the norm of η over Q. Hence $t^2-pu^2=4$. Thus $t^2=pu^2+4$. If we set $a=pu^2$, $b=tu$, then we obtain

$$\eta^2=\frac{1}{4}(t^2+pu^2+2tu\sqrt{p})=\frac{1}{2}(a+b\sqrt{p})+1,$$

because a equation $t^2=pu^2+4=a+4$ holds. Thus $\frac{1}{2}(a+b\sqrt{p})+1$ is the desired unit of $Q(\sqrt{p})$ and so the proof is complete. Q. E. D.

Now we construct a unit of $R(A_n)$ which is not of finite order.

Let $[m_1,\cdots,m_r]$, $m_1+\cdots+m_r=n$ be a self-associated frame and let $(q_1, q_2,\cdots,q_k)$ be a conjugacy class of S_n which is assigned to $[m_1,\cdots,m_r]$; ($q_1=2m_1-1$, $q_2=2m_2-3,\cdots$).

Let $\mathfrak{C}'$, $\mathfrak{C}''$ be the two conjugacy classes of A_n into which $(q_1, q_2,\cdots,q_k)$ splits. We set $p=q_1 q_2\cdots q_k$. In addition, we assume that $p\equiv 1$ (mod. 4) and p is not the square of a number. Let

$\frac{1}{2}(a+b\sqrt{p})+1$, $a, b\in Z$ $(p\mid a,\ b\neq 0)$ be the unit of $Q(\sqrt{p})$ which is stated in Lemma 3.3. Then we have Theorem 3.4.

THEOREM 3.4. *There exists a unit ψ of $R(A_n)$ such that*

$$\psi(x)=1 \quad for\ x\in A_n,\ x\notin\mathfrak{C}',\ \mathfrak{C}''.$$
$$\psi(c')=\frac{1}{2}(a+b\sqrt{p})+1,\ \psi(c'')=\frac{1}{2}(a-b\sqrt{p})+1$$

where c', c'' are the representatives of $\mathfrak{C}'$, $\mathfrak{C}''$ respectively.

PROOF. First we note that a self-associated character θ of S_n which corresponds to $[m_1,\cdots,m_r]$, is the sum of two irreducible characters φ_1, φ_2 of A_n, when we consider θ as a character of A_n.

By Theorem 2.3, we assume that

$$\varphi_1(c')=\frac{1}{2}(1+\sqrt{p}),\quad \varphi_1(c'')=\frac{1}{2}(1-\sqrt{p})$$
$$\varphi_2(c')=\frac{1}{2}(1-\sqrt{p}),\quad \varphi_2(c'')=\frac{1}{2}(1+\sqrt{p})$$
$$\varphi_1(x)=\varphi_2(x)\in Z \quad for\ x\in A_n,\quad x\notin\mathfrak{C}',\ \mathfrak{C}''$$

Let χ_1 (a principal character),$\cdots$, χ_s be all other irreducible characters of A_n. Then $\chi_i(c')=\chi_i(c'')\in Z$ $(i=1,\cdots,s)$. Here we show that the class function ψ which is stated in this theorem, is actually written as a linear combination of χ_i and φ_j $(i=1,\cdots,s\,;j=1,2)$ with integral coefficients. Now we pay attention to the fact that $|\mathfrak{C}'|=|\mathfrak{C}''|=|A_n|/p$ (See Lemma 3.1) and that

$$(\psi-\chi_1)(x)=0 \text{ for } x\in A_n,\ x\notin\mathfrak{C}',\ \mathfrak{C}''.$$
$$(\psi-\chi_1)(c')=\frac{1}{2}(a+b\sqrt{p}),\ (\psi-\chi_1)(c'')=\frac{1}{2}(a-b\sqrt{p})$$

We denote by (λ, μ) the inner product of two class functions λ, μ of A_n. That is,

$$(\lambda,\mu)=\frac{1}{|A_n|}\Sigma_{g\in A_n}\lambda(g)\overline{\mu(g)}.$$

Here we compute several inner products as follows

$$(\psi-\chi_1,\chi_i)=\frac{1}{|A_n|}\{|\mathfrak{C}'|(\psi-\chi_1)(c')\overline{\chi_i(c')}+ |\mathfrak{C}''|(\psi-\chi_1)(c'')\overline{\chi_i(c'')}\}= \frac{1}{p}(\frac{a+b\sqrt{p}}{2}+\frac{a-b\sqrt{p}}{2})\chi_i(c')=\frac{a}{p}\chi_i(c')\in Z$$

, because $\chi_i(c')=\chi_i(c'')\in Z$ and a divides by p.

$$(\psi-\chi_1, \varphi_1)=\frac{1}{|A_n|}\{|\mathfrak{C}'|(\psi-\chi_1)(c')\overline{\varphi_1(c')}+ |\mathfrak{C}''|(\psi-\chi_1)(c'')\overline{\varphi_1(c'')}\}$$

$$=\frac{1}{p}\left(\frac{a+b\sqrt{p}}{2}\ \frac{1+\sqrt{p}}{2}+\frac{a-b\sqrt{p}}{2}\ \frac{1-\sqrt{p}}{2}\right)=\frac{1}{2p}(a+bp)\in Z,$$ because $a\equiv b$ (mod. 2), p is an odd number and a divides by p. Analogously we have

$$(\psi-\chi_1, \varphi_2)=\frac{1}{2p}(a-bp)\in Z.$$

Therefore we obtain

$$\psi=\chi_1+\frac{a}{p}\sum_{i=1}^{s}\chi_i(c')\chi_i+\frac{a+bp}{2p}\varphi_1+\frac{a-bp}{2p}\varphi_2\in R(A_n)$$

Now we denote by ψ' the class function of A_n which satisfies

$$\psi'(x)=1 \text{ for } x\in A_n,\ x\notin\mathfrak{C}',\ \mathfrak{C}''$$

$$\psi'(c')=\frac{1}{2}(a-b\sqrt{p})+1,\quad \psi'(c'')=\frac{1}{2}(a+b\sqrt{p})+1.$$

Then we obtain by the same method,

$$\psi'=\chi_1+\frac{a}{p}\sum_{i=1}^{s}\chi_i(c')\chi_i+\frac{a-bp}{2p}\varphi_1+\frac{a+bp}{2p}\varphi_2\in R(A_n)$$

By the proof of Lemma 3. 3, we can see that $\eta^2=\frac{1}{2}(a+b\sqrt{p})+1$, $N\eta=1$, where η is a unit of $Q(\sqrt{p})$. Since $N(\eta^2)=$

$$\left(\frac{a+b\sqrt{p}}{2}+1\right)\left(\frac{a-b\sqrt{p}}{2}+1\right)=1,$$

we have $\psi\psi'=\chi_1$. Therefore ψ is a unit of $R(A_n)$ which is not of finite order. This completes the proof of Theorem 3. 4. Q. E. D.

4. rank $U(R(A_n))/\{\pm 1\}$

Let $\Gamma_1,\cdots,\Gamma_{c(n)}$ be the self-associated frames such that the conditions (i), (ii) in Definition 2. 4. hold. (See Definition 2. 4 about $c(n)$). To each Γ_i, a conjugacy class $\mathfrak{C}_i$ of S_n is assigned and it splits into two conjugacy classes $\mathfrak{C}'_i, \mathfrak{C}''_i$ of A_n. Let c'_i, c''_i be the representatives of $\mathfrak{C}'_i, \mathfrak{C}''_i$ respectively. By Theorem 3. 4, there is a unit ψ_i of $R(A_n)$ which is not of finite order, with respect to $\Gamma_i(i=1,\cdots,c(n))$, and we have

$$\psi_i(x)=1 \text{ for } x\in A_n,\ x\notin\mathfrak{C}_i',\ \mathfrak{C}_i''$$
$$\psi_i(c_i')=\frac{1}{2}(a_i+b_i\sqrt{p_i})+1,\ \psi_i(c'')=\frac{1}{2}(a_i-b_i\sqrt{p_i})+1$$
$$\psi_j(c_i')=\psi_j(c_i'')=1 \quad (i\neq j)$$

where $\frac{1}{2}(a_i\pm b_i\sqrt{p_i})+1$ are units of $Q(\sqrt{p_i})$ as stated in the theorem.

We fix $\psi_1,\cdots,\psi_{c(n)}$ and denote by $\langle\psi_1,\cdots,\psi_{c(n)}\rangle$ an abelian subgroup of $U(R(A_n))$, which is generated by $\psi_1,\cdots,\psi_{c(n)}$. Then we have Lemma 4.1.

LEMMA 4.1. *rank* $\langle\psi_1,\cdots,\psi_{c(n)}\rangle=c(n)$.

PROOF. We keep the above notations. Suppose that $\psi_1^{e_1}\cdots\psi_{c(n)}^{e_{c(n)}}=\chi_1(e_1,\cdots,e_{c(n)}\in Z)$. Then we have

$$1=\chi_1(c_i')=(\psi_1^{e_1}\cdots\psi_{c(n)}^{e_{c(n)}})(c_i')=(\psi_i(c_i'))^{e_i}=(\frac{1}{2}(a_i+b_i\sqrt{p_i})+1)^{e_i}.$$

Hence $e_i=0$ $(i=1,\cdots,c(n))$. Therefore we obtain rank $\langle\psi_1,\cdots,\psi_{c(n)}\rangle=c(n)$. The lemma is proved. Q.E.D.

Finally we can obtain the following main theorem.

THEOREM 4.2. *rank* $U(R(A_n)/\{\pm1\}=c(n)$.

PROOF. We keep the above notations. Let ε_i be a fundamental unit in $Q(\sqrt{p_i})$. By the proof of Lemma 3.3, we can see that $\psi_i(c_i')>0$, $\psi_i(c_i'')>0$ and $\psi_i(c_i')\psi_i(c_i'')=1$. Hence we can assume that there exists a natural mumber h_i such that

$$\psi_i(c_i')=\varepsilon_i^{h_i},\quad \psi_i(c_i'')=\varepsilon_i^{-h_i} \qquad (i=1,\cdots,c(n)).$$

Here we pay attention to the fact that for an imaginary quadratic field K, a unit group $U(K)$ is $\{\pm1\}$, except for the case $K=Q(i)$, $K=Q(\sqrt{-3})$. And in the case $K=Q(i)$, $U(K)=\{\pm1, \pm i\}$ and in the case $K=Q(\sqrt{-3})$, $U(K)=\{\pm1, \pm\rho, \pm\rho^2\}$, $\rho=\frac{1}{2}(-1+\sqrt{-3})$.

For any $\mu\in U(R(A_n))$, $\mu(x)=\pm1$ or $\mu(x)$ is a unit in an imaginary quadratic field for $x\in A_n$, $x\notin\mathfrak{C}_i$, $\mathfrak{C}_i''$. $(i=1,\cdots,c(n))$.

And $\mu(c_i')$, $\mu(c_i'')$ are units in $Q(\sqrt{p_i})$ such that $\mu(c_i'')$ is a conjugate element of $\mu(c_i')$ over Q. And so if $\mu(c_i')=\pm\varepsilon_i^{k_i}$, then $\mu(c_i'')=\pm\varepsilon_i^{-k_i}$. By the above attention, we have $\mu^{12}(x)=1$ for $x\in A_n$, $x\notin\mathfrak{C}_i'$, $\mathfrak{C}_i''$ $(i=1,\cdots,c(n))$. Therefore we can see that

$\mu^{12h_1\cdots h_{c(n)}} \in \langle \psi_1, \cdots, \psi_{c(n)} \rangle$.

Hence we have

$$\text{rank } U(R(A_n))/\{\pm 1\} = \text{rank } \langle \psi_1, \cdots, \psi_{c(n)} \rangle = c(n).$$

This completes the proof of Theorem 4.2. Q. E. D.

Summarizing the results which we have obtained, we have

THEOREM 4.3. *Let S_n and A_n be a symmetric group and an alternating group on n symbols respectively. Then we have*

(i) $U(R(S_n)) = U_f(R(S_n)) = \{\pm\chi_1, \pm\chi_2\}$
where χ_2 is an alternating character, that is,
$\chi_2(\sigma) = 1$ *if σ is an even permutation,*
$\chi_2(\sigma) = -1$ *if σ is an odd permutation.*

(ii) *A_3 and A_4 have three linear characters χ_1, χ_2, χ_3 and*
$U(R(A_3)) = U_f(R(A_3)) = \{\pm\chi_1, \pm\chi_2, \pm\chi_3\}$.
$U(R(A_4)) = U_f(R(A_4)) = \{\pm\chi_1, \pm\chi_2, \pm\chi_3\}$.

For a natural number $n \geqq 5$, we have
If $c(n)=0$, then $U(R(A_n)) = U_f(R(A_n)) = \{\pm\chi_1\}$.
If $c(n) \neq 0$, then the units of $R(A_n)$ have the form

$$\pm \mu_1^{e_1} \cdots \mu_{c(n)}^{e_{c(n)}} \qquad (e_i \in Z,\ i=1, \cdots, c(n))$$

for some fixed $c(n)$ units $\mu_1, \cdots, \mu_{c(n)}$ of $R(A_n)$.

PROOF. It suffices to prove $U(R(S_n)) = \{\pm\chi_1, \pm\chi_2\}$. For any irreducible C-character χ of S_n, $\chi(x) \in Z$ for $x \in S_n$. Hence for any element ψ of $R(S_n)$, $\psi(x) \in Z$ for $x \in S_n$. Let μ be any unit of $R(S_n)$. Then we can see that $\mu(x) = \pm 1$ for $x \in S_n$, by the proof of Theorem 2.1. Therefore μ is a unit of finite order. Hence we have $\mu = \pm\chi_1$ or $\pm\chi_2$, by Corollary 1.2, because S_n has two linear characters χ_1, χ_2. Thus the proof is complete. Q. E. D.

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