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Transversely piecewise linear foliation by planes and cylinders; PL version of a theorem of E. Ghys

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§ 0. Introduction

Let Σ be a closed oriented surface of genus ≥ 2 and $p: E \rightarrow \Sigma$ a oriented S^1-bundle over Σ. Then there exists the cohomology class $e(E) \in H^2(\Sigma, \boldsymbol{Z})$ which is known as the Euler class. We define the Eular number $eu(E) \in \boldsymbol{Z}$ by the formula

$$eu(E) = \langle e(\mathrm{E}), [\Sigma] \rangle.$$

Here $[\Sigma] \in H_2(\Sigma, \boldsymbol{Z})$ denotes the fundamental class of Σ.

The S^1-bundle E has a codimension-one foliation $\mathscr{F}$ transverse to each fiber of it if and only if it satisfies the inequality

$$|eu(E)| \leq |\chi(\Sigma)|,$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ (see [Mil], [Wo]).

Recently E. Ghys found the influence of the qualitative properties of $\mathscr{F}$ on the Euler number $eu(E)$ of E in his paper [Gh]. In order to see this, we will explain the minimal set of $\mathscr{F}$, and the classification of it first.

Let M be a closed manifold and $\mathscr{G}$ a codimension-one foliation of M. A subset S of M is *saturated* if it is a union of leaves of $\mathscr{G}$. Non-empty, closed, saturated subset $\mathscr{M}$ of M is called *minimal* if it is minimal about these properties. Since M is compact, there exists a minimal set $\mathscr{M}$ of M. Any minimal set $\mathscr{M}$ is one of the following three types:

(1) a closed leaf,
(2) M (in this case, the foliation $\mathscr{G}$ is called *minimal*),
(3) an exceptional minimal set (that is, for any point $x \in \mathscr{M}$, there exists a compact arc T through x in M such that $\mathscr{M} \cap T$ is a Cantor set).

Now let E, $\mathscr{F}$, Σ be as above. As is well known, if $\mathscr{F}$ has a closed leaf, then $eu(E) = 0$. And for any integer n with $|n| \leq |\chi(\Sigma)|$, there exists a transversely projective foliated S^1-bundle $(E_n, \mathscr{F}_n)$ over Σ such that $eu(E_n) = n$. The result of E. Ghys ([Gh]) mentioned above is as follows: if $\mathscr{F}$ is of class C^2 and it has an exceptional minimal set, then $|eu(E)|<|\chi(\Sigma)|$.

Now we are interested in the following problem: whether the last inequality holds for transversely piecewise C^2 or transversely $C^{1+\alpha}(0<\alpha<1)$ foliations (see [Gh], [H-K]). One of the reasons why this problem is important is that it has something to do with a topological invariance of Anosov foliations of the unit tangent bundle $T_1\Sigma$ of Σ. In this paper, we consider this problem in the category of transversely piecewise linear foliations and get the following theorem:

THEOREM. *Let Σ, E, $\mathscr{F}$ be as above. If $\mathscr{F}$ is a transversely piecewise linear foliation and has an exceptional minimal set, then we have*

$$|eu(E)|<|\chi(\Sigma)|.$$

§ 1. Reduction of theorem.

Let Σ, E, $\mathscr{F}$ be as in Section 0. E. Ghys proved the following theorem in [Gh. Th.3] for C^0 foliated S^1-bundles (E, Σ).

THEOREM 1.1. *If $|eu(E)|=|\chi(\Sigma)|$, then every leaf of $\mathscr{F}$ is homeomorphic to T^2 or $S^1\times\boldsymbol{R}$ or $\boldsymbol{R}^2$.*

On the other hand, for any connected manifold M, let $\varepsilon(M)$ denote the number of the endset of M. That is

$$\varepsilon(M)=\sup_{K\in\mathscr{K}}\#\left\{U\left|\begin{array}{l}U\text{ is a connected compoent of }M-K\\ \text{and is not relatively compact in }M\end{array}\right.\right\},$$

where $\mathscr{K}$ denotes the family of all compact subsets of M. For example, $\varepsilon(T^2)=0$, $\varepsilon(S^1\times\boldsymbol{R})=2$ and $\varepsilon(\boldsymbol{R}^2)=1$. The following theorem is due to G. Duminy (unpublished, but see [C-C 2]).

THEOREM 1.2. *Let $\mathscr{G}$ be a codimenison-one, C^2 foliation of a closed manifold M. If $\mathscr{G}$ has an exceptional minimal set $\mathscr{M}$, then every semi -proper leaf L in $\mathscr{M}$ has an infinitely many ends, that, is, $\varepsilon(L)=\infty$,*

Then, if $\mathscr{F}$ is of class C^2 and has an exceptional minimal set, then Ghys inequality $|eu(E)|<|\chi(\Sigma)|$ follows from Theorem 1.1 and Theorem 1.2 as above. But Theorem 1.2 is so powerful result that it is stronger than what we need to prove the Ghys inequality. Indeed, whether Theorem 1.2 holds for a piecewise linear (PL) foliation is an open problem. So we consider the following problem which is sufficient to prove a Ghys inequality for PL-foliations:

PROBLEM 1.3. *Let $\mathscr{G}$ be a codimension-one foliation of a closed 3-manifold. If every leaf of $\mathscr{G}$ is homeomorphic to $S^1 \times \boldsymbol{R}$ or $\boldsymbol{R}^2$, then is it minimal ?*

If $\mathscr{G}$ is class C^2, then the answer to PROBLEM 1.3 is a affirmative by THEOREM 1.2. But it is negative for C^1 foliations. Indeed, let $(T^2, \mathscr{G}_D)$ be a Denjoy flow of class C^1 (see [Her]) and $\pi: T^3 = T^2 \times S^1 \to T^2$ a natural projection. Then $\pi^* \mathscr{G}_D$ is a C^1 foliation by cylinders of T^3 which has an exceptional minimal set. The following theorem is one of the main results of this paper which will be proved in Section 2:

THEOREM 1.4. *Let $\mathscr{G}$ be a codimension-one, transversely oriented, transversely PL foliation of a closed manifold M. If every leaf of $\mathscr{G}$ is homeomorphic to $S^1 \times \boldsymbol{R}$ or $\boldsymbol{R}^2$, then it is a minimal foliation.*

The definition of a transversely PL foliation is given is Section 3.

PROOF OF THEOREM IN SECTION 0: Theorem 1.1 and 1.4 implies the theorem.

§ 2. Transversely piecewise linear foliations.

Let $\mathscr{G}$ be a codimension-one, transversely oriented, transversely PL foliation of a 3-dimensional closed manifold M and $\mathscr{T}$ a one-dimensional foliation of M transverse to $\mathscr{G}$. That is, there exists a finite family $\{(U_i, \varphi_i)\}_{i=1}^n$ which satisfies the following four conditions:

(1) $\{U_i\}_{i=1}^n$ is an open cover of M.

(2) $\psi_i: U_i \to (-1, 1)^2 \times (a_i, b_i)$ is a homeomorphism such that

$$\mathscr{G}|_{U_i} = \psi_i^* \{(-1, 1)^2 \times \{t\}\}_{t \in (a_i, b_i)} \text{ and } \mathscr{T}|_{U_i} = \psi_i^* \{\{x\} \times (a_i, b_i)\}_{x \in (-1,1)^2}.$$

(3) If $U_i \cap U_j \neq \emptyset$ $(1 \leq i, j \leq n)$, then there exists a simple foliation chart (U, ψ) such that $U \supset \bar{U}_i \cup \bar{U}_j$. Here, a foliation chart (U, ψ) is simple if it satisfies the condition (2) above.

(4) For every coordinate transformation $\psi_i \circ \psi_j = (f_{ij}, \gamma_{ij})$, there exists an element $g \in PL_+(\boldsymbol{R})$ such that $\gamma_{ij} = g$ on the domain of γ_{ij}. Here, $PL_+(\boldsymbol{R})$ denotes the group of all orientation preserving piecewise linear homeomorphisms of $\boldsymbol{R}$ whose non-differentiable point set has no accumulation point in $\boldsymbol{R}$.

From now on, let M, $\mathscr{G}$, $\mathscr{T}$, $\{(U_i, \psi_i)\}_{i=1}^n$, γ_{ij} be as above and fix them. For every γ_{ij}, its graph has at most finitely many non-differentiable points, which we denote by

$$(x_1^{ij}, y_1^{ij}), \cdots, (x_{l_{ij}}^{ij}, y_{l_{ij}}^{ij}).$$

We define a compact set K by

$$K=\bar{K}_0, \quad K_0=\Big(\bigcup_{\substack{1\le i,j\le n\\ 1\le l\le l_{ij}}} \phi_i^{-1}((-1,1)^2\times\{y_l^{ij}\})\Big)\cup\Big(\bigcup_{\substack{1\le i,j\le n\\ 1\le l\le l_{ij}}} \phi_j^{-1}((-1,1)^2\times\{x_l^{ij}\})\Big).$$

REMARK 2.1. We note that $\mathscr{G}|_{M-K}$ is a transversely affine foliation of $M-K$. Here, a transversely affine foliation means that every transverse trasition function γ_{ij} is a restriction of an affine homeomorphism of $\boldsymbol{R}$.

A holonomy associated to a loop in a leaf of $\mathscr{G}$ can be written as a composite of γ_{ij}. We have the following fundamental proposition due to T. Kanayama ([Kan]):

PROPOSITION 2.2. *For any connected $\mathscr{F}$ saturated open set U, its Dippolito completion $(\hat{U}, \hat{\iota})$ ($\hat{\iota}: \hat{U} \to M$ is the natural immersion induced by the inclusion $\iota: U \hookrightarrow M$.) has a nucleus-arm decomposition $\hat{U}=N\cup A_1 \cup\cdots\cup A_l$ (for some integer $l\ge 0$) such that $(A_i, \hat{\iota}^*\mathscr{F}|A_i)(i=1, \cdots, l)$ is a trivially foliated I-product.*

A leaf L of $\mathscr{G}$ is *semi-proper* if, for any point x of M, there exists a non-degenerate compact arc T which is contained in a leaf of $\mathscr{T}$ such that $L\cap T=\{x\}$. A semi-proper leaf L is *stable* on the proper side if there exists a foliation preserving topological immersion $f:(L\times[0,1), \{L\times\{t\}\}_{t\in[0,1)}) \to (M, \mathscr{G})$ such that $f(x,0)=x$, $f|_{L\times(0,1)}$ is an embedding and $f(L\times(0,1))\cap L=\emptyset$. The following two propositions are easily proved by Remark 2.1 and Proposition 2.2.

PROPOSITION 2.3. *For any semi-proper leaf $L\in\mathscr{G}$, the holonomy associated to any loop in L-K is trivial. Especially L is without holonomy if it is homeomorphic to S^2-Z, where Z is a totally disconnected closed set of S^2.*

PROPOSITION 2.4. *If a semi-proper leaf $L\in\mathscr{G}$ is without holonomy on the proper side, then L is stable on this side. Especially L is stable on the proper side if is homemorphic to S^2-Z, where Z is a totally disconnected closed set of S^2.*

The following proposition is important to think about the stucture of a foliation by planes and cylinders:

PROPOSITION 2.5. *Let U be a connected $\mathscr{G}$-saturated set of M and $(\hat{U}, \hat{\imath})$ a Dippolito completion of U. If $\hat{\imath}^*\mathscr{G}$ has no compact leaf and every leaf of $\hat{\imath}^*\mathscr{G}$ has a finite number of ends, then we have*

(1) *The pair $(\hat{U}, \hat{\imath}^*\mathscr{G})$ is a trivially foliated I-product over a boundary leaf.*

(2) *For every leaf $L \in \mathscr{G}|_U$, its closure* $\mathrm{cl}_M(L)$ *contains* $\mathrm{cl}_M(\hat{\imath}(\partial\hat{U}))$.

PROOF: We take a boundary leaf $\hat{L} \subset \partial\hat{U}$ and a compact 2-dimensional submanifold $\hat{K}$ of $\hat{L}$ with boundary which contains $\hat{\imath}^{-1}(K) \cap \hat{L}$. Let $\hat{B}$ be a non-compact connected component of $\hat{L}$-$\mathrm{int}_{\hat{L}}\hat{K}$. By exchanging $\hat{K}$ for a bigger one if necessary, we can assume that $\partial\hat{B}$ is homeomorphic to S^1 and that $\hat{B}$ is contained in some arm A_i of a nucleus -arm decomposition of $\hat{U}$ as in Proposition 2.2. Now we take a base point x_0 in $\partial\hat{B}$ and let T_* be a leaf of $\hat{\imath}^*\mathscr{T}$ through x_0. If a leaf $L \in \hat{\imath}^*\mathscr{G}$ has non-empty intersection with $\mathrm{int}(T_*)$, then it is a proper leaf without holonomy. Indeed, if not, L or a leaf L' near L contains infinitely many copies of $\hat{B}$ and $\varepsilon(L)=\infty$ or $\varepsilon(L')=\infty$, a contradiction. So L is stable by Proposition 2.4. Then we can construct a topological embedding h: $\hat{L} \times T_* \to \hat{U}$ such that $h(\hat{L} \times \partial T_*) \subset \partial\hat{U}$, $h^* i^* \mathscr{G} = \{\hat{L} \times \{t_*\}\}_{t_* \in T_*}$ and $h^*\hat{\imath}^*\mathscr{T} = \{\{\hat{y}\} \times T_*\}_{\hat{y} \in \hat{L}}$. We can easily see that the image $h(\hat{L} \times T_*)$ is open and closed. Since $\hat{U}$ is connected, then h is a homeomorphism. This completes the proof.

COROLLARY 2.6. *If $\mathscr{G}$ has no compact leaf and every leaf in $\mathscr{G}$ has finite ends, then $\mathscr{G}$ has a unique minimal set.*

PROOF: Let $\mathscr{M}$ be a minimal set of $\mathscr{G}$. If $\mathscr{M} = M$ then $\mathscr{G}$ has a unique minimal set M. Since $\mathscr{M}$ can not be a compact leaf, we may suppose that $\mathscr{M}$ is a exceptional minimal set. For any leaf $L \in \mathscr{G}|_{M-\mathscr{M}}$, its closure $\mathrm{cl}_M(L)$ contains $\mathscr{M}$ by Proposition 2.5. Then $\mathscr{M}$ must be a unique minimal set of $\mathscr{G}$.

§ 3. Foliations by cylinders and planes.

Let M, $\mathscr{G}$, $\mathscr{T}$, K, etc.... be as in Section 2. In this section, we assume that every laef of $\mathscr{G}$ is homeomorphic to $S^1 \times \boldsymbol{R}$ or $\boldsymbol{R}^2$.

If $\mathscr{G}$ is without holonomy, then there exists a Novikov transformation $\phi : \pi_1(M) \to PL_+(\boldsymbol{R})$ (for definition, see [No], [H-H]). In this case, $\phi(\pi_1(M))$ is fixed point free, free abelian group and of rank ≥ 2. Here, fixed point free means that any element $f \neq \mathrm{id}_{\boldsymbol{R}}$ of $\phi(\pi_1(M))$ has no fixed point. By taking a conjugation by a element of $PL_+(\boldsymbol{R})$ if necessary, we can assume that $\phi(\pi_1(M))$ contains a translation by 1. Then it induces a fixed point free, free abelian subgroup $H \subset PL_+(S^1)$ of rank ≥ 1. Then the natural action of H on S^1 is minimal (see [Min]). Thus we have the following theorem:

THEOREM 3.1. *If $\mathscr{G}$ is without holonomy, then it is a minimal foliation.*

Now let L be a leaf of $\mathscr{G}$ and X a topological space. Given a continuous map $c: X \to L$, $f: X \times [a, b] \to M$ is called a *fence* associated with c, if it satisfies the following four conditions:

(1) $0 \in (a, b)$.
(2) $f(x, 0) = c(x)$ for any $x \in X$.
(3) For any $t \in [a, b]$, $f(X \times \{t\})$ is contained in a leaf of $\mathscr{G}$.
(4) For any $x \in X$, f is a topological embedding from $\{x\} \times [a, b]$ into a leaf of $\mathscr{T}$.

As is well known, if $X = [0, 1]^2$, then there always exists a fence $f: [0, 1]^2 \times [a, b] \to L$ $(a < 0 < b)$ associated with any continuous map $c: [0, 1]^2 \to L$ (see [Ni]).

THEOREM 3.2. *If $\mathscr{G}$ is with holonomy, then it is also minimal foliation.*

Before the proof, we prepare the following two theorems. One is about an invariant measure due to S. Goodman and J. Plante ([G-P]) and the other about a holonomy of a semi-proper leaf in a codimension-one foliation of a compact manifold with boundary essentially due to G. Duminy, G. Hector, J. Cantwell and L. Conlon ([He 1], [He 2], [C-C 2]).

THEOREM 3.3. *Let N be a paracompact manifold, $\mathscr{F}$ a continuous foliation of N of arbitrary codimension, S a compact $\mathscr{F}$-saturated set, and $L \subset S$ a non-proper leaf with trivial holonomy such that $H_c^1(L, \mathbf{Z}) \neq 0$ is finitely generated. Then there exists a nontrivial invariant measure in S.*

THEOREM 3.4. *Let N be a compact manifold with corner $\angle N$ and $\mathscr{F}$ a codimension-one foliation of N. Suppose that the boundary ∂N is a union of the compact two manifolds $\partial_{tan} N$ and $\partial_{tr} N$ such that $\partial_{tan} N \cap \partial_{tr} N = \angle N$ and that $\mathscr{F}$ is tangent to $\partial_{tan} N$ and transverse to $\partial_{tr} N$. If $\mathscr{F}$ is a transversely C^2 foliation, than every non-proper semi-proper leaf $L \in \mathscr{F}$, whose closure $\mathrm{cl}_N(L)$ has no intersection with $\partial_{tan} N$, has a non-trivial holonomy.*

PROOF OF THEOREM 3.2: We prove this theorem by a contradiction.

Indeed, we show that the hypothesis of an existence of an exceptional minimal set $\mathscr{M}$ leads to a contradiction.

By Proposition 2.6, $\mathscr{M}$ is a unique minimal set of $\mathscr{G}$. There exists a cylindrical leaf with non-trivial affinely contracting holonomy in $\mathscr{M}$ by hypothesis, Remark 2.1 and Proposition 2.5. This implies that $\mathscr{M}$ has no nontrivial invariant measure. Then by Proposition 2.3 and Theorem 3.3, every semi-proper leaf in $\mathscr{M}$ is homeomorphic to $\boldsymbol{R}^2$. For, if not, a leaf L in $\mathscr{M}$ satisfies the condition of Theorem 3.3 and $\mathscr{M}$ has a non-trivial invariant measure, a contradiction. Using Proposition 2.5, we see that any leaf L in $M-\mathscr{M}$ is homeomorphic to $\boldsymbol{R}^2$. Thus, using Theorem 3.3, we can see that every cylindrical leaf in non-proper on its both sides, and is contained in $\mathscr{M}$, and moreover it has a non-trivial affinely contracting holonomy.

We take a neighborhood of K as follows. Clearly K intersects only finite leaves which we denote by

$$L_1,\cdots,L_k,$$

and define the sets $K_i(i=1,\cdots k)$ by $K_i=K\cap L_i$. If $L_i\cong\boldsymbol{R}^2$, then we take a topological embedding $c_i:[0,1]^2\to L_i$ such that $c_i((0,1)^2)\supset K_i$. And if $L_i\cong S^1\times\boldsymbol{R}$, then we take a cotinuous map $c_i:[0,1]^2\to L_i$ with $c_i(s,0)=c_i(s,1)$ for any $s\in[0,1]$ such that $c_i|_{(0,1)^2}$ is a topological embedding, $c_i((0,1)\times[0,1])\supset K_i$ and the loop $c_{i,0}(t)=c_i(0,t)$ is not null homotopic in L_i. If we take a fence $f_i:[0,1]^2\times[a_i,b_i]\to L_i$ associated with c_i so small, then $S(f_i)=f_i([0,1]^2\times[a_i,b_i])(i=1,\cdots,k)$ is disjoint to each other. We consider the compact set $S=\cup_{1\leq i\leq k}S(f_i)$. Since cylindrical leaf has a non-trivial affinely contracting holonomy, then all leaves in S is topologically classified as follows:

(1) $S^1\times[0,1]$.
(2) $[0,1]^2$.
(3) $[0,1]\times\{y\in\boldsymbol{R}|y\geq0\}$.

Moreover there exists only finite numbers of leaves of type(1) and they are contained in different cylindrical leaves of $\mathscr{G}$. Then for any leaf $L\cong\boldsymbol{R}^2$, the subset $L-S$ is still connected and for any leaf $L'\cong S^1\times\boldsymbol{R}$, the subset $L'-S$ consists of at most two components, which are not relatively compact in L'. This means that for any leaf L in $\mathscr{M}$ the closure $\mathrm{cl}_{M-\mathrm{int}(s)}(L-S)$ coincides with $\mathrm{cl}_M(L)\cap(M=\mathrm{int}(S))$. By exchanging the fence f_i with small one if necessary, we can assume that $\mathscr{M}$ intersects transversely with S at each intersecting point. Then $(M-\mathrm{int}(S),\ \mathscr{G}'=\mathscr{G}|_{M-\mathrm{int}(s)})$ is a transversely affine foliation and has an exceptinal minimal set $\mathscr{M}'=\mathscr{M}-\mathrm{int}(S)$ with $\mathscr{M}'\pitchfork\partial(M-\mathrm{int}(S))$. By Theorem 3.4, any semi

-proper leaf of $\mathscr{M}'$ has a non-trivial holonomy. Therefore any semi -proper leaf of $\mathscr{G}$ must have a non-trivial holonomy. But every semi -proper leaf of $\mathscr{G}$ is homeomorphic to $\boldsymbol{R}^2$, which is a contradiction. This completes the proof.

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