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On bicommutators of modules over H-separable extension rings

Kozo SUGANO

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Abatract

Let a ring A be an H-separable extension of a subring B, and assume that A is left B-finitely generated projective. The aim of this paper is to show under this condition that for any left A-module M the bicommutator A^* of ${}_AM$ is an H-separable extension of the bicommutator B^* of ${}_BM$ such that $V_{A^*}(V_{A^*}(B^*))=B^*$, and that A^* is left B^*-finitely generated projective (Theorem 1). We will also show that, under the additional condition that B is a left (or right) B-direct summand of A, ${}_AM$ has the double centralizer property if and only if ${}_BM$ does. Theorem 1 is a generalization of (4) Theorem 3.3 [7], but it is interesting for itself. This paper contains the correction to an error in [7]. In the proof of 3.3 [7] we put $\Lambda=\mathrm{End}(I)$, $\Delta=\mathrm{End}({}_AI)$ and $\Gamma=\mathrm{End}({}_BI)$, where I is a faithful minimal left ideal of A. We let $\Gamma(\supseteq\Delta)$ opperate on I to the right, and regarded all of A, B, Γ and Δ as subrings of Λ. This is an error. If we had let all of them opperate on I to the left then we were right. In Proposition 1 and Theorem 1 we will give the correct proof of Theorem 3.3 [7] in more general form.

Throughout this paper all rings will have the identities, and all modules over rings will be unitary. Let A be a ring. For any subset S of A, $V_A(S)$ will mean the centralizer of S in A, manely,

$$V_A(S)=\{a \in A : sa=as \text{ for any } s \in S\}$$

For an A-A-module M we will write

$$M^A=\{m \in M : am=ma \text{ for any } a \in A\}$$

Therefore if ${}_AM_R$ and ${}_AN_R$ are A-R-modules for another ring R, $\mathrm{Hom}({}_AM, {}_AN)$ becomes an R-R-module, and we have $[\mathrm{Hom}({}_AM, {}_AN)]^R=\mathrm{Hom}({}_AM_R, {}_AN_R)$. Let M be a left A-module and $\Delta=\mathrm{End}({}_AM)$. We will let Δ operate to the left on M, and make M a left A-Δ-bimodule. We call $\mathrm{End}({}_\Delta M)$ the bicommutator of ${}_AM$, and denote it by $\mathrm{Bic}({}_AM)$. Put $A^*=\mathrm{Bic}({}_AM)$. There exists a natural homomorphism ι of A to A^* such that $\iota(a)(m)=am$, for $a \in A$, $m \in M$. If M is A-faithful, ι is an injection. In this case we will identify A with $Im\iota$, and regard A as a subring of A^*.

Hereafter A will always be a ring with the identity 1 and B a subring of A containing 1, and C and D will be the center of A and the centralizer of B in A, respectively. Note that for each left A-module M $B^*=\mathrm{Bic}({}_BM)$ is a subring of $A^*=\mathrm{Bic}({}_AM)$, and the canonical map of B to B^* is the restriction of the one of A to A^* on B. A is an H-separable extension of B if and only if for any A-A-module M the map g_M of $D\otimes_C M^A$ to M^B defined by $g_M(d\otimes m)=dm$, for $d \in D$, $m \in M^A$, is an isomorphism. As for the fundamental property of H-separable extensions of rings see [2], [4] and [5].

PROPOSITION 1. *Let A be an H-separable extension of B and M a left A-module. Put $A^*=Bic({}_AM)$ and $B^*=Bic({}_BM)$, and let C^* be the center of A^* and D^* the centralizer of B^* in A^*. Futhermore put $\bar{A}=\iota(A)$ and $\bar{B}=\iota(B)$. Then we have $B^*=V_{A^*}(V_{A^*}(B^*))$, $V_{A^*}(\bar{A})=C^*$ and $D^*=V_{A^*}(\bar{B})\cong D\otimes_C C^*$. If furthermore M is faithful as A-module, we have $V_A(V_A(B))=A\cap B^*$, regarding A as a subring of A^*.*

PROOF. Put $\Lambda=\mathrm{End}(M)$, $\Delta=\mathrm{End}({}_AM)$ and $\Gamma=\mathrm{End}({}_BM)$. We will regard M as a left Λ-module. Of course A, Δ and Γ are subrings of Λ, and we have $\Delta=\Lambda^A=V_\Lambda(A)$, $\Gamma=\Lambda^B=V_\Lambda(B)$, $A^*=V_\Lambda(\Delta)$ and $B^*=V_\Lambda(\Gamma)$. Since A is an H-separable extension of B, regarding A^* and Λ as A-A-modules, we have the following two isomorphisms

$$V_{A^*}(\bar{B})=A^{*B}\cong D\otimes_C A^{*A}=D\otimes_C V_{A^*}(\bar{A})$$
$$\Gamma=\Lambda^B\cong D\otimes_C\Lambda^A=D\otimes_C\Delta$$

By the latter isomorphism we have $B^*=\mathrm{End}({}_\Gamma M)=\mathrm{End}({}_{D\text{-}\Delta}M)=[\mathrm{End}({}_\Delta M)]^D=A^{*D}=V_{A^*}(\bar{D})$, where $\bar{D}=\iota(D)$. Then $V_{A^*}(V_{A^*}(B^*))=V_{A^*}(V_{A^*}(V_{A^*}(\bar{D})))=V_{A^*}(\bar{D})=B^*$. On the other hand since $B^*=V_\Lambda(V_\Lambda(\bar{B}))$, we have $V_\Lambda(B^*)=V_\Lambda(V_\Lambda(V_\Lambda(\bar{B})))=V_\Lambda(\bar{B})$ and $V_{A^*}(B^*)=A^*\cap V_\Lambda(B^*)=A^*\cap V_\Lambda(\bar{B})=V_{A^*}(\bar{B})$. Furthermore we see that $C^*=V_\Delta(\Delta)=\mathrm{End}({}_{A-\Delta}M)=[\mathrm{End}({}_\Delta M)]^A=A^{*A}=V_{A^*}(\bar{A})$. Then, $V_{A^*}(B^*)=V_{A^*}(\bar{B})=D\otimes_C V_{A^*}(\bar{A})=D\otimes_C C^*$. The last assertion can be stated more generally. By Proposition 1.5 [4] we have $V_{\bar{A}}(\bar{B})\cong D\otimes_C\bar{C}$, where $\bar{C}$ is the center of $\bar{A}$. Then $V_{\bar{A}}(V_{\bar{A}}(\bar{B}))=V_{\bar{A}}(\bar{D}\,\bar{C})=V_{\bar{A}}(\bar{D})=\bar{A}\cap V_{A^*}(\bar{D})=\bar{A}\cap B^*$.

COROLLARY 1. *Let A be an H-separable extension of B. If there exists a faithful left A-module such that ${}_BM$ has the double centralizer property, that is, $B\cong Bic({}_BM)$, then we have $B=V_A(V_A(B))$.*

PROOF. Regarding A as a subring of A^*, we have $V_A(V_A(B))=A\cap B^*=A\cap B=B$ by the last part of Proposition 1.

COROLLARY 2. *Let A be an H-separable extension of B, and assume that A is left (or right) B-finitetly generated projective. Then if there exists a left A-module M such that ${}_BM$ has the double centralizer property, we have $B=V_A(V_A(B))$.*

PROOF. Put $\mathfrak{a}=\mathrm{Ann}({}_AM)$, the annihilator of ${}_AM$. Then $\mathfrak{a}\cap B=\mathrm{Ann}({}_BM)=0$, since $B=\mathrm{Bic}({}_BM)$. But our assumption implies $\mathfrak{a}=(\mathfrak{a}\cap B)A$ (or $\mathfrak{a}=A(\mathfrak{a}\cap B)$) (See Theorem 3.1 [5]). In either case we have $\mathfrak{a}=0$, which means that M is faithful as A-module. Now apply Corollary 1.

The next lemma has been proved in [6] by the same author. Here we will state it without proof.

LEMMA 1 (Proposition 1 [6]). *In the case where $V_A(V_A(B))=B$, the following conditions are equivalent;*

(i) *A is an H-separable extension of B and left B-finitely generated projective*

(ii) *A is a left $D\otimes_C A^\circ$-generator, and D is C-finitely generated projective.*

Now we can obtain our main theorem, which includes Theorem 3.3 (4) [7].

THEOREM 1. *Let A be an H-separable extension of B. If A is left (resp. right) B-finitely generated projective, then for any left A-module M, $A^*=\mathrm{Bic}({}_AM)$ is an H-separable extension of $B^*=\mathrm{Bic}({}_BM)$ such that $B^*=V_{A^*}(V_{A^*}(B^*))$, and A^* is left (resp. right) B^*-finitely generated projective. If B is a left (resp. right) B-direct summand of A, then B^* is a left (resp. right) B^*-direct summand of A^*.*

PROOF. Put $D^*=V_{A^*}(B^*)$. Then $D^*\cong D\otimes_C C^*$ by Proposition 1. Since D is C-finitely generated projective, D^* is C^*-finitely generated projective. Next, since $D\otimes_C A^\circ\cong\mathrm{End}({}_BA)$, and A is left B-finitely generated projective, A is a left $D\otimes_C A^\circ$-generator. This means that $D\otimes_C A<\oplus(A\oplus A\oplus\cdots\oplus A)$ as D-A-module. Then $D\otimes_C A^*\cong D\otimes_C A\otimes_A A^*<\oplus(A\oplus A\oplus\cdots\oplus A)\otimes_A A^*\cong A^*\oplus A^*\oplus\cdots\oplus A^*$ as D-A^*-module, which means that A^* is a lft $D\otimes_C A^{*\circ}$-generator, while $D\otimes_C A^{*\circ}\cong D\otimes_C C^*\otimes_{C^*}A^{*\circ}\cong D^*\otimes_{C^*}A^{*\circ}$. Furthermore we have $V_{A^*}(V_{A^*}(B^*))=B^*$ by Proposition 1. Now we can apply Lemma 1 to have that A^* is an H-separable extension of B^* and left B^*-finitely generated projective. Now assume that B is a left B-direct summand of A. Then the map $D\otimes_C A\longrightarrow A$, defined by $d\otimes a\longrightarrow da$ for $d\in D$ and $a\in A$, sklits as D-A-map (See Proposition 3.2 [2]). Then the map $D\otimes_C A^*\longrightarrow A^*$, defined by the same way, splits as D-A^*-map, which implies that $B^*=A^{*D}$ is a left B^*-direct summand of A^*. By the left and right symmetry we can prove the assertion in the case A is right B-f•g•projective.

For any ring A and its subring B, if the map π of $A\otimes_B A$ to A such that $\pi(a\otimes b)=ab$, for $a, b \in A$, is an isomorphism, we will write simply $A\otimes_B A\cong A$. In this case we have $C=V_A(B)$, since $C\cong \mathrm{Hom}({}_AA_A, {}_AA_A)\cong \mathrm{Hom}({}_AA\otimes_B A_A, {}_AA_A)\cong V_A(B)$.

LEMMA 2. *$A\otimes_B A\cong A$ if and only if A is an H-separable extension of B such that $C=D$.*

PROOF. Suppose that A is H-separable over B and $C=D$. Then we have an isomorphism η of $A\otimes_B A$ to $\mathrm{Hom}({}_CD, {}_CA)$ such that $\eta(x\otimes y)(d)=xdy$ for $x, y \in A$ and $d \in D$. But $\mathrm{Hom}({}_CD, {}_CA)=\mathrm{Hom}({}_CC, {}_CA)\cong A$. The composition of η and the above isomorphism is equal to π. Thus we have $A\otimes_B A\cong A$. The converse is obvious.

PROPOSITION 2. *Let A be an H-separable extension of B such that A is left B-finitely generated projective and M a left A-module. Suppose that ${}_BM$ has the double centralizer property, and let A^*, B^*, C^*, D^*, and ι be as in Proposition 1. Then ι is an injection, and regarding A as a subring of A^*, we have $B^*=B=V_A(V_A(B))$, $C^*=C$ and $D^*=D$. Furthermore, A^* is left A-finitely generated projective and $A^*\otimes_A A^*\cong A^*$.*

PROOF. For the same reason as Corollary 2 ι is an injection, and we can identify A with $\iota(A)$ and B with B^*. Then since A^* is H-separable over B^* and A is separable over B^*, A^* is H-separable over A. But $C^*=V_{A^*}(A)$ by Proposition 1. Hence we have $A^*\otimes_A A^*\cong A^*$ by Lemma 2. Next, we have $C\subset V_{A^*}(A)=C^*$ and $C^*\subset V_{A^*}(V_{A^*}(B^*))=B^*\subset A$ by Proposition 1. That $C^*\subset A$ implies $C^*\subset C$, and we have $C^*=C$. Then $D^*=DC^*=D$ by Proposition 1. That $B=V_A(V_A(B))$ is due to Corollary 2.

THEOREM 2. *Let A be an H-separable extension of B such that A is left B-finitely generated projective and M a left A-module. Assume furthermore that B is a left (or right) B-direct summand of A. Then, ${}_AM$ has the double centralizer property if and only if ${}_BM$ does.*

PROOF. Suppose $A=A^*$. Then by Proposition 1 we have $V_A(V_A(B))=B^*\cap A=B^*\cap A^*=B^*$, while we have $B=V_A(V_A(B))$ by Proposition 1.2 [4]. Thus we have $B=B^*$. Conversely suppose that $B=B^*$. Then by proposition 2 we have $A\subset A^*$ and $A^*\otimes_A A^*\cong A^*$. On the other hand since B is a left (resp. right) B-direct summand of A, B^* is a left (resp. right) over B^*, A is a left (resp. right) A-direct summand of A^* by Lemma 4.4 [2]. By this fact together with $A^*\otimes_A A^*\cong A^*$, we have $A=A^*$.

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Department of Mathematics
Faculty of Science
Hokkaido university
Sapporo 060, Japan