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On the units in a character ring

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1. Introduction

Throughout this paper, G denotes always a finite group, Z the ring of rational integers, Q the rational field, C the complex number field. And we denote a character ring of G by $R(G)$. Let n be the exponent of G and let ζ be a primitive n-th root of unity. Then $K=Q(\zeta)\subset C$ is a splitting field for G.

In particular, if G is a finite abelian group and A is the ring of algebraic integers in K, then any unit of finite order in the group ring AG has the form εg for some $g\in G$ and some unit ε in A. (See p263, Theorem 37.4 of [**1**].) This result yields an interesting theorem. That is, if G and G' are finite abelian groups such that $ZG\cong ZG'$, then $G\cong G'$. (See p264, Theorem 37.7 of [**1**].)

We denote the algebraic closure of Q in C by $\bar{Q}$ and the ring of algebraic integers in $\bar{Q}$ by $\bar{Z}$.

In this paper, applying the theory of characters, we intend to study the units of finite order in the ring $\bar{Z}\otimes R(G)$, where G is a finite group and for any ring B, we use a symbol "$B\otimes R(G)$" in place of "$B\otimes_Z R(G)$" for convenience.

Afterward we shall show that any unit of finite order in $\bar{Z}\otimes R(G)$ has the form $\varepsilon\chi$ for some linear character χ and some unit ε in $\bar{Z}$ (SeeTheorem 2.1.), and then we shall apply this result to conclude that if G and G' are finite groups such that $R(G)\cong R(G')$ as rings, then $G/D(G)\cong G'/D(G')$, where $D(G)$ and $D(G')$ are commutator subgroups of G and G' respectively.

The theorems concerning the units of finite order in a character ring are stated in [**3**], [**4**] and [**5**] (See Theorem 1 of [**3**], Theorem 1 of [**4**], and Lemma 6.1 of [**5**].), and Theorem 2.1 is an extension of these results. We also present a short proof of Theorem 2.1.

2. A study of units of finite order

We keep the notation in section 1 and in addition, use the following notation.

$\chi_1(=1_G),\ldots,\chi_{h-1}$ and χ_h denote the irreducible C-characters of G.

For $\alpha\in C$, $\bar{\alpha}$ denotes a conjugate complex number of α, and $|\alpha|$ an absolute value of α.

For any ring B, $U_f(B)$ denotes the set of units of finite order in B, and $\hat{G}$ the group of linear characters of G.

For θ, $\eta\in\bar{Z}\otimes R(G)$, we set

$$<\theta, \eta>=1/|G|\sum_{g\in G}\theta(g)\overline{\eta(g)}.$$

Then we have the following theorem about the units of finite order in $\bar{Z}\otimes R(G)$.

THEOREM 2.1. $U_f(\bar{Z}\otimes R(G))=U_f(\bar{Z})\times\hat{G}$ (a direct product).

PROOF. For $u=\sum_{i=1}^{h}a_i\chi_i\in\bar{Z}\otimes R(G)$, $a_i\in\bar{Z}$, we set $\bar{u}=\sum_{i=1}^{h}\bar{a}_i\bar{\chi}_i$, where $\bar{\chi}_i$ denotes a conjugate character of $\chi_i(i=1,\ldots,h)$. Suppose that $u\in U_f(\bar{Z}\otimes R(G))$. Then $u(g)$ is a root of unity for all $g\in G$. Hence $|u(g)|^2=u(g)\overline{u(g)}=1$. Therefore we have $u\bar{u}=1_G$. From this equation, it follows that

$$\sum_{i=1}^{h}|a_i|^2=<u, u>=<u\bar{u}, 1_G>=1\cdots\cdots(2.1)$$

For any $\sigma\in G(\bar{Q}/Q)$, we set $u^\sigma=\sum_{i=1}^{h}a_i^\sigma\chi_i^\sigma$. Since χ_i^σ also is an irreducible character of G, we have $u^\sigma\in U_f(\bar{Z}\otimes R(G))$.

By the equation of (2.1), we have

$$\sum_{i=1}^{h}|a_i^\sigma|^2=1 \text{ for all } \sigma\in G(\bar{Q}/Q).$$

Hence for each i, $|a_i^\sigma|\leqq 1$ for all $\sigma\in G(\bar{Q}/Q)$. Therefore a_i is either *0* or a root of unity. ($i=1,\ldots,h$). That is, it follows that $u=\varepsilon_i\chi_i$ for some i, where ε_i is a root of unity. Since $|\chi_i(1)|=|\varepsilon_i^{-1}u(1)|=1$, χ_i must be a linear character of G. This completes the proof. Q. E. D.

As a consequence of Theorem 2.1, we can easily obtain the following corollary.

COROLLARY 2.2. $U_f(R(G))=\{\pm 1\}\times\hat{G}$ (a direct product).

THEOREM 2.3. *If $R(G)\cong R(G')$ as rings for two finite groups G, G', then we have*

$$G/D(G)\cong G'/D(G').$$

PROOF. Since $R(G)\cong R(G')$, we see that $U_f(R(G))\cong U_f(R(G'))$. By Corollary 2.2, we have $\{\pm 1\}\times\hat{G}\cong\{\pm 1\}\times\hat{G'}$. By the fundamental theorem of finite abelian groups, we obtain $\hat{G} \cong \hat{G}'$. Hence we have

$$G/D(G) \cong \hat{G} \cong \hat{G}' \cong G'/D(G').$$

This completes the proof. Q. E. D.

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