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A property of spectrums of measures on certain transformation groups

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§ 1 Introduction.

Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the Banach space of continuous functions on X which vanish at infinity, and let $M(X)$ be the Banach space of complex-valued bounded regular Borel measures on X with the total variation norm. Let $M^+(X)$ be the set of nonnegative measures in $M(X)$. For $\mu\in M(X)$ and $f\in L^1(|\mu|)$, we often write $\mu(f)=\int_X f(x)d\mu(x)$. Let X' be another locally compact Hausdorff space, and let $S: X\rightarrow X'$ be a continuous map. For $\mu\in M(X)$, let $S(\mu)\in M(X')$ be the continuous image of μ under S. We denote by $\mathscr{B}(X)$ the σ-algebra of Borel sets in X. $\mathscr{B}_0(X)$ means the σ-algebra of Baire sets in X. That is, $\mathscr{B}_0(X)$ is the σ-algebra generated by compact G_δ-sets in X.

Let G be a LCA group with dual $\hat{G}$. $M(G)$ and $L^1(G)$ denote the measure algebra and the group algebra respectively. For $\mu\in M(G)$, $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ. m_G denotes the Haar measure of G. Let $M_a(G)$ be the set of measures in $M(G)$ which are absolutely continuous with respect to m_G. Then by the Radon-Nikodym theorem we can identify $M_a(G)$ with $L^1(G)$. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. For a closed subgroup H of G, $H^\perp$ stands for the annihilator of H.

Let (G, X) be a (topological) transformation group, in which G is a compact abelian group and X is a locally compact Hausdorff space. That is, suppose that there exists a continuous map $(g, x)\rightarrow g\cdot x$ from $G\times X$ onto X with the following properties:

(1.1) $x\rightarrow g\cdot x$ is a homeomorphism on X for each $g\in G$ and $0\cdot x=x$, where 0 is the identity element in G;

(1.2) $g_1\cdot(g_2\cdot x)=(g_1+g_2)\cdot x$ for g_1, $g_2\in G$ and $x\in X$.

We note that $(g, x)\rightarrow f(g\cdot x)$ is a Baire function on $G\times X$ for each Baire function f on X. For $\lambda\in M(G)$ and $\mu\in M(X)$, define $\lambda*\mu\in M(X)$ by

$$(1.3)\qquad \lambda*\mu(h)=\int_X\int_G h(g\cdot x)\,d\lambda(g)\,d\mu(x)=\int_G\int_X h(g\cdot x)\,d\mu(x)\,d\lambda(g)$$

for $h\in C_0(X)$. Let $J(\mu)$ be the collection of all $f\in L^1(G)$ with $f*\mu=0$.

DEFINITION 1.1. For $\mu\in M(X)$, define the spectrum $\mathrm{sp}(\mu)$ of μ by $\mathrm{sp}(\mu)=\bigcap_{f\in J(\mu)}\hat{f}^{-1}(0)$.

Let $\pi: X\to X/G$ be the canonical map. For $x\in X$, let $B_x: G\to G\cdot x(\subset X)$ be the continuous map defined by $B_x(g)=g\cdot x$. For $\dot{x}=\pi(x)$, define $m_{\dot{x}}\in M^+(X)$ by $m_{\dot{x}}=B_x(m_G)$. Let $M_{aG}(X)$ be an L-subspace of $M(X)$ defined by

$$(1.4)\qquad M_{aG}(X)=\left\{\mu\in M(X): \begin{array}{l}\mu\ll\rho*\nu \text{ for some } \rho\in L^1(G)\cap M^+(G)\\ \text{and } \nu\in M^+(X)\end{array}\right\}.$$

Put $M_{aG}(X)^\perp=\{\nu\in M(X):\nu\perp\mu \text{ for all } \mu\in M_{aG}(X)\}$. Then $M_{aG}(X)^\perp$ is also an L-subspace of $M(X)$, and $M(X)=M_{aG}(X)\oplus M_{aG}(X)^\perp$. That is, for every $\mu\in M(X)$, it can be uniquely represented as follows:

$$(1.5)\qquad \mu=\mu_{aG}+\mu_{sG},$$

where $\mu_{aG}\in M_{aG}(X)$ and $\mu_{sG}\in M_{aG}(X)^\perp$. In [16], the author obtained the following theorem as an extension of the F. and M. Riesz theorem of Helson and Lowdenslager type.

THEOREM 1.1. ([16, Theorem 2.1]).
Let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact Hausdorff space. Let P be a semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Let σ be a positive Radon measure on X that is quasi-invariant. Let $\mu\in M(X)$, and let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to σ. Suppose $sp(\mu)\subset P$. Then both $sp(\mu_a)$ and $sp(\mu_s)$ are also contained in P. If, in addition, $P\cap(-P)=\{0\}$ and $\pi(|\mu|)\ll\pi(\sigma)$, then $sp(\mu_s)\subset P\backslash\{0\}$, where $\pi: X\to X/G$ is the canonical map.

In this paper, we shall prove the following theorem.

THEOREM 1.2. *Let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact Hausdorff space. Let P be a semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Let μ be a measure in $M(X)$ with $sp(\mu)\subset P$. Then both $sp(\mu_{aG})$ and $sp(\mu_{sG})$ are also contained in P. If, in addition, $P\cap(-P)=\{0\}$, then $sp(\mu_{sG})\subset P\backslash\{0\}$.*

REMARK 1.1. (i) Let μ be a measure in $M(X)$. It follows from [15, Proposition 5.1] that $\mu\in M_{aG}(X)$ if and only if μ translates G-

continuously (i. e., $\lim_{g\to 0}\|\mu-\delta_g*\mu\|=0$, where δ_g is the point mass at $g\in G$).

(ii) Let (G, X) be as in Theorem 1.2. Let E be a Riesz set in $\hat{G}$ (i. e., $M_E(G)\subset L^1(G)$). Then, for any measure $\mu\in M(X)$ with $\mathrm{sp}(\mu)\subset E$, we have $\mu\in M_{aG}(X)$, by [16, Theorem 2.3].

(iii) Let σ be a positive Radon measure on X that is quasi-invariant, and let μ be a measure in $M(X)$. If $\mu\ll\sigma$, then $\mu\in M_{aG}(X)$. In fact, since μ is bounded regular, we may assume that σ is bounded (i. e., $\sigma\in M^+(X)$). It follows from [15, Lemma 1.1] that σ and $m_G*\sigma$ are mutually absolutely continuous. Hence we have $\mu\ll m_G*\sigma$, and so $\mu\in M_{aG}(X)$.

Let G be a LCA group and H a compact subgroup of G. Then we have a transformation group (H, G) such that H acts freely on G.

COROLLARY 1.1. *Let G be a LCA group, and let P be an open semigroup in $\hat{G}$ such that $P\cup(-P)=\hat{G}$. Put $\Lambda=P\cap(-P)$ and $H=\Lambda^{\perp}$. Let μ be a measure in $M_P(G)$. Then*

(i) *μ_{aH}, $\mu_{sH}\in M_P(G)$, and*
(ii) *$\hat{\mu}_{sH}(\gamma)=0$ on $P\cap(-P)$.*

REMARK 1.2. We obtain [11, Corollary 3 (b)] in a consequence of Corollary 1.1.

REMARK 1.3. Suppose G is a compact abelian group. In Corollary 1.1, let $\mu=\mu_a+\mu_s$ be the Lebesgue decomposition of μ with respect to the Haar measure of G. If $\mu\in M_P(G)$, then we have μ_a, $\mu_s\in M_P(G)$ (cf. [14, Corollary]). Moreover, if $P\cap(-P)=\{0\}$, then we have $\hat{\mu}_s(0)=0$ (cf. [13, 8.2.3. Theorem]). However, if $P\cap(-P)\neq\{0\}$, we can not expect that $\hat{\mu}_s(0)=0$ in general.

REMARK 1.4. Suppose G is a compact abelian group. In Corollary 1.1, if $P\cap(-P)=\{0\}$, then $M_{aH}(G)=L^1(G)$ and $M_{aH}(G)^{\perp}=M_s(G)$. Hence, in this case, Corollary 1.1 is the F. and M. Riesz theorem of Helson and Lowdenslager type ([13, 8.2.3. Theorem]).

In section 2, we shall prove Theorem 1. 2 and Corollary 1.1.

§2 Proofs of Theorem 1.2 and Corollary 1.1.

We first state two conditions (D. I) and (D. II).

(D. I) Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact Hausdorff space. For any $\mu\in M^+(X)$, put $\eta=\pi(\mu)$, where $\pi: X\to X/G$ is the canonical map. Then there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ of measures in $M^+(X)$ with the following properties:

(2.1) $\dot{x}\to\lambda_{\dot{x}}(f)$ is η-measurable for each bounded Baire function f on X,

(2.2) $\|\lambda_{\dot{x}}\|=1$,

(2.3) $\operatorname{supp}(\lambda_{\dot{x}})\subset\pi^{-1}(\dot{x})$,

(2.4) $\mu(f)=\int_{X/G}\lambda_{\dot{x}}(f)\,d\eta(\dot{x})$ for each bounded Baire function f on X.

(D. II) Let (G, X) and π be as in (D. I). Let $\nu\in M^+(X/G)$. Suppose $\{\lambda_{\dot{x}}^1\}_{\dot{x}\in X/G}$ and $\{\lambda_{\dot{x}}^2\}_{\dot{x}\in X/G}$ are families of measures in $M(X)$ with the following properties:

(2.5) $\dot{x}\to\lambda_{\dot{x}}^i(f)$ is ν-integrable for each bounded Baire function f on $X(i=1,2)$,

(2.6) $\operatorname{supp}(\lambda_{\dot{x}}^i)\subset\pi^{-1}(\dot{x})$ $(i=1,2)$,

(2.7) $\int_{X/G}\lambda_{\dot{x}}^1(f)\,d\nu(\dot{x})=\int_{X/G}\lambda_{\dot{x}}^2(f)\,d\nu(\dot{x})$ for each bounded Baire function f on X.

Then $\lambda_{\dot{x}}^1=\lambda_{\dot{x}}^2$ ν-a. a. $\dot{x}\in X/G$.

Let $\mu\in M(X)$ and $\eta\in M^+(X/G)$. By an η-disintegration of μ, we mean a family $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ of measures in $M(X)$ satisfying (2.1)′ $\dot{x}\to\lambda_{\dot{x}}(f)$ is η-integrable for each bounded Baire function f on X and (2.3)-(2.4) in (D. I). If, in addition, $\eta=\pi(|\mu|)$ and $\|\lambda_{\dot{x}}\|=1$ for all $\dot{x}\in X/G$, then we call $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ a canonical disintegration of μ. Thus condition (D. I) says that each $\mu\in M^+(X)$ has a canonical disintegration $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ with $\lambda_{\dot{x}}\in M^+(X)$.

REMARK 2.1. Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact metric space. Then (G, X) satisfies conditions (D. I) and (D. II) (cf. [15, Remark 6.1]).

LEMMA 2.1. *Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact Hausdorff space. Suppose (G, X) satisfies conditions (D. I) and (D. II). Let μ_1, $\mu_2\in M^+(X)$, and let $\eta\in M^+(X/G)$. Let $\{\mu_{\dot{x}}^k\}_{\dot{x}\in X/G}$ be an η-disintegration of μ_k with $\mu_{\dot{x}}^k\in M^+(X)$ $(k=1,2)$. Then the following are equivalent:*

(i) $\mu_1\ll\mu_2$;

(ii) $\mu_{\dot{x}}^1\ll\mu_{\dot{x}}^2$ *η-a. a.* $\dot{x}\in X/G$.

PROOF. (i)$\Rightarrow$(ii): Since $\mu_1 \ll \mu_2$, there exists a nonnegative real-valued Baire function F on X such that $\mu_1 = F\mu_2$. Define $\lambda_{\dot{x}} \in M^+(X)$ by $\lambda_{\dot{x}} = F\mu_{\dot{x}}^2$. Then we have

(1) $\dot{x} \to \lambda_{\dot{x}}(f)$ is η-integrable for each bounded Baire function f on X,

(2) supp $(\lambda_{\dot{x}}) \subset \pi^{-1}(\dot{x})$, and

(3) $\int_{X/G} \mu_{\dot{x}}^1(f)\, d\eta(\dot{x}) = \mu_1(f) = \int_{X/G} \lambda_{\dot{x}}(f)\, d\eta(\dot{x})$ for each bounded Baire function f on X.

By condition (D. II), we have

$$\mu_{\dot{x}}^1 = \lambda_{\dot{x}} \quad \eta\text{-a. a. } \dot{x} \in X/G,$$

which yields $\mu_{\dot{x}}^1 \ll \mu_{\dot{x}}^2$ η-a. a. $\dot{x} \in X/G$.

(ii)$\Rightarrow$(i): Let B be a Baire set in X with $\mu_2(B) = 0$. Then

$$0 = \mu_2(B) = \int_{X/G} \mu_{\dot{x}}^2(B)\, d\eta(\dot{x}),$$

hence

$$\mu_{\dot{x}}^2(B) = 0 \quad \eta\text{-a. a. } \dot{x} \in X/G.$$

Accordingly, by the hypothesis, we have

$$\mu_1(B) = \int_{X/G} \mu_{\dot{x}}^1(B)\, d\eta(\dot{x}) = 0,$$

which together with [15, Proposition 1. 3] yields $\mu_1 \ll \mu_2$. This completes the proof.

LEMMA 2. 2. *Let (G, X) be as in the previous lemma. Let $\mu \in M^+(X)$ and $\eta \in M^+(X/G)$. Let $\{\mu_{\dot{x}}\}_{\dot{x} \in X/G}$ be an η-disintegration of μ with $\mu_{\dot{x}} \in M^+(X)$. If $\mu_{\dot{x}} \perp m_{\dot{x}}$ η-a. a. $\dot{x} \in X/G$, then μ belongs to $M_{aG}(X)^\perp$.*

PROOF. We may assume that $\mu \neq 0$. Let $\mu = \mu_{aG} + \mu_{sG}$, where $\mu_{aG} \in M_{aG} \cap M^+(X)$ and $\mu_{sG} \in M_{aG}(X)^\perp \cap M^+(X)$. Since $\mu_{aG} \leq \mu$, there exists a Baire measurable function F on X such that $0 \leq F \leq 1$ and $\mu_{aG} = F\mu$. Define $\lambda_{\dot{x}} \in M^+(X)$ by $\lambda_{\dot{x}} = F\mu_{\dot{x}}$. Then $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ is an η-disintegration of μ_{aG}. Since $\mu_{\dot{x}} \perp m_{\dot{x}}$ η-a. a. $\dot{x} \in X/G$, we have

(1) $\lambda_{\dot{x}} \perp m_{\dot{x}}$ η-a. a. $\dot{x} \in X/G$.

On the other hand, it follows from [16, Lemma 4. 1] that

(2) $\mu_{aG} \ll m_G * \mu_{aG}$.

We note that $\{m_G * \lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ is an η-disintegration of $m_G * \mu_{aG}$ with $m_G * \lambda_{\dot{x}} \in M^+(X)$. Hence, by (2) and Lemma 2.1, we have

$$\lambda_{\dot{x}} \ll m_G * \lambda_{\dot{x}} \quad \eta\text{-a. a. } \dot{x} \in X/G,$$

which together with [15, Lemma 1.3] yields

$$\lambda_{\dot{x}} \ll K(\dot{x}) m_{\dot{x}} \quad \eta\text{-a. a. } \dot{x} \in X/G,$$

where $K(\dot{x}) = \lambda_{\dot{x}}(X)$. Hence

(3) $\lambda_{\dot{x}} \ll m_{\dot{x}}$ η-a. a. $\dot{x} \in X/G$.

By (1) and (3), we have $\lambda_{\dot{x}} = 0$ η-a. a. $\dot{x} \in X/G$. Since $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ is an η-disintegration of μ_{aG}, we get $\mu_{aG} = 0$, and so $\mu = \mu_{sG} \in M_{aG}(X)^{\perp}$. This completes the proof.

LEMMA 2.3. *Let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact Hausdorff space. Let μ be a measure in $M^+(X)$. If $\mu \perp m_G * \mu$, then μ belongs to $M_{aG}(X)^{\perp}$*

PROOF. Let $\mu = \mu_{aG} + \mu_{sG}$, where $\mu_{aG} \in M_{aG}(X) \cap M^+(X)$ and $\mu_{sG} \in M_{aG}(X)^{\perp} \cap M^+(X)$. Since $\mu_{aG} \le \mu$ and $m_G * \mu_{aG} \le m_G * \mu$, we have, by the hypothesis,

$$\mu_{aG} \perp m_G * \mu_{aG}.$$

On the other hand, it follows from [16, Lemma 4.1] that $\mu_{aG} \ll m_G * \mu_{aG}$. Hence $\mu_{aG} = 0$, and so $\mu = \mu_{sG} \in M_{aG}(X)^{\perp}$. This completes the proof.

Let G be a LCA group and H a compact subgroup of G. Then we have a transformation group (H, G) such that H acts freely on G. For $\mu \in M(G)$, let $\mathrm{sp}(\mu)$ be the spectrum of μ defined in Definition 1.1.

LEMMA 2.4. *Let G be a LCA group and P an open semigroup in $\hat{G}$ such that $P \cup (-P) = \hat{G}$. Let $\Lambda = P \cap (-P)$ and $H = \Lambda^{\perp}$. Let $\pi_{\Lambda}: \hat{G} \to \hat{G}/\Lambda$ be the natural homomorphism, and put $\tilde{p} = \pi_{\Lambda}(P)$. Then, for $\mu \in M(G)$, the following are equivalent.*

(i) $\mu \in M_P(G)$;
(ii) $sp(\mu) \subset \tilde{P}$.

PROOF. (i)$\Rightarrow$(ii): For $\tilde{\gamma} \in \tilde{P}^c$, choose $\gamma \in \hat{G} \backslash P$ so that $\tilde{\gamma} = \pi_{\Lambda}(\gamma)$. Then $(\gamma + \Lambda) \cap P = \phi$. Hence, for $f \in C_0(G)$, we have

$$\tilde{\gamma}*\mu(f)=\int_G\int_H f(h+g)\tilde{\gamma}(h)\,dm_H(h)\,d\mu(g)$$
$$=\int_G\int_H f(h+g)(h,\gamma)\,dm_H(h)\,d\mu(g)$$
$$=(\gamma m_H)*\mu(f)$$
$$=0,$$

which shows $\tilde{\gamma}\notin \mathrm{sp}(\mu)$. Hence we have $\mathrm{sp}(\mu)\subset\tilde{P}$.

(ii) $\Rightarrow$ (i): For any $\gamma\in\hat{G}\backslash P$, put $\tilde{\gamma}=\pi_\Lambda(\gamma)$. Then $\tilde{\gamma}\notin\mathrm{sp}(\mu)$. Hence we have

$$(\gamma m_H)*\mu=\tilde{\gamma}*\mu=0,$$

which yields $\hat{\mu}(\gamma)=0$. Hence $\mu\in M_P(G)$, and the proof is complete.

LEMMA 2.5. *Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact Hausdorff space. If (G, X) satisfies conditions (D. I) and (D. II), then the conclusion of Theorem* 1.2 *holds.*

PROOF. Let μ be a measure in $M(X)$ with $\mathrm{sp}(\mu)\subset P$. Let $\pi: X\to X/G$ be the canonical map, and put $\eta=\pi(|\mu|)$. By condition (D. I), $|\mu|$ has a canonical disintegration $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ with $\lambda_{\dot{x}}\in M^+(X)$. Let h be a unimodular Baire function on X such that $\mu=h|\mu|$. Put $\mu_{\dot{x}}=h\lambda_{\dot{x}}$. Then $\{\mu_{\dot{x}}\}_{\dot{x}\in X/G}$ is a canonical disintegration of μ. Since $\mathrm{sp}(\mu)\subset P$, it follows from [15, Lemma 2.6] that

(1) $\mathrm{sp}(\mu_{\dot{x}})\subset P$ η-a. a. $\dot{x}\in X/G$.

For $x\in X$, put $\dot{x}=\pi(x)$ and $G_x=\{g\in G: g\cdot x=x\}$. Then G_x is a closed subgroup of G. Define a map $\tilde{B}_x: G/G_x\to G\cdot x$ by $\tilde{B}_x(g+G_x)=g\cdot x$. Then $\tilde{B}_x$ is a homeomorphism. Since $\mathrm{supp}(\mu_{\dot{x}})\subset\pi^{-1}(\dot{x})$, there exists a measure $\xi_{\dot{x}}\in M(G/G_x)$ such that $\tilde{B}_x(\xi_{\dot{x}})=\mu_{\dot{x}}$. Then, by (1) and [15, Proposition 1.2], we have

(2) $\xi_{\dot{x}}\in M_{P\cap G_x^\perp}(G/G_x)$ η-a. a. $\dot{x}\in X/G$,

which together with [14, Corollary] yields

(3) $\xi_{\dot{x}}^a,\ \xi_{\dot{x}}^s\in M_{P\cap G_x^\perp}(G/G_x)$ η-a a. $\dot{x}\in X/G$,

where $\xi_{\dot{x}}=\xi_{\dot{x}}^a+\xi_{\dot{x}}^s$ is the Lebesgue decomposition of $\xi_{\dot{x}}$ with respect to m_{G/G_x}. Let $\mu_{\dot{x}}=\mu_{\dot{x}}^a+\mu_{\dot{x}}^s$ be the Lebesgue decomposition of $\mu_{\dot{x}}$ with respect to $m_{\dot{x}}$. Then by [15, Proposition 1.5] we have $\tilde{B}_x(\xi_{\dot{x}}^a)=\mu_{\dot{x}}^a$ and $\tilde{B}_x(\xi_{\dot{x}}^s)=\xi_{\dot{x}}^s$. It follows from (3) and [15, Proposition 1.2] that

(4) $\mathrm{sp}(\mu_{\dot{x}}^a)$, $\mathrm{sp}(\mu_{\dot{x}}^s)\subset P$ η-a. a. $\dot{x}\in X/G$.

Let $\lambda_{\dot{x}}=\lambda_{\dot{x}}^a+\lambda_{\dot{x}}^s$ be the Lebesgue decomposition of $\lambda_{\dot{x}}$ with respect to $m_{\dot{x}}$. It follows from [15, Lemma 2.8] that $\dot{x}\to\lambda_{\dot{x}}^a(f)$ and $\dot{x}\to\lambda_{\dot{x}}^s(f)$ are η-measurable for all bounded Baire functions f on X. Define λ_1, $\lambda_2\in M^+(X)$ by

$$(5)\quad \begin{aligned}\lambda_1(f)&=\int_{X/G}\lambda_{\dot{x}}^a(f)\,d\eta(\dot{x}),\\ \lambda_2(f)&=\int_{X/G}\lambda_{\dot{x}}^s(f)\,d\eta(\dot{x})\end{aligned}$$

for $f\in C_0(X)$. We note that (5) holds for all bounded Baire functions f on X. Put $\sigma=m_G*|\mu|$. Then σ is quasi-invariant and $\pi(\sigma)=\pi(|\mu|)=\eta$. By (5), we have $\pi(\lambda_1)\ll\eta=\pi(\sigma)$. Hence [15, Lemma 2.5] yields $\lambda_1\ll\sigma=m_G*|\mu|$, which shows

(6) $\lambda_1\in M_{aG}(X)$.

By Lemma 2.2, we have

(7) $\lambda_2\in M_{aG}(X)^\perp$

Since $\mu_{\dot{x}}^a=h\lambda_{\dot{x}}^a$ and $\mu_{\dot{x}}^s=h\lambda_{\dot{x}}^s$, $\dot{x}\to\mu_{\dot{x}}^a(f)$ and $\dot{x}\to\mu_{\dot{x}}^s(f)$ are η-measurable for each bounded Baire function f on X. Define μ_1, $\mu_2\in M(X)$ by

$$(8)\quad \begin{aligned}\mu_1(f)&=\int_{X/G}\mu_{\dot{x}}^a(f)\,d\eta(\dot{x}),\\ \mu_2(f)&=\int_{X/G}\mu_{\dot{x}}^s(f)\,d\eta(\dot{x})\end{aligned}$$

for $f\in C_0(X)$. Then, by (5)-(7), we have $\mu_1\in M_{aG}(X)$ and $\mu_2\in M_{aG}(X)^\perp$, and so $\mu_1=\mu_{aG}$ and $\mu_2=\mu_{sG}$. For $\gamma\notin P$, (4) and [15, Remark 1.1 (II.1)] yield $\gamma*\mu_{\dot{x}}^a=0$ η-a. a. $\dot{x}\in X/G$. It follows from [15, Lemma 2.3 (II)] that

$$\gamma*\mu_{aG}(f)=\gamma*\mu_1(f)=\int_{X/G}\gamma*\mu_{\dot{x}}^a(f)\,d\eta(\dot{x})=0$$

for $f\in C_0(X)$. Hence $\gamma*\mu_{aG}=0$, which together with [15, Remark 1.1 (II.1)] yields $\gamma\notin\mathrm{sp}(\mu_{aG})$. Thus we get $\mathrm{sp}(\mu_{aG})\subset P$. By [15, Remark 1.1 (II.2)], we also have $\mathrm{sp}(\mu_{sG})=\mathrm{sp}(\mu-\mu_{aG})\subset P$.

Next we prove the latter half. Suppose $P\cap(-P)=\{0\}$. Then, by (3) and [13, 8.2.3 Theorem], we get $\hat{\xi}_{\dot{x}}^s(0)=0$ η-a. a. $\dot{x}\in X/G$, and so $0\notin\mathrm{sp}(\mu_{\dot{x}}^s)$ η-a. a. $\dot{x}\in X/G$. Hence $1*\mu_{\dot{x}}^s=0$ η-a. a. $\dot{x}\in X/G$, where 1 is the constant function on G with value one. Hence, by a similar argument above, we have $1*\mu_{sG}=1*\mu_2=0$, which shows $0\notin \mathrm{sp}(\mu_{sG})$. Thus $\mathrm{sp}(\mu_{sG})\subset P\backslash\{0\}$, and the proof is complete.

LEMMA 2.6. *Let (G, X) be a transformation group, in which G is a compact abelian group and X is a σ-compact, locally compact metric space. Set $H=\{g\in G: g\cdot x=x$ for all $x\in X\}$. Then H is a compact subgroup of G such that G/H is metrizable. Moreover, we have a transformation group $(G/H, X)$ by the action $(g+H)\cdot x=g\cdot x$ for $g+H\in G/H$ and $x\in X$.*

PROOF. It is easy to see that H is a closed subgroup of G. Hence H is a compact subgroup of G. Let $\{f_n\}_{n=1}^{\infty}$ be a countable dense subset of $C_0(X)$. Then

$$(1)\qquad H=\bigcap_{k,n=1}^{\infty}\{g\in G: \|f_n\circ g-f_n\|_{\infty}<\frac{1}{k}\},$$

where $f_n\circ g(x)=f_n(g\cdot x)$. Since $\{g\in G: \|f_n\circ g-f_n\|_{\infty}<\frac{1}{k}\}$ is an open set in G, it follows from (1) that H is a G_δ-set. Hence G/H is metrizable. Since (G, X) is a transformation group, it is easy to verify that $(G/H, X)$ becomes a transformation group by the action $(g+H)\cdot x=g\cdot x$. This completes the proof.

Let (G, X) and H be as in Lemma 2.6. Let μ be a measure in $M(X)$. For $\lambda\in M(G/H)$, we can define a convolution $\lambda\underset{G/H}{*}\mu\in M(X)$ by

$$(2.8)\qquad \lambda\underset{G/H}{*}\mu(h)=\int_X\int_{G/H}h(\dot{g}\cdot x)\,d\lambda(\dot{g})\,d\mu(x)$$

for $h\in C_0(X)$. Set $J_{G/H}(\mu)=\{f\in L^1(G/H): f\underset{G/H}{*}\mu=0\}$, and define the spectrum $\mathrm{sp}_{G/H}(\mu)$ of μ by

$$(2.9)\qquad \mathrm{sp}_{G/H}(\mu)=\bigcap_{f\in J_{G/H}(\mu)}\hat{f}^{-1}(0).$$

LEMMA 2.7. *Let (G, X) and H be as in Lemma 2.6. Let $\Lambda=H^{\perp}$. Let $\gamma\in\hat{G}$, and let μ be a measure in $M(X)$. Then*

(i) *$\gamma*\mu=0$ if $\gamma\in\hat{G}\backslash\Lambda$, and*

(ii) *$\gamma*\mu=\gamma\underset{G/H}{*}\mu$ if $\gamma\in\Lambda$.*

In particular, $\mathrm{sp}(\mu)=\mathrm{sp}_{G/H}(\mu)$.

PROOF. For $g\in G$, $\dot{g}$ denotes the coset $g+H$. For $\gamma\in\hat{G}\backslash\Lambda$, we have

$$\begin{aligned}\gamma * \mu(h) &= \int_X \int_G h(g \cdot x)\gamma(g)\, dm_G(g)\, d\mu(x) \\ &= \int_X \int_{G/H} \int_H h((g+u)\cdot x)\gamma(g+u)\, dm_H(u)\, dm_{G/H}(\dot{g})\, d\mu(x) \\ &= \int_X \int_{G/H} h(\dot{g}\cdot x) \int_H \gamma(g+u)\, dm_H(u)\, dm_{G/H}(\dot{g})\, d\mu(x) \\ &= 0 \qquad\qquad (\gamma|_H \neq 0)\end{aligned}$$

for all $h \in C_0(X)$. Thus we have (i).

Next we prove (ii). For $\gamma \in \Lambda$, we have

$$\begin{aligned}\gamma * \mu(h) &= \int_X \int_G h(g \cdot x)\gamma(g)\, dm_G(g)\, d\mu(x) \\ &= \int_X \int_{G/H} h(\dot{g}\cdot x) \int_H \gamma(g+u)\, dm_H(u)\, dm_{G/H}(\dot{g})\, d\mu(x) \\ &= \int_X \int_{G/H} h(\dot{g}\cdot x)\gamma(\dot{g})\, dm_{G/H}(\dot{g})\, d\mu(x) \\ &= \gamma \underset{G/H}{*} \mu(h)\end{aligned}$$

for all $h \in C_0(X)$. Thus (ii) follows.

By (i), (ii) and [15, Remark 1.1 (II. 1)], we have $\mathrm{sp}(\mu) = \mathrm{sp}_{G/H}(\mu)$. This completes the proof.

LEMMA 2.8. *Let (G, X) and H be as in Lemma 2.6. Then $M_{aG}(X) = M_{aG/H}(X)$.*

PROOF. Let $q_H : G \to G/H$ be the canonical map. Let μ be a measure in $M(X)$. We note that $\delta_g * \mu = \delta_{q_H(g)} \underset{G/H}{*} \mu$ for $g \in G$. Hence we have

$$\begin{aligned}\mu \in M_{aG}(X) &\Longleftrightarrow \lim_{g \to 0} \|\mu - \delta_g * \mu\| = 0 \qquad \text{(by Remark 1.1 (i))} \\ &\Longleftrightarrow \lim_{q_H(g) \to 0} \|\mu - \delta_{q_H(g)} \underset{G/H}{*} \mu\| = 0 \\ &\Longleftrightarrow \mu \in M_{aG/H}(X). \qquad \text{(by Remark 1.1 (i))}\end{aligned}$$

This completes the proof.

PROPOSITION 2.1. *Let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact metric space. Then the conclusion of Theorem 1.2 holds.*

PROOF. Since a measure in $M(X)$ is bounded regular, we may assume that X is σ-compact. Put $H = \{g \in G : g \cdot x = x \text{ for all } x \in X\}$. It follows from Lemma 2.6 that H is a compact subgroup of G such that G/H is metrizable. Moreover, by Lemma 2.6, we have a transformation group $(G/H, X)$ by the action $(g+H)\cdot x=g\cdot x$ for $g\in G$ and $x\in X$. Hence the conclusion of Theorem 1.2 follows from Remark 2.1 and Lemmas 2.5, 2.7 and 2.8. This completes the proof.

The following lemma is due to [16].

LEMMA 2.9 (cf. [16, Lemma 3.1]).
Let (G, X) be a transformation group, in which G is a compact abelian group and X is a σ-compact, locally compact Hausdorff space. Let μ_1 be a nonzero measure in $M(X)$, and let μ_2 and σ_2 be mutually singular measures in $M^+(X)$. Then there exists an equivalence relation "$\sim$" on X with the following properties:

(i) *$X/\sim$ is a σ-compact metrizable, locally compact Hausdorff space with respect to the quotient topology;*
(ii) *$(G, X/\sim)$ becomes a transformation group by the action $g\cdot\tau(x)=\tau(g\cdot x)$ for $g\in G$ and $x\in X$;*
(iii) $\tau(\mu_1)\neq 0$;
(iv) $\tau(\mu_2)\perp\tau(\sigma_2)$,

where τ: $X\rightarrow X/\sim$ is the canonical map.

Now we prove Theorem 1.2. We may assume that X is σ-compact (cf. the proof of [16, Theorem 2.1]). Let μ be a measure in $M(X)$ with $\mathrm{sp}(\mu)\subset P$. In order to prove the first assertion, it suffices to show that $\mathrm{sp}(\mu_{sG})\subset P$. We may assume that $\mu_{sG}\neq 0$. Suppose there exists $\gamma_0\in\hat{G}\backslash P$ with $\gamma_0\in\mathrm{sp}(\mu_{sG})$. Then $\gamma_0*\mu_{sG}\neq 0$. Since $|\mu_{sG}|\perp m_G*|\mu_{sG}|$, it follows from Lemma 2.9 that there exists an equivalence relation "$\sim$" on X satisfying (i)-(iv) in Lemma 2.9 with $\mu_1=\gamma_0*\mu_{sG}$, $\mu_2=|\mu_{sG}|$ and $\sigma_2=m_G*|\mu_{sG}|$. Hence we have

(2.10) $\tau(\gamma_0*\mu_{sG})\neq 0$, and
(2.11) $\tau(|\mu_{sG}|)\perp\tau(m_G*|\mu_{sG}|)$,

where $\tau: X\rightarrow X/\sim$ is the canonical map. By [16, Lemma 2.1] and (2.11), we have $\tau(|\mu_{sG}|)\perp m_G*\tau(|\mu_{sG}|)$. It follows from Lemma 2.3 that $\tau(|\mu_{sG}|)\in M_{aG}(X/\sim)^{\perp}$. By [16, Lemma 2.1], we have $\tau(\mu_{aG})\in M_{aG}(X/\sim)$. Thus, since $\tau(\mu)=\tau(\mu_{aG})+\tau(\mu_{sG})$, we have

$$\tau(\mu)_{aG}=\tau(\mu_{aG}) \text{ and } \tau(\mu)_{sG}=\tau(\mu_{sG}).$$

By [16, Lemma 2.2], we have $\mathrm{sp}(\tau(\mu))\subset\mathrm{sp}(\mu)\subset P$. Hence, by Proposition 2.1, we have $\mathrm{sp}(\tau(\mu_{sG}))=\mathrm{sp}(\tau(\mu)_{sG})\subset P$. On the other hand, (2.10) implies that $\gamma_0*\tau(\mu_{sG})=\tau(\gamma_0*\mu_{sG})\neq 0$, and so $\gamma_0\in\mathrm{sp}(\tau(\mu_{sG}))$. Hence we have $\gamma_0 \in P$, which contradicts the choice of γ_0. Thus we have $\mathrm{sp}(\mu_{sG}) \subset P$.

Next we prove the second half of Theorem 1.2. It is sufficient to prove that $0 \notin \mathrm{sp}(\mu_{sG})$. Suppose $0 \in \mathrm{sp}(\mu_{sG})$. Then $1 * \mu_{sG} \neq 0$, where 1 is the constant function on G with value one. Since $|\mu_{sG}| \perp m_G * |\mu_{sG}|$, it follows from Lemma 2.9 that there exists an equivalence relation " $\approx$ " on X such that

(2.12) $X/\approx$ is a σ-compact metrizable, locally compact Hausdorff space with respect to the quotient topology,

(2.13) $(G, X/\approx)$ becomes a transformation group by the action $g \cdot \tau'(x) = \tau'(g \cdot x)$ for $g \in G$ and $x \in X$,

(2.14) $\tau'(1 * \mu_{sG}) \neq 0$, and

(2.15) $\tau'(|\mu_{sG}|) \perp \tau'(m_G * |\mu_{sG}|)$,

where $\tau' : X \to X/\approx$ is the canonical map. Then, as seen in the first half, we have $\tau'(\mu) = \tau'(\mu_{aG}) + \tau'(\mu_{sG})$, $\tau'(\mu)_{aG} = \tau'(\mu_{aG})$ and $\tau'(\mu)_{sG} = \tau'(\mu_{sG})$. Since $\mathrm{sp}(\tau'(\mu)) \subset \mathrm{sp}(\mu) \subset P$, it follows from Proposition 2.1 that $\mathrm{sp}(\tau'(\mu_{sG})) \subset P \backslash \{0\}$, which yields

$$1 * \tau'(\mu_{sG}) = 0.$$

Since $\tau'(1 * \mu_{sG}) = 1 * \tau'(\mu_{sG})$, this contradicts (2.14). Hence $0 \notin \mathrm{sp}(\mu_{sG})$, and the proof of Theorem 1.2 is complete.

Finally we prove Corollary 1.1. We note that H is a compact subgroup of G. We first prove (i). Let μ be a measure in $M_P(G)$. Let $\pi_\Lambda : \hat{G} \to \hat{G}/\Lambda$ be the natural homomorphism, and put $\tilde{P} = \pi_\Lambda(P)$. Then $\tilde{P}$ is a semigroup in $\hat{G}/\Lambda$ such that $\tilde{P} \cup (-\tilde{P}) = \hat{G}/\Lambda$ and $\tilde{P} \cap (-\tilde{P}) = \{0\}$. By Lemma 2.4, we have $\mathrm{sp}(\mu) \subset \tilde{P}$, which together with Theorem 1.2 yields $\mathrm{sp}(\mu_{aH}) \cup \mathrm{sp}(\mu_{sH}) \subset \tilde{P}$. Hence (i) follows from Lemma 2.4. Next we we prove (ii). By Lemma 2.4 and the latter half of Theorem 1.2, we have $\mathrm{sp}(\mu_{sH}) \subset \tilde{P} \backslash \{0\}$. Let γ be any element in $P \cap (-P)$. Then, by the argument in the proof of Lemma 2.4 and [15, Remark 1.1 (II)], we have

$$\begin{aligned} (\gamma m_H) * \mu_{sH} &= \pi_\Lambda(\gamma) * \mu_{sH} \\ &= 1 * \mu_{sH} && (\text{by } \gamma \in P \cap (-P)) \\ &= 0. && (\text{by } \mathrm{sp}(\mu_{sH}) \subset \tilde{P} \backslash \{0\}) \end{aligned}$$

This shows that $\hat{\mu}_{sH}$ vanishes on $P \cap (-P)$ since $(\gamma m_H)^\wedge$ is the characteristic function of $P \cap (-P)$. Hence we get (ii), and the proof of Corollary 1.1 is complete.

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