Hokkaido Mathematical Journal Vol. 21 (1992) p. 335-347

Extension of local direct product structures of normal complex spaces

Akihiro SAEKI
(Received April 23, 1991)

§ 0. Introduction

In [GM], Gômez-Mont defined foliations by curves on complex spaces. Let X be a complex space of pure dimension n and non-singular in codimension one, i. e. the singular locus Sing$X$ is of codimension strictly greater than one. $X\backslash$Sing$X$ is a (not necessarily connected) complex manifold of dimension n. We consider a pair $(\mathscr{F}_{A}, A)$, where A is an analytic set in X of codimension strictly greater than one and containing Sing$X$ and $\mathscr{F}_{A}$ is a holomorphic foliation of complex dimension one on $X\backslash A$. (Note that $X\backslash A$ is an n-dimensional complex manifold.) Two pairs $(\mathscr{F}_{A}, A)$, $(\mathscr{F}_{A'}, A')$ are said to be equivalent if there exists an analytic set B of X of codimension strictly greater than one which contains $A \cup A'$, and if two foliations $\mathscr{F}_{A}|_{X\backslash B}$ and $\mathscr{F}_{A'}|_{X\backslash B}$ on $X\backslash B$ coincide with each other. A foliation $\mathscr{F}$ by curves on X is an equivalence class of such a pair $(\mathscr{F}_{A}, A)$. Note that if X is normal and connected then X is pure dimensional and non-singular in codimension one.

After Gômez-Mont's definition, we may say that the "simplest" foliation by curves on X is defined by a pair $(\mathscr{F}_{A}, A)$ such that $\mathscr{F}_{A}$ is a direct product, i. e. for an $(n-1)$-dimensional complex manifold M and a Riemann surface S,

$$X\backslash A\simeq M\times S$$

as complex manifolds.

In this parer, we confine our interest to the local case and investigate the following problem:

PROBLEM 0. 0.
Let X be a complex space and $x_0\in X$. Suppose that there are an open neighborhood $U\subset X$ of x_0, an analytic set A in U, a complex space V and a Riemann surface W satisfying the following conditions:

a) $x_0\in A$,

b) $codim A>1$ *(in U) and*

c) $U\backslash A \simeq V\times W$ (*as complex spaces*).
Then do there exist an open neighborhood $U_0\subset U$ *of* x_0, *a complex space* V_0 *and a Riemann surface* W_0 *with the following properties?*

α) $U_0\simeq V_0\times W_0$ (*as complex spaces*) *and*

β) *the foliation on* U *defined by α*) *is identical with the restriction to* U_0 *of the foliation defined by c*).

In § 3, we review holomorphic vector fields on complex spaces. In § 4, Gómez-Mont's results about foliations by curves [GM] are reviewed and an application of our main theorem is given.

We recall the Open Mapping Theorem and the Riemann Extension Theorem on complex spaces, which are of essential importance. (See e.g. [G-R-2].)

OPEN MAPPING THEOREM 0.1.
Let $(X, \mathcal{O}_X)$ *be a reduced complex space,* $p\in X$ *and* $f\in\Gamma(X, \mathcal{O}_X)$. *If* f *is not constant near* p, *then the map* $f: X\to \boldsymbol{C}$ *is open at* p.

A closed subset Z of a complex space X is called *thin in* X if every point $x\in X$ has an open neighborhood U such that $Z\cap U$ is contained in a nowhere dense analytic set A in U. For an integer $k\geqq 1$, we call Z *thin of order* k if we can choose U and A for any x in such a way that the inequality $\dim_z A\leqq \dim_z X-k$ is satisfied at each point $z\in Z\cap U$.

RIEMANN EXTENSION THEOREM 0.2.
Let $(X, \mathcal{O}_X)$ *be a normal complex space,* A *a thin set in* X *and* $f_A\in\Gamma(X\backslash A, \mathcal{O}_X)$. *If* f_A *is bounded near* A *or* A *is thin of order* 2, *then* f_A *has a unique holomorphic extension* $f\in\Gamma(X, \mathcal{O}_X)$.

§ 1. Statement of the main theorem

MAIN THEOREM 1.0.
Problem 0.0 *has an affirmative answer if and only if there exist* U, A, V *and* W *satisfying the following* 1) *and* 2) *as well as a*), *b*) *and c*):

1) *U is normal and*

2) *the projection* $g_A: U\backslash A\to W$ *extends to* U
i.e. there exists a Riemann surface W', *an open embedding* $i: W\to W'$ *and a holomorphic mapping* $g: U\to W'$ *such that the following diagram commutes:*

$$\begin{array}{ccc} U\backslash A & \xrightarrow{g_A} & W \\ \cap & & \cap \; i. \\ U & \xrightarrow[g]{} & W' \end{array}$$

From this theorem, we can easily deduce the following.

THEOREM 1.1.
Let X be a normal complex space, $x_0 \in X$, $d \geqq 1$ an integer and U a connected open neighborhood of x_0. Assume that A is an analytic set in U containing x_0 and that satisfies one of the following conditions:

I) *A is irreducible and of codimension strictly greater than one in U.*

II) *A is of codimension strictly greater than d in U.*

If there are a complex space V and a bounded open polydisk W of dimension d satisfying $U\backslash A \simeq V \times W$, then we can choose an open neighborhood $U_0 \subset U$ of x_0, an open complex subspace V_1 of V and an open polydisk $W_0 \subset W$ such that there exists a complex space V_0 with an open embedding ι: $V_1 \subset V_0$ making the following diagram commute:

$$\begin{array}{ccc} U\backslash A \simeq & V \times W & \\ \cup & \cup & \\ U_0\backslash A \simeq & V_1 \times W_0 & \\ \cap & \cap & \iota \times id_{W_0} \\ U_0 \simeq & V_0 \times W_0 & \end{array}$$

§ 2. Proof of the theorems

First, we prove Theorem 1.0. The proof progresses in several steps. We adopt the notations in Problem 0.0 and Theorem 1.0. We may assume that U is connected. The first step is the following observation.

LEMMA 2.0.
Under the condition 1), *the condition* 2) *is equivalent to the following*:

2)' *W is a domain in $\boldsymbol{C}$.*

PROOF. Assume that U, A, V and W satisfy 1) and 2). Let g: $U \to W'$ be an extension of the projection g_A: $U\backslash A \to W$ and $w_0 \in W'$ the image of x_0 by g. Taking an open coordinate neighborhood $B \subset W'$ of w_0, $U_1 = g^{-1}(B) \subset U$ is an open neighborhood of x_0. Considering $g_1 = g|_{U_1}$ as $g_1 \in \mathcal{O}_X(U_1)$, the open mapping theorem tells us that g_1: $U_1 \to B$ is open and that we can shrink the neighborhood of x_0 so that 2)', as well as a), b), c) and 1), is satisfied. Conversely, suppose that there exist U, A, V and W satisfying 1) and 2)'. Regarding $g_A: U\backslash A \rightarrow W$ as $g_A \in \mathcal{O}_X(U\backslash A)$, g_A extends to $g \in \mathcal{O}_X(U)$ by Riemann extension theorem. The open mapping theorem assures us that the image $g(U)$ is a Riemann surface, thus 2) holds. ■

LEMMA 2.1.
Let $g: U \rightarrow W'$ be an extension of $g_A: U\backslash A \rightarrow W$, $w_0 = g(x_0)$ and consider that $W = g(U\backslash A) \subset W'$. Then $w_0 \in W \subset W'$.

PROOF. By Lemma 2.0, we may assume that W is a domain in $\boldsymbol{C}$ and that $g_A \in \mathcal{O}_X(U\backslash A)$. Then the extension g is considered as $g \in \mathcal{O}_X(U)$ and uniquely determined. Suppose that $w_0 \notin W$. Then $g^{-1}(w_0)$ would be contained in the analytic set A. But $g^{-1}(w_0)$ is an analytic set in U of codimension $\leqq 1$ whereas $\operatorname{codim} A > 1$. This is a contradiction. ■

Thus we can assume that

i) *U is normal and connected,*

ii) there exists a closed embedding $\iota: U \hookrightarrow D$, *where $D = D_1 \times \ldots \times D_N$ is a bounded open polydisk in $\boldsymbol{C}^N$ centred at $\iota(x_0) = 0 \in \boldsymbol{C}^N$,*

iii) *W is a bounded open disk in $\boldsymbol{C}$ centred at 0 and*

iv) *$w_0 = g(x_0) = 0 \in \boldsymbol{C}$.*

Furthermore, we have the following

LEMMA 2.2.
The extension g of $g_A: U\backslash A \rightarrow W$ is a holomorphic map onto W:
$g: U \rightarrow W$.

PROOF. By the open mapping theorem, $g(U) \subset \boldsymbol{C}$ is open. Since
$W \subset g(U) \subset$ *the closure of W in $\boldsymbol{C}$*
and W is a bounded open disk in $\boldsymbol{C}$, $W' = g(U) = W$ holds. ■

$V' = g^{-1}(0)$ is an analytic set in U and $x_0 \in V'$. Considering it as a complex space with the reduced induced structure, we denote by $\xi: V_0 \rightarrow V'$ the normalization of V'. For any complex space $(Z, \mathcal{O}_Z)$, we denote the underlying topological space by $\operatorname{sp}(Z)$.

LEMMA 2.3.
The continuous map $\xi: sp(V_0) \rightarrow sp(V')$ subordinate to the morphism $\xi: V_0 \rightarrow V'$ is a homeomorphism.

PROOF. The isomorphism $U\backslash A \simeq V \times W$ induces $V'\backslash A \simeq V \times \{0\} \simeq V$ and V' is the closure of V in U (with the reduced induced structure). Since U is normal and connected and A is a thin analytic set in U, $U_A =$

$U\backslash A$ is normal and connected. W is an open disk in $\boldsymbol{C}$ and we can deduce that $V'\backslash A\simeq V$ is also normal and connected. This means that ξ: $\xi^{-1}(V'\backslash A)\to V'\backslash A$ is biholomorphic and that $\xi^{-1}(V'\backslash A)$ is connected. Since, as analytic sets in U, $\operatorname{codim} V'=1$ and $\operatorname{codim} A>1$, $V'\cap A$ is a thin analytic set in the complex space V'. It follows that $\xi^{-1}(V'\cap A)=V_0\backslash\xi^{-1}(V'\backslash A)$ is a thin analytic set in V_0 and that $\xi^{-1}(V'\backslash A)$ is an open dense complex subspace of V_0. Thus the normalization V_0 is connected and the continuous map ξ: sp $(V_0)\to$sp(V') subordinate to the morphism ξ: $V_0\to V'$ is a homeomorphism. ■

By Lemma 2. 3, $v_0=\xi^{-1}(x_0)$ is uniquely determined. This allows us to assume

v) *there exists a closed embedding* η: $V_0\hookrightarrow E$, *where* $E=E_1\times\ldots\times E_m$ *is a bounded open polydisk in* $\boldsymbol{C}^m$ *centred at* η $(v_0)=0\in\boldsymbol{C}^m$.

By the isomorphisms $\xi^{-1}(V'\backslash A)\simeq V'\backslash A\simeq V\times\{0\}\simeq V$, we consider that $V\subset V_0$ i. e. an open complex subspace. Let $Y=V_0\times W$, $Y_A=V\times W$ and $y_0=(v_0, 0)\in Y$. By v), we have

v)' *there exists a closed embedding* $I=\eta\times id_w$: $Y=V_0\times W\hookrightarrow E\times W$ *and* $I(y_0)=0$.

Y_A is an open complex subspace of Y and isomorphic to $U_A=U\backslash A$ by φ_A: $U_A\xrightarrow{\sim}Y_A=V\times W$. We claim the following lemma, which completes the proof of our main theorem 1. 0.

LEMMA 2. 4.

There exists an isomorphism φ: $U\xrightarrow{\sim}Y$ *such that the folowing digram commutes*:

$$\begin{array}{ccc} U_A & \xrightarrow[\simeq]{\varphi_A} & Y_A \\ \cap & & \cap \\ U & \xrightarrow[\varphi]{\simeq} & Y. \end{array}$$

The rest of this section is devoted to the proof of Lemma 2. 4. For each $j=1,\ldots,m$, we define a holomorphic map φ_{Aj}: $U_A\to E_j$ by

$$
\begin{array}{ccccl}
U_A & \xrightarrow[\sim]{\varphi_A} & Y_A = & V \times W & \\
& \searrow & & \cap & \\
& & Y = & V_0 \times W & \\
& \varphi_{Aj} \quad \searrow & & \uparrow & I = \eta \times id_W. \\
& & & E \times W & \\
& & \searrow & \downarrow & \\
& & & E_j &
\end{array}
$$

Since U_A is normal and E_j is a bounded open disk in $\boldsymbol{C}$, φ_{Aj} extends, as in Lemma 2.2, to a holomorphic map φ_j: $U \rightarrow E_j$ by Riemann extension theorem. φ_j's and g define a holomorphic map Φ: $U \rightarrow E \times W$ which makes the following diagrams commute:

$$
\begin{array}{ccc}
U & \xrightarrow{\Phi} & E \times W \\
& {\scriptstyle \varphi_j} \searrow & \downarrow \\
& & E_j
\end{array}
\qquad\qquad
\begin{array}{ccc}
U & \xrightarrow{\Phi} & E \times W \\
& {\scriptstyle g} \searrow & \downarrow \\
& & W
\end{array}
$$

We remark the following

LEMMA 2.5.
The morphism Φ: $U \rightarrow E \times W$ *factors through the closed complex subspace* Y, *i.e. there exists a uniquely determined morphism* φ: $U \rightarrow Y$ *such that the following diagram commutes*:

$$
\begin{array}{ccc}
U & \xrightarrow{\varphi} & Y \\
& {\scriptstyle \Phi} \searrow & \uparrow I. \\
& & E \times W
\end{array}
$$

Furthermore, the restriction of φ *to the open complex subspace* $U_A = U \backslash A \subset U$ *is identical with the isomorphism* φ_A: $U_A \xrightarrow{\sim} Y_A$, *i.e. the following diagram is commutative*:

$$
\begin{array}{ccc}
U_A & \xrightarrow[\sim]{\varphi_A} & Y_A \\
\cap & & \cap \\
U & \xrightarrow[\varphi]{} & Y.
\end{array}
$$

PROOF. Since these complex spaces are reduced because of their normality, we have only to show the subordinate continuous map Φ: sp$(U)\to$ sp$(E\times W)$ factors through sp(Y), i. e. $\Phi(\mathrm{sp}(U))\subset\mathrm{sp}(Y)$. But, considered as $\varphi_j\in\Gamma(U, \mathcal{O}_X)$ and $g\in\Gamma(U, \mathcal{O}_X)$, they are the extensions of $\varphi_{Aj}\in\Gamma(U_A, \mathcal{O}_X)$ and $g_A\in\Gamma(U_A, \mathcal{O}_X)$, respectively. Since φ_{Aj}'s and g_A define the morphism φ_A: $U_A\xrightarrow{\sim}Y_A$, the image $\Phi(\mathrm{sp}(U))$ is contained in sp(Y), which is the closure of sp(Y_A) in sp$(E\times W)$. Thus Φ factors through Y and the latter statement is also proved. ■

Let $\psi_A=\varphi_A^{-1}$: $Y_A\xrightarrow{\sim}U_A$. For each $k=1,\ldots,N$, we can define a morphism ψ_{Ak}: $Y_A\to D_k$ by

$$\begin{array}{ccc} Y_A & \xrightarrow[\sim]{\psi_A} & U_A\subset U \\ & \searrow^{\psi_{Ak}} & \uparrow\iota \\ & & D \\ & & \downarrow \\ & & D_k \end{array}$$

ψ_{Ak} extends to ψ_k: $Y\to D_k$ as above. ψ_k's define a unique morphism Ψ: $Y\to D$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\Psi} & D \\ & \searrow^{\psi_k} & \downarrow \\ & & D_k \end{array}$$

Similarly, we have the following lemma.

LEMMA 2.6.
The morphism Ψ: $Y\to D$ factors through the closed complex subspace U, i. e. there exists a uniquely determined morphism ψ: $Y\to U$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & U \\ & \searrow^{\Psi} & \uparrow\iota \\ & & D \end{array}$$

Furthermore, the restriction of ψ to the open complex subspace $Y_A\subset Y$ is identical with the isomorphism ψ_A: $Y_A\xrightarrow{\sim}U_A$, i. e. the following diagram is *commutative* :

$$\begin{array}{ccc} Y_A & \xrightarrow[\sim]{\psi_A} & U_A \\ \cap & & \cap \\ Y & \xrightarrow[\psi]{} & U. \end{array}$$

Considering the composite morphisms with the projections $D \to D_k$'s, $E \times W \to E_j$'s and $E \times W \to W$, we define ι_k: $U \to D_k$ and I_j: $Y \to E_j$ by

$$\begin{array}{ccc} U & \overset{\iota}{\hookrightarrow} & D \\ & {\scriptstyle \iota_k} \searrow & \downarrow \\ & & D_k \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y & \overset{I}{\hookrightarrow} & E \times W, \\ & {\scriptstyle I_j} \searrow & \downarrow \\ & & E_j \end{array}$$

for $k=1,\ldots,N$ and $j=1,\ldots,m+1$. Here we set $E_{m+1}=W$ and $\varphi_{m+1}=g$: $U \to E_{m+1}$.

By Lemma 2.5 and Lemma 2.6, we have two commutative diagrams

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & Y \\ {\scriptstyle \iota} \downarrow & \swarrow {\scriptstyle \Psi} & \\ D & & \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\psi} & U \\ {\scriptstyle I} \downarrow & \swarrow {\scriptstyle \Phi} & \\ E \times W & & \end{array}$$

and the following statements hold:

$$(\Psi \cdot \varphi)|_{U_A} = \iota|_{U_A}$$
$$(\Phi \cdot \psi)|_{Y_A} = I|_{Y_A}$$

These mean that

$$(\psi_k \cdot \varphi)|_{U_A} = \iota_k|_{U_A} \qquad (k=1,\ldots,N)$$

and that

$$(\varphi_j \cdot \psi)|_{Y_A} = I_j|_{Y_A} \qquad (j=1,\ldots, m+1).$$

Since $U_A \subset U$ and $Y_A \subset Y$ is open dense, we can deduce that

$$\psi_k \cdot \varphi = \iota_k \qquad (k=1,\ldots, N)$$

and that

$$\varphi_j \cdot \psi = I_j \qquad (j=1,\ldots, m+1).$$

Thus

$$\psi \cdot \varphi = id_U$$

and

$$\varphi \cdot \psi = id_Y.$$

Lemma 2. 4 is proved and the proof of the theorem is completed.

Though we introduced the nornalization $\xi: V_0 \to V'$, the following holds:

REMARK 2. 7.

$V_0 \rightrightarrows V'$ and $U \rightrightarrows V' \times W$ as complex spaces.

Now we prove Theorem 1. 1.

PROOF of THEOREM 1. 1.

Let X, x_0, U, A, V and W be as stated in Theorem 1. 1. In case that the condition II) is satisfied, the statement directiy follows from Theorem 1. 0. Suppose that A satisfies the condition I). We proceed by induction on the dimension d of W. Assume $d \geqq 2$ and $W = W_1 \times \ldots \times W_d$, where each W_j is a bounded open disk in $\boldsymbol{C}$ centred at $0 \in \boldsymbol{C}$. We denote by $g_j: U \backslash A \to W_j$ the composite map $U \backslash A \rightrightarrows W$ followed by $W \to W_j$, for each $j = 1, \ldots, d$. As in Lemma 2. 2, these holomorphic maps extend to U, which we also call $g_j: U \to W_j$. Let $V_j = g_j^{-1}(0)$. By theorem 1. 0 and the above remark, we may assume $U \simeq V_j \times W_j$. Note V_j's are connected normal complex spaces. In the complex space V_j, $\operatorname{codim}_{x_0} A \cap V_j \geqq 1$. We claim that $\operatorname{codim}_{x_0} A \cap V_j > 1$ for some j. By the irreducibility of A, $\operatorname{codim}_{x_0} A \cap V_j = 1$ implies that A is contained in V_j. If $\operatorname{codim}_{x_0} A \cap V_k = 1$ for $k \neq j$, then A is contained in $V_j \cap V_k$. However, as we see in the following, if A is contained in $V_j \cap V_k$, then $\operatorname{codim}_{x_0} A \cap V_j > 1$ in V_j, Let $Z = V \times \prod_{i \neq j, i \neq k} W_i$. Since $U \backslash A \simeq Z \times W_j \times W_k$, $A \neq V_j \cap V_k$. Note the following

LEMMA 2. 8.

$V_j \cap V_k$ is irreducible.

From this lemma and the irreducibility of A, we deduce that A is thin in $V_j \cap V_k$. Since $V_j \cap V_k$ itself is thin in V_j, $\operatorname{codim}_{x_0} A \cap V_j > 1$ in V_j. Thus there exists j such that $\operatorname{codim}_{x_0} A \cap V_j > 1$ in V_j and by induction hypotheses V_j is a direct product at x_0. ■

PROOF of LEMMA 2. 8.

Note that $V_j \cap V_k \cap (U \backslash A)$ is an open dense complex subspace of $V_j \cap V_k$ and that $V_j \cap V_k \cap (U \backslash A) \simeq Z$ is a connected normal complex space. Let $\xi: Z_0 \rightarrow V_j \cap V_k$ be the normalization of $V_j \cap V_k$. Then, as in Lemma 2. 3, the underlying continuous map $\xi: \mathrm{sp}(Z_0) \rightarrow \mathrm{sp}(V_j \cap V_k)$ is a homeomorphism and $V_j \cap V_k$ is irreducible. ■

§ 3. Holomorphic vector fields

Let X be an arbitrary complex space. A holomorphic vector field λ on an open complex subspace $U \subset X$ is, by definition, a $\mathcal{O}_U$-valued derivation of $\mathcal{O}_U$ over $\boldsymbol{C}$, i. e. a morphism of sheaves of $\boldsymbol{C}$-vector spaces on U satisfying the following condition:

For any open $V \subset U$, $\boldsymbol{C}$*-linear map*

$$\lambda = \lambda(V): \mathcal{O}_U(V) \rightarrow \mathcal{O}_U(V)$$

satisfies

$$\lambda(f\, g) = f \cdot \lambda(g) + \lambda(f) \cdot g \quad \textit{for all } f,\ g \in \mathcal{O}_U(V). \tag{3-0}$$

We denote by Θ_X the sheaf of germs of holomorphic vector fields on X, which has a natural structure of $\mathcal{O}_X$-module. Moreover, Θ_X is a coherent $\mathcal{O}_X$-module. Let $p \in X$ and $U \subset X$ an open neighbourhood of p. Assume that U has a closed embedding $i: U \hookrightarrow D$, where D is a domain in $\boldsymbol{C}^N$. Let $\mathcal{I}$ be the $\mathcal{O}_D$-ideal defining U. For any germ $\lambda_p \in \Theta_{X,p}$, there exists a germ $\Lambda_p \in \Theta_{D,p}$ of holomorphic vector field on D at $p = i(p)$ such that

$$\Lambda_p(\mathcal{I}_p) \subset \mathcal{I}_p \tag{3-1}$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_{D,p} & \xrightarrow{\Lambda_p} & \mathcal{O}_{D,p} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X,p} & \xrightarrow[\lambda_p]{} & \mathcal{O}_{X,p} \end{array} \tag{3-2}$$

Conversely, any germ $\Lambda_p \in \Theta_{D,p}$ of holomorphic vector field on D at p satisfying (3-1) determines such a germ $\lambda_p \in \Theta_{X,p}$ of holomorphic vector field on X at p that makes the diagram (3-2) commute.

We denote by T_pX the (*analytic*) *Zariski tangent space*, i. e. a tangent vector $v \in T_pX$ is a $\boldsymbol{C}$-valued derivation of $\mathcal{O}_{X,p}$ over $\boldsymbol{C}$. There is a natural injective $\boldsymbol{C}$-linear map $T_pX \to T_pD$, by which we consider $T_pX \subset T_pD$:

$$T_pX = \{v \in T_pD \mid v(\mathcal{I}_p) = 0\}.$$

Any germ $\lambda_p \subset \Theta_{X,p}$ of holomorphic vector field determines canonically a unique tangent vector $\lambda_p(p) \in T_pX$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_{X,p} & \xrightarrow{\lambda_p} & \mathcal{O}_{X,p} \\ & \underset{\lambda_p(p)}{\searrow} & \downarrow \\ & & \boldsymbol{C} \simeq \mathcal{O}_{X,p}/m_{X,p} \end{array}$$

This defines an $\mathcal{O}_{X,p}$-module homomorphism $\Theta_{X,p} \to T_pX$. Note that it is not necessarily surjective. (e.g. at $0 \in \{y^2 - x^3 = 0\}$.) Let $U \subset X$ an open complex subspace and $x \in U$. The composite $\boldsymbol{C}$-linear mapping

$$\Gamma(U, \Theta_X) \to \Theta_{X,x} \to T_xX$$

is defined, and the image of $\lambda \in \Gamma(U, \Theta_X)$ by this mapping is called the value of λ at x and denoted by $\lambda(x)$. For any open $U \subset X$ and $\lambda \in \Gamma(U, \Theta_X)$, $\{x \in U \mid \lambda(x) = 0\}$ is a well-defined analytic set in U.

On holomorphic vector fields on complex spaces, the following theorem is fundamental (see e.g. [F] pp. 91-92.):

THEOREM 3.3.

Let X be an arbitrary complex space, $p \in X$ and $\lambda_p \in \Theta_{X,p}$, If $\lambda_p(p) \neq 0$ then there exist an open neighbourhood $U \subset X$ of p, a representative vector field $\lambda_U \in \Gamma(U, \Theta_X)$, a complex space V and an open disk $W \subset \boldsymbol{C}$ such that

(3-4) $\quad U \simeq V \times W$

and at any piont $x \in U$,

(3-5) $\quad 0 \neq \lambda_U(x) \in T_xW \subset T_xU.$

Let $p \in X$. The $\boldsymbol{C}$-dimension $\dim_{\boldsymbol{C}} T_pX$ is called the embedding dimension of X at p and denoted by $\operatorname{emdim}_p X$. On the other hand, the dimension of X at p is denoted by $\dim_p X$ and satisfies $\dim_p X = \dim \mathcal{O}_{X,p}$, where $\dim \mathcal{O}_{X,p}$ is the Krull dimension of $\mathcal{O}_{X,p}$, It is well known that an inequality $\dim_p X \leqq \operatorname{emdim}_p X$ always holds and that $p \in X$ is a regular point if and only if $\dim_p X = \mathrm{emdim}_p X$. This fact and Theorem 3.3. imply the following corollary.

COROLLARY 3.6.

Let p be a reduced point of a complex space X. Then the $\boldsymbol{C}$-linear map $\Theta_{X,p} \to T_pX$ is surjective if and only if p is a regular point of X.

§ 4. An application of the main theorem

Gômez-Mont investigated holomorphic foliations by curves on complex spaces and obtained the following result. (See [GM], p. 131, Theorem 5.)

THEOREM 4.0 (Gômez-Mont).

Let X be a normal complex sapace, $\mathscr{F}$ a holomorphic foliation by curves on X and λ a holomorphic vector field which is tangent to the leaves of $\mathscr{F}$ wherever it does not vanish. If $\{x \in Y \mid \lambda(x)=0\}$ is of codimension strictly greater than one, then for any holomorphic vector field $\eta \in \Gamma(X, \Theta_X)$ which is tangent to the leaves of $\mathscr{F}$ wherever it does not vanish, there exists a never-vanishing $\alpha \in \mathscr{O}_X(X)$ such that $\eta = \alpha \cdot \lambda$.

From this and our main theorem, we can deduce the following proposition.

PROPOSITION 4.1.

Let X be a normal complex space, $p \in X$, $\mathscr{F}$ a foliation by curves on X and $\lambda \in \Gamma(X, \Theta_X)$ tangent to the leaves of $\mathscr{F}$. If $\lambda(p)=0$ and if the analytic set $Z=\{x \in X \mid \lambda(x)=0\}$ is of codimension strictly greater than one then we can never find an open neighbourhood U of p, an analytic set A in U of codimension strictly greater than one and containing $Z \cap U$, a complex space V and an open disk $W \subset \boldsymbol{C}$ such that $U \backslash A \simeq V \times W$ and that the restriction of $\mathscr{F}$ to U is identical with the foliation defined by $U \backslash A \simeq V \times W$.

PROOF. Assume that there were U, A, V and W satisfying the conditions. Then, by our main theorem, there would exist U_0, V_0 and W_0 as described in the theorem. Furthermore, shrinking U_0 if necessary, we may assume $V_0 \hookrightarrow E$, where E is a domain in $\boldsymbol{C}^m$. $U_0 = V_0 \times W_0 \hookrightarrow D = E \times W_0$. Let w be a coordinate in W_0 and $\eta \in \Theta_X(U_0)$ the vector field defined by $\partial/\partial w \in \Theta_D(D)$. η is a vector field tangent to the leaves of $\mathscr{F}$ and never vanishes on U_0. By the theorem of Gômez-Mont, there should be a never-vanishing holomorphic function $\alpha \in \Gamma(U_0, \mathscr{O}_X)$ such that $\lambda|_{U_0} = \alpha \cdot \eta$ But since $\lambda(p)=0$, we have a contradiction. ■

References

[F] FISCHER, G.: Complex analytic geometry. Lect. Notes in Math., vol. 538, Springer-Verlag (1976).

[GM] GÓMEZ-MONT, X.: Foliations by curves of complex analytic spaces. Contemporary Math., **58**, 123-141 (1987).

[G-R-1] GRAUERT, H, and REMMERT, R.: Analytische Stellenalgebren. Springer-Verlag (1971).

[G-R-2] ———: Coherent analytic sheaves. Springer-Verlag (1984).

Department of Mathematics
Fauclty of Science
University of Tokyo
Hongo, Tokyo 113, Japan