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On cyclic tournaments

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1.

Let V be the set of integers $1, 2, \ldots, v$ and $S(v)$ the symmetric group on V. Put $C=(1, 2, \ldots, v)$. Let $W(v)$ be the set of all subgroups of $S(v)$ of odd orders containing C.

A complete asymmetric digraph A whose set of vertices is V is also called a tournament. We identify a digraph with its adjacency matrix. We also identify a permutation with its matrix representation.

Let A and B be two tournaments of order v. Then B is equivalent to A if there exists a permutation matrix P such that $B=P^{t}AP$, where t denotes the transposition. This is a true equivalence relation. If $B=A$, then P is called an automorphism of A. The set $G(A)$ of all automorphisms of A forms a group, the automorphism group of A.

A tournament A is called cyclic if $G(A)$ contains C. Let A be a cyclic tournament of order v. We may regard the first row vector $O(1)$ of A as the out-neighborhood of the vertex 1. Since A is cyclic, A is completely determined by $O(1)$. Put $v=2k+1$ and $i^{*}=v-i+1$ for $2 \leqq i \leqq k+1$. We call $\{i, i^{*}\}$ a complementary pair for $2 \leqq i \leqq k+1$. Choose one element from each complementary pair. This procedure determines $O(1)$ and hence A. Thus there exist 2^{k} cyclic tournaments. Let $C(v)$ be the set of all cyclic tournaments of order v.

Let G be an element of $W(v)$ and H the stabilizer of 1 in G. If we want to construct a cyclic tournament A such that $G(A)$ contains G, then we have a restriction on the choice of elements from complementary pairs imposed by H. Namely if i and j^{*} belong to the same orbit of H, then both of i and j^{*} or none of them have to be chosen. If we do so, then we see that every maximal element G of $W(v)$ is of the form $G=G(A)$ for some element A of $C(v)$.

Let $v=p$ be a prime and $u(p)$ the odd portion of $p-1$, namely $p-1=2^{e}u(p)$. Then $G(p)$ denotes the metacyclic group of order $pu(p)$ on V. Since $G(p)$ is maximal in $W(p)$, $G(p)=G(A)$ for some tournament A of order p.

In the present paper we show the following:

(i) If any v-cycle of $G(A)$ is a power of C, we can determine the size of the equivalence class of A in $C(v)$;

(ii) Any maximal element of $W(v)$ is of the form $G(p_1)^\circ G(p_2)^\circ \dots {}^\circ G(p_r)$, where $v = p_1 p_2 \dots p_r$ is a prime decomposition of v and $^\circ$ denotes the Polya composition. For this see [1];

(iii) An element of $W(v)$ of the largest order is uniquely (up to the conjugacy in $W(v)$) determined by a certain linear order of odd primes; and

(iv) Any element of $W(v)$ is of the form $G(A)$ for a certain element A of $C(v)$.

2.

PROPOSITION 1. *Let A and B be two equivalent cyclic tournaments such that $B = P^t AP$, where P is a permutation matrix. Assume that any v-cycle of $G(A)$ is a power of C. Then P belongs to the normalizer $N(\langle C \rangle)$ of $\langle C \rangle$. Put $N(\langle C \rangle) = \langle C \rangle N(v)$, where $N(v)$ is the stabilizer of 1 in $N(\langle C \rangle)$. Then $N(v)$ is Abelian of order $\varphi(v)$, where φ denotes the Euler totient function. Let $\varphi(A)$ be the order of $N(v) \cap G(A)$. Then the size of the equivalence class in $C(v)$ to which A belongs equals $\varphi(v)/\varphi(A)$.*

PROOF. We have that $A = PBP^t = PC^t BCP^t = PC^t P^t APCP^t$. So PCP^t belongs to $G(A)$. By assumption PCP^t is a power of C. The rest is obvious.

REMARK 1. The assumption on $G(A)$ in proposition 1 is satisfied, in particular, if $G(A) = \langle C \rangle$ or v is square-free. So for certain v it is possible to have a formula for the number of equivalence classes of cyclic tournaments.

(i) If v is a Fermat prime, $v = 2^m + 1$, then each equivalence class has size $v-1$ and hence there exist $2^{2^m - 1 - m}$ classes.

(ii) If v and $(v-1)/2 = k$ are primes, then, since any tournament A of order v such that $G(A)$ has order vk is equivalent to the tournament of quadratic residue (or non-residue) type ([2]), there exist $(2^{k-1} - 1/k) + 1$ classes.

3.

PROPOSITION 2. *Let G be a maximal element of $W(v)$. Then G is similar to $G(p_1)^\circ G(p_2)^\circ \dots {}^\circ G(p_r)$, where $^\circ$ denotes the Polya composition and $v = p_1 p_2 \dots p_r$ is a prime decomposition.*

PROOF. If v is a prime, then our assertion holds good by a theorem of Burnside ([4], (11.7)). So assume that v is not a prime. Since G contains C, by a theorem of Schur ([4], (25.3)) G is imprimitive. Let M be a maximal subgroup of G of index m containing G_1, the stabilizer of 1 in G. Let $core(M)$ denote the largest normal subgroup of G contained in M. Then $G/core(M)$ is a permutation group of degree m and of odd order containing an m-cycle. Since M is maximal in G, by a theorem of Schur ([4], (25.3)) we have that m is a prime. Now we apply an induction argument with respect to the degree. For the rest we refer to ([3], 10.5.5).

REMARK 2. We notice that, under the assumption that G contains a v-cycle, we have shown the solvability of G without invoking the Feit-Thompson theorem.

4.

Let P be the set of all odd primes. We introduce a new order in P as follows: $p \gg q$ if and only if $(qu(q))^{p-1} > (pu(p))^{q-1}$.

LEMMA 1. *$\gg$ is a lincar order.*

PROOF. If $p \neq q$, then $(qu(q))^{p-1} \neq (pu(p))^{q-1}$. Now assume that $p \gg q$ and $q \gg r$. Then we have that $(qu(q))^{p-1} > (pu(p))^{q-1}$ and $(ru(r))^{q-1} > (qu(q))^{r-1}$. So it follows that $(qu(q))^{(p-1)(r-1)} > (pu(p))^{(q-1)(r-1)}$ and that $(ru(r))^{(q-1)(p-1)} > (qu(q))^{(r-1)(p-1)}$. Hence we have that $(ru(r))^{p-1} > (pu(p))^{r-1}$, namely $p \gg r$.

REMARK 3. The following is the sequence of odd primes under 100 in the increasing order using $\gg$: 3, 7, 5, 11, 13, 19, 23, 31, 29, 17, 43, 37, 47, 41, 59, 67, 61, 71, 79, 83, 73, 89, 97.

PROPOSITION 3. *Let G be an element of $W(v)$ of the largest order. Then $G = G(p_1)^\circ G(p^2)^\circ \ldots {}^\circ G(p_r)$, where $v = p_1 p_2 \ldots p_r$ is a prime decomposition such that $p_1 \geqq p_2 \geqq \ldots \geqq p_r$.*

PROOF. The case $r=1$ is trivial. Assume that $r=2$. Then the orders of $G(p_1)^\circ G(p_2)$ and $G(p_2)^\circ G(p_1)$ are equal to $p_1 u(p_1)(p_2 u(p_2))^{p_1}$ and $p_2 u(p_2)(p_1 u(p_1))^{p_2}$ respectively. So if $p_1 \gg p_2$, then the order of $G(p_1)^\circ G(p_2)$ is larger than that of $G(p_2)^\circ G(p_1)$. Now assume that $r \geqq 3$ and put $v = p_1 p_2 z$. Then by an induction argument on r it is sufficient to compare the orders of $G_1 = G(p_1)^\circ G(p_2)^\circ G(z)$ and $G_2 = G(p_2)^\circ G(p_1)^\circ G(z)$, where $G(z) = G(p_3)^\circ \ldots {}^\circ G(p_r)$. In particular, we may assume that $p_1 \neq p_2$. Let $g(z)$ denote the order of $G(z)$. Now the orders of G_1 and G_2 are equal to $p_1 u(p_1)(p_2 u(p_2) g(z)^{p_2})^{p_1}$ and $p_2 u(p_2)(p_1 u(p_1) g(z)^{p_1})^{p_2}$ respectively. So exactly as in the case where $r=2$, we see that the order of G_1 is larger than that of G_2.

REMARK 4. Though G is unique up to the conjugacy in $S(v)$, there may exist many inequivalent A's such that $G(A)=G$.

5.

Let $O(i)$ denote the out-neighborhood of i, $1\leqq i\leqq v$.

LEMMA 2. *Let A be a cyclic tournament of order v such that the out-neighborhood $O(1)$ of the vertex 1 consists of $2, 3,\dots, k$, where $v=2k+1$. Then $G(A)=\langle C\rangle$.*

PROOF. It is enough to notice that $O(1)\cap O(i)$ contains $k-i+1$ vertices for $1\leqq i\leqq k+1$, which implies that the stabilizer of 1 in $G(A)$ is trivial.

REMARK 5. We remark that we have $G(A)=\langle C\rangle$ for most cyclic tournaments A.

LEMMA 3. *Let X and Y be elements of $W(v)$ such that X contains Y properly. Let $X(1)$ and $Y(1)$ be the stabilizers of 1 in X and Y respectively. Then $X(1)$ and $Y(1)$ have distinct orbit decompositions on $V-\{1\}$.*

PROOF. We apply an induction argument on the order v. If v is a prime, then, by a theorem of Burnside [4, 11.7], $X(1)$ and $Y(1)$ are semiregular on $V-\{1\}$ and $X(1)$ contains $Y(1)$ properly. So the assertion is obvious. If v is not a prime, then, by a theorem of Schur [4, 25.3] X is imprimitive. Let D be a non-trivial block and $X(D)$ and $Y(D)$ the global stabilizers of D in X and Y respectively. Since Y contains C, $Y(D)$ is transitive on D. $X(1)$ and $Y(1)$ are the stabilizers of 1 in $X(D)$ and $Y(D)$ respectively. Then by induction hypothesis the orbit decomposition of $Y(1)$ is a proper refinement of that of $X(1)$ on $D-\{1\}$.

PROPOSITION 4. *Let W be an element of $W(v)$. Then there exists a cyclic tournament A of order v such that $W=G(A)$.*

PROOF. In §1 we described a procedure to construct a cyclic tournament A° such that $G(A^\circ)$ contains W. Now assume that $G(A^\circ)$ contains W properly. Let $G(A^\circ)(1)$ and $W(1)$ be the stabilizers of 1 in $G(A^\circ)$ and W respectively. Then by Lemma 3 the orbit decomposition of $V-\{1\}$ by $W(1)$ is a proper refinement of that by $G(A^\circ)(1)$. So by the procedure described in §1 we can construct a cyclic tournament $A^{\circ\circ}$ such that $G(A^{\circ\circ})$ contains W and $G(A^{\circ})$ contains $G(A^{\circ\circ})$ properly. We may repeat this process. So eventually we obtain a cyclic tournament A such that $G(A)=W$.

References

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