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Some properties of Fourier transform for operators on homogeneous Banach spaces

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Abstract.

The Fourier transform of linear operator on a general homogeneous Banach space B in $L^1(G)$ for locally compact abelian group G is defined and characterized. It is proved that the Fourier transform of a linear operator is an operator valued continuous function on $\widehat{G}$, th dual group of G, and vanishing at infinity. Convolution of function and operator is studied. Some linear operator on B is characterized as an integration of its Fourier transform over $\widehat{G}$.

1. Introduction and preliminaries

Throughout the paper let G be a locally compact as well as a σ-compact abelian group, and let $\widehat{G}$ be its dual group. A *homogeneous Banach space* B is a dense subspace of $L^1(G)$ such that

(i) B is a Banach space under another norm $\| \ \|_B$ which is stronger than $L^1(G)$-norm $\| \ \|_1$.
(ii) The norm $\| \ \|_B$ is translation invariant and $\|R_x f-f\|_B \to 0$ as $x \to 0$ in G where $R_x f(y)=f(y-x)$ for all x and y in G.

Some special homogeneous Banach spaces are investigated in Larsen [6], Lai [7-10]. For example, the spaces

$$A^p(G)=\{f \in L^1(G) : \hat{f} \in L^p(\widehat{G}),\ 1 \leq p \leq \infty\}$$

with norm $\|f\|_{A^p(G)}=\|f\|_1+\|\hat{f}\|_p$
and $A_{1,p}(G)=L^1 \cap L^p(G)$ with norm $\|f\|=\|f\|_1+\|f\|_p$

are homogeneous Banach spaces.

A homogeneous Banach space B may not admit multiplication by character $\gamma \in \widehat{G}$, and even if it does, it may not be isometry under the norm $\| \ \|_B$ (see Reiter [15]). If for any $\gamma \in \widehat{G}$, the operator

$$M_\gamma : f \in B \to \gamma \cdot f \in B$$

is closed on B, we call B an *invariant homogeneous Banach space*. In order to define the Fourier transform of linear operators T in $\mathscr{L}(B)$, the space of all linear operators on B, we will assume throughout that B is invariant.

Let $\mathscr{L}_b(B)$ be the space of bounded linear operators on B. $T\in\mathscr{L}_b(B)$ is said to be *almost invariant* if

$$\lim_{x\to 0}\|R_x T - TR_x\|=0$$

where the norm is the uniform norm of $\mathscr{L}_b(B)$.

In [1] and [2], DeLeeuw investigated the harmonic analysis for almost invariant operators on B in the case of $G=T$, the circle group and Tewari and Madan [16] in the case of compact group G. Recently, Yu [17] considered the homogeneous Banach space $B\subset L^1(G)$ for $G=\boldsymbol{R}$. All of these works have partially shown that some basic properties on harmonic analysis hold for operator $T\in\mathscr{L}_b(B)$. Essentially, DeLeeuw defines the Fourier transform of $T\in\mathscr{L}_b(B)$ for $G=T$ by

$$\hat{T}(n)f=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inx}R_{-x}TR_x f\,dx,\ \text{for } f\in B$$

and proved some results as follows:

(a) $$\lim_{N\to+\infty}\sum_{-N}^{N}\left(1-\frac{|n|}{N+1}\right)\hat{T}(n)f=Tf \text{ in } B.$$

Moreover if T is an almost invariant operator on B, then

$$\sum_{-N}^{N}\left(1-\frac{|n|}{N+1}\right)\hat{T}(n)\to T \text{ uniformly in } \mathscr{L}_b(B).$$

(b) $$\lim_{|n|\to+\infty}\|\hat{T}(n)f\|_B=0 \text{ for any } f\in B.$$

If $T\in\mathscr{L}_b(B)$ is almost invariant, then

$$\lim_{|n|\to+\infty}\|\hat{T}(n)\|=0 \text{ in } \mathscr{L}_b(B).$$

(c) For any bounded regular measure $\mu\in M(T)$ and $T\in\mathscr{L}_b(B)$, the convolution

$$(\mu * T)f=\int_{-\pi}^{\pi}R_x TR_{-x}f\,d\mu(x) \text{ for } f\in B$$

is well defined, and $\mu * T\in\mathscr{L}_b(B)$ such that

$$(\mu * T)\hat{}(n)f=\hat{\mu}(n)\hat{T}(n)f \text{ for all } f\in B.$$

In this paper we will treat these results for operators on $B\subset L^1(G)$ with locally compact abelian group G. It is a new approach different from the Fourier analysis for functions in $L^1(G)$.

We denote by $\mathscr{L}(B)$ the space of all linear operators on B, and let $\mathscr{L}_s(B)$ be the space of $\mathscr{L}(B)$ with the strong operator topology $\sigma(\mathscr{L}(B), B)$ induced from B. Define a subspace of $\mathscr{L}_b(B)$ by

$$\mathscr{L}_s^1(B)=\{T\in\mathscr{L}_b(B): \int_G\|TR_xf\|_B dx<+\infty,\ f\in B\}$$

namely the space of *strongly right translated itegrable operators* on B. Evidently,

$$\mathscr{L}_s^1(B)\subset\mathscr{L}_b(B)\subset\mathscr{L}_s(B)\subset\mathscr{L}(B)$$

and $\mathscr{L}_s^1(B)\neq\mathscr{L}_b(B)$ except G is compact.

For example if $T\in\mathscr{L}_b(B)$ is *translation invariant*, that is, $TR_x=R_xT$, then $T\notin\mathscr{L}_s^1(B)$ except G is compact. In fact,

$$\int_G\|TR_xf\|_B dx=\int_G\|TR_xTf\|_B dx=\int_G\|Tf\|_B\ dx=\|Tf\|_B\lambda(G)<\infty \text{ iff } \lambda(G)<\infty,$$

that is, only G is compact.

We will explore some basic properties for harmonic analysis on $\mathscr{L}_s^1(B)$. It is different in comparison with $L^1(G)$. In Section 2, we will prove that the Fourier transform of $T\in\mathscr{L}_s^1(B)$ is an operator-valued continuous function on $\hat{G}$ which vanishes at infinity. This result extends the DeLeeuw's result shown by (b). In Section 3, we introduce a positive kernel in $L^1(G)$ and show that for any $T\in\mathscr{L}_s^1(B)$, there exists a net of operators in $L^1(G)*\mathscr{L}_s^1(B)$ which converges to T in $\mathscr{L}_s^1(B)$; so that $\mathscr{L}_s^1(B)$ is an essential $L^1(G)$-module under convolution. This is a generalization of the result (a). The Fourier transform for convolution of function and operator becomes the pointwise product of functions. This extends the result (c) to $\mathscr{L}_s^1(B)$ for locally compact abelian group G. Every operator $T\in\mathscr{L}_s^1(B)$ can be represented by the integration of its Fourier transform if it is strongly integrable over $\hat{G}$. Finally we show that the bounded regular measure algebra $M(G)$ is embedded as a subspace of the multiplier space for $\mathscr{L}_s^1(B)$.

2. Fourier transform for linear operators on B.

If $T\in\mathscr{L}_s^1(B)$, then $TR_{(\cdot)}f\in L^1(G,B)$ for any $f\in B$. One can easily show that the mappings:

$$x\to TR_{-x}f,\ f\in B$$

and $F: x\to(-x,\gamma)TR_{-x}f,\ \gamma\in\hat{G}$ and $f\in B$

are continuous, and that

$$\int_G \|F(x)\|_B dx = \int_G \| TR_{-x}f\|_B dx < +\infty.$$

This shows that the integration

$$\int_G (-x,\gamma)\, TR_{-x}f \ dx \qquad \text{for all } f\in B$$

is well defined. Thus it incurs the following definition.

DEFINITION. For $T\in\mathscr{L}_s^1(B)$, define

$$\hat{T}(\gamma)f=\int_G (-x,\gamma)\, TR_{-x}f \ dx \quad \text{for } \gamma\in\hat{G} \text{ and } f\in B, \tag{2.1}$$

and call $\hat{T}$ the *Fourier transform* of T.

By difinition, $\hat{T}$ is a mapping from $\hat{G}$ to $\mathscr{L}_s(B)$. It is remarkable that if $T=I$, the identity operator on B, then $\hat{I}(\gamma)f(0)=\hat{f}(\gamma)$. But $I\notin\mathscr{L}_s^1(B)$ except G is compact since otherwise (2.1) can not make sense for Bochner integral (see Diestel and Uhl [3] in detail, cf. also Lai [11−14] and Dunford and Schwartz [4]).

If G is compact, then $\mathscr{L}_s^1(B)=\mathscr{L}_b(B)$ and the identity operator $I\in\mathscr{L}_s^1(B)$ has Fourier transform

$$\begin{aligned}
[\hat{I}(\gamma)f](y) &= [\int_G ((-x,\gamma)R_{-x}f \ dx](y), \ y\in G \\
&= \int_G (-x,\gamma)R_{-x}f(y)dx \\
&= \int_G (-x,\gamma)f(x+y)dx \\
&= \int_G (y,\gamma)(-x-y,\gamma)f(x+y)dx \\
&= \int_G (y,\gamma)(-z,\gamma)f(z)dz \\
&= (y,\gamma)\hat{f}(\gamma).
\end{aligned}$$

As $y=0$, $[\hat{I}(\gamma)f](0)=\hat{f}(\gamma)$.

The following proposition follows immediately by calculation.

PROPOSITION 2.1. *For $\varphi\in L^1(G)\cap L^\infty(G)$, the multiplication operator T_φ on $L^1(G)$, defined by $T_\varphi f=\varphi * f$ (resp. $T_\varphi f=\varphi\cdot f$) for $f\in L^1(G)$, has Fourier transform*:

$$[\hat{T}_\varphi(\gamma)f](y)=(y,\gamma)\hat{\varphi}(\gamma)\hat{f}(\gamma)$$
$$(\textit{resp.}\ [\hat{T}_\varphi(\gamma)f](y)=(y,\gamma)\varphi(y)\hat{f}(\gamma)).$$

The following theorem is essential in this section. It was partly shown in [17, Theorem 3.3] in the case $G=\boldsymbol{R}$. But the proof is not available for a general LCA group. Whence we need a rigorous proof for LCA group $G\neq\boldsymbol{R}$ which we state and prove as in the following theorem.

THEOREM 2.2. *Let $T\in\mathscr{L}_s^1(B)$. Then $\hat{T}(\cdot):\hat{G}\to\mathscr{L}_s(B)$ is a bounded continuous function such that*

$$\|\hat{T}(\gamma)f\|_B\leqq\|TR_{(\cdot)}f\|_{1,B} \quad \textit{for any } \gamma\in\hat{G}$$

where $\|g\|_{1,B}=\int_G\|g(x)\|_B\,dx$ *for* $g\in L^1(G,B)$. *Moreover* $\hat{T}\in C_0\ (\hat{G},\mathscr{L}_s(B))$.

PROOF. For any γ_1,γ_2 in $\hat{G}$, $f\in B$ and $T\in\mathscr{L}_s^1(B)$ we have

$$\begin{aligned}\|\hat{T}(\gamma_1)f-\hat{T}(\gamma_2)f\|_B&=\|\int_G[(-x,\gamma_1)TR_{-x}f-(-x,\gamma_2)TR_{-x}f]dx\|_B\\&\leqq\int_G|(-x.\gamma_1-\gamma_2)-1|\ \|TR_{-x}f\|_B dx\\&\leqq\int_G 2\|TR_{-x}f\|_B\,dx.\\&<+\infty.\end{aligned}$$

It follows from the dominated convergence theorem that

$$\lim_{\gamma_1\to\gamma_2}\|\hat{T}(\gamma_1)f-\hat{T}(\gamma_2)f\|_B\leq\int_G\lim_{\gamma_1\to\gamma_2}|(-x,\gamma_1-\gamma_2)-1|\ \|TR_{-x}f\|_B dx=0,$$

and

$$\begin{aligned}\|\hat{T}(\gamma)f\|_B&=\|\int_G(-x,\gamma)TR_{-x}f\ dx\|_B\\&\leq\int_G\|TR_{-x}f\|_B dx\\&=\|TR_{(\cdot)}f\|_{1,B}\quad\text{for all }\gamma\in\hat{G}.\end{aligned}$$

Hence $\hat{T}:\hat{G}\to\mathscr{L}_s(B)$ is a bounded continuous function. It remains to show that $\hat{T}$ vanishes at infinity with value in $\mathscr{L}_s(B)$. We have only to show that for any $\varepsilon>0$ and $f\in B$, there exists a compact subset K in $\hat{G}$ such that

$$\|\hat{T}(\gamma)f\|_B<\varepsilon \qquad \text{whenever } \gamma\in\hat{G}\backslash K.$$

Since G and $\hat{G}$ are σ-compact and the B-valued function $F=TR_{(\cdot)}f\in L^1(G,B)$, thus for any $\varepsilon>0$ there exists a simple function

$$F_n(x)=\sum_{i=1}^{n} s_i\chi_{E_i}(x)$$

such that

$$\int_G \|F_n(x)-F(x)\|_B dx<\varepsilon/2 \tag{1}$$

where $s_i\in B$, $E_i\subset G$ is measurable with Haar measure $|E_i|<\infty$ and χ_E denotes the characteristic function of E. Next for any $\gamma\in\hat{G}$

$$\begin{aligned}\|\hat{T}(\gamma)f\|_B&=\|\int_G(-x,\gamma)TR_{-x}f\ dx\|_B\\ &\leqq\|\int_G(-x,\gamma)[F(x)-F_n(x)]dx\|_B+\|\int_G(-x,\gamma)F_n(x)dx\|_B\\ &\leqq\int_G\|F(x)-F_n(x)\|_B dx+\|\sum_{i=1}^{n}s_i\int_G(-x,\gamma)\chi_{E_i}(x)dx\|_B \end{aligned}\tag{2}$$

Since $\chi_{E_i}\in L^1(G),\hat{\chi}_{E_i}\in C_0(\hat{G})$ for each i. Thus there is a compact set $K_i\subset \hat{G}$ such that

$$|\hat{\chi}_{E_i}(\gamma)|<\varepsilon/2(n\|s_i\|_B)\quad \text{for } \gamma\in\hat{G}\backslash K_i.$$

Let $K=\bigcup_{i=1}^{n}K_i$ (note that $\hat{G}$ is σ-compact). Then K is compact and

$$\sum_{i=1}^{n}\|s_1\|_B|\hat{\chi}_{E_i}(\gamma)|<\varepsilon/2 \tag{3}$$

Substituting (1) and (3) into (2), we obtain that

$$\|\hat{T}(\gamma)f\|_B<\varepsilon \quad \text{for } \gamma\in\hat{G}\backslash K.$$

This proves that $\hat{T}(\cdot)\in C_0(\hat{G},\mathscr{L}_s(B))$. □

After this theorem, an open problem arises naturally that

Question: Does the set $\mathscr{L}_s^1(B)\hat{}$ of all Fourier transform for $\mathscr{L}_s^1(B)$ be dense of first category in $C_0(\hat{G},\mathscr{L}_s(B))$?

3. Convolution of functions and operators.

It is known that any homogeneous Banach space $B\subset L^1(G)$ is also a Segal algebra with convolution as the ring multiplication (see Reiter [15]), thus B is a dense ideal of $L^1(G)$. We will define the convolution of functions on G and linear operators T in $\mathscr{L}_s^1(B)$ as follows.

Denote $M(G)$ the space of all bounded regular measures on G. Note that $B\subset L^1\subset M(G)$ and $L^1(G)$ is equivalent to the absolutely continuous part of $M(G)$. So for any measure $\mu\in M(G)$ there is a density function $h\in L^1(G)$ such that $d\mu(x)=h(x)dx$.

DEFINITION. The *convolution* of $\mu\in M(G)$ and $T\in\mathscr{L}_s^1(B)$ is defined by

$$(\mu * T)f=\int_G TR_x f d\mu(x) \quad \text{for } f\in B. \tag{3.1}$$

In particular, if $h\in L^1(G)$, then define

$$(h * T)f=\int_G h(x)TR_x f dx, \ f\in B. \tag{3.2}$$

Since the mapping $x\in G\to TR_x f$ is continuous, (3.1) implies that

$$\|(\mu * T)f\|_B \leqq \int_G \|TR_x f\|_B d|\mu|(x)$$
$$\leqq \|\mu\| \, \|T\| \, \|f\|_B \quad \text{for all } f\in B$$

and $$\|\mu * T\| \leq \|\mu\| \, \|T\|$$

where $\|\mu\|$ is the total variation of $\mu\in M(G)$ on G. It follows that the convolution $\mu * T$ defines an element of $\mathscr{L}_s^1(B)$. Indeed

$$\int_G (\mu * T)R_x f dx=\int_G\int_G TR_y R_x f d\mu(y)dx$$
$$=\int_G (\int_G TR_{x+y} f dx)d\mu(y)$$

and $$\int_G \|(\mu * T)R_x f\|_B dx \leq \|\mu\| \int_G \|TR_z f\| dz < +\infty.$$

If G is compact and $T=I$ is the identity operator on B, then

$$(\mu * I)f=\int_G R_x f d\mu(x)=\mu * f.$$

From Lai [7, Theorem 1], we see that the Segal algebra $A^p(G)$ ($p\leq 1$) has an approximate identity which is also the bounded approximate identity for $L^1(G)$ and whose Fourier transform has compact support in $\hat{G}$. Such approximate idntity is not uniform bounded in $A^p(G)$-norm. Thus it incurs us to assume a net $\{e_\alpha(\cdot)\}_{\alpha\in\Lambda}$ of functions $e_\alpha\in L^1(G)$ satisfying the following conditions

(A1) For each $x\in G$, $e_\alpha(x)\geq 0$ with $\|e_\alpha\|_1=1$ for all α.

(A2) For any $\varepsilon>0$ and any symmetric compact neighborhood V of identity in G, there is an $\alpha_0\in\Lambda$ such that $\int_{G\setminus V} e_{\alpha_0}(x)dx<\varepsilon$.

(A3) For each $\alpha\in\Lambda$, $\hat{e}_\alpha\in L^1(\hat{G})$.

We call this net $\{e_\alpha(\cdot)\}_{\alpha\in\Lambda}$ a *positive kernel* of $L^1(G)$.

For example if $G=\boldsymbol{R}$, the functions defined by

$$e_\alpha(x)=\begin{cases}\dfrac{2}{\pi}\dfrac{\sin^2(\alpha x/2)}{\alpha x^2} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases} \tag{3.3}$$

for $\alpha>0$ (see Katznelson [5, p. 124]), form a positive kernel in $L^1(\boldsymbol{R})$.

In fact, the net $\{e_\alpha\}_{\alpha\in \boldsymbol{R}^+}$ defined by (3.3) is evidently satisfying the condition (A1). For (A2) we choose $V=\{x:|x|<\delta\}$ for any $\delta>0$, then

$$\lim_{\alpha\to+\infty}\int_{|x|\geq\delta}e_\alpha(x)dx=0.$$

While (A3), because the Fourier transform of the function

$$\varphi_\alpha(\gamma)=\begin{cases}\dfrac{1}{2\pi}\left(1-\dfrac{|\gamma|}{\alpha}\right) & \text{for } |\gamma|\leqq\alpha\\ 0 & \text{for } |\gamma|>\alpha\end{cases}$$

is given by

$$\begin{aligned}\hat{\varphi}_\alpha(-x)&=\int_{\boldsymbol{R}}\varphi_\alpha(\gamma)e^{i\gamma x}d\gamma=\int_{-\alpha}^{\alpha}\frac{1}{2\pi}\left(1-\frac{|\gamma|}{\alpha}\right)e^{i\gamma x}d\gamma\\ &=\frac{2}{\pi}\frac{\sin^2(\alpha x/2)}{\alpha x^2}=e_\alpha(x)\end{aligned}$$

and $e_\alpha\in L^1(G)$, it follows that $\varphi_\alpha(\gamma)=\hat{e}_\alpha(\gamma)\in L^1(\hat{G})$.

Actually this positive kernel of $L^1(G)$ is also an approximate identity for the group algebra $L^1(G)$.

PROPOSITION 3.1. *The positive kernel $\{e_\alpha\}$ of $L^1(G)$ plays an approximate identity for $L^1(G)$.*

PROOF. For $f\in L^1(G)$, the continuity of translation operator on $L^1(G)$ implies that for any $\varepsilon>0$ there exists a symmetric compact neighborhood V of identity 0 in G such that

$$\|R_yf-f\|_1<\varepsilon/2 \quad \text{whenever } y\in V.$$

To this V, there is an $\alpha_0\in I$ such that

$$\int_{G\setminus V}e_{\alpha_0}(y)dy<\varepsilon/4\|f\|_1,$$

It follows that

$$\begin{aligned}\|e_{\alpha_0}*f-f\|_1&\leqq\int_V\|R_yf-f\|_1e_{\alpha_0}(y)dy+\int_{G\setminus V}\|R_yf-f\|_1e_{\alpha_0}(y)dy\\ &<\frac{\varepsilon}{2}+2\|f\|_1\cdot\varepsilon/4\|f\|_1\\ &=\varepsilon.\end{aligned}$$

Hence $\{e_\alpha\}$ is an approximate identity for $L^1(G)$. □

The following theorem is essential in this section. Part of this theorem was shown by [17, Theorem 3.4 to 4.1] only in the case $G=T$, the circle group. We prove here by a general formation for a LCA group.

THEOREM 3.2.
(1) *$\mathscr{L}_s^1(B)$ is an essential $L^1(G)$-module under convolution, so that*

$$L^1(G)*\mathscr{L}_s^1(B)=\mathscr{L}_s^1(B).$$

(2) *For any $h\in L^1(G)$ and $T\in\mathscr{L}_s^1(B)$,*

$$(h*T)\hat{}=\hat{h}\hat{T} \text{ in } C_0(\hat{G}, \mathscr{L}_s(B)).$$

(3) *For any $T\in\mathscr{L}_s^1(B)$, there exists a net of operators T_α in $\mathscr{L}_s^1(B)$ such that*

$$\int_{\hat{G}}\hat{T}_\alpha(\gamma)fd\gamma\rightarrow Tf \text{ in } B \text{ for all } f\in B.$$

(4) *If $\hat{T}(\cdot)f\in L^1(\hat{G},B)$ for any $f\in B$, then $\int_{\hat{G}}\hat{T}(\gamma)f\,d\gamma=Tf$.*
(5) *If $\hat{T}(\gamma)f=0$ for all $\gamma\in\hat{G}$ and $f\in B$, then $T=0$.*

PROOF. (1) Let $\{e_\alpha\}$ be a positive kernel of $L^1(G)$. For any $T\in\mathscr{L}_s^1(B)$, define the operator T_α by

$$T_\alpha=e_\alpha*T \quad \text{for all } \alpha.$$

Then $T_\alpha\in\mathscr{L}_s^1(B)$ will converge to T in $\mathscr{L}_s^1(B)$. To claim this fact, we proceed from

$$(e_\alpha*T)f-Tf=\int_G e_\alpha(x)T(R_xf-f)dx \quad \text{for } f\in B,$$

then obtain that

$$\|(e_\alpha*T)f-Tf\|_B\leq\|T\|\int_G\|R_xf-f\|_Be_\alpha(x)dx.$$

Since $\|R_xf-f\|_B\rightarrow0$ as $x\rightarrow0$, thus for $\varepsilon>0$, there is a symmetric compact neighborhood V of 0 in G such that $\|R_xf-f\|_B<\varepsilon/2\|T\|$ whenever $x\in V$. For this V there is an $\alpha_0\in I$ such that

$$\int_{G\backslash V}e_\alpha(x)dx<\varepsilon/2\|T\| \quad \text{whenever } \alpha>\alpha_0.$$

It follows that for $\alpha>\alpha_0$,

$$\|T_\alpha f-Tf\|_B\leq\|T\|\Big(\int_{G\setminus V}+\int_V+\Big)<\varepsilon.$$

Hence $\{T_\alpha\}$ converges to T in $\mathscr{L}^1_s(B)$. Therefore $\mathscr{L}^1_s(B)$ is an essential L^1-module and hence

$$L^1(G)*\mathscr{L}^1_s(B)=\mathscr{L}^1_s(B).$$

(2) For any $h\in L^1(G)$ and $T\in\mathscr{L}^1_s(B)$, $h*T\in\mathscr{L}^1_s(B)$. Thus $(h*T)R_{(\cdot)}f$ is continuous on G for any $f\in B$, and so it is strongly measurable (see Dunford and Schwartz [4]) and

$$\int_G\|(h*T)R_yf\|_Bdy\leq\|h\|_1\ \|TR_{(\cdot)}f\|_{1,B}<+\infty.$$

It follows that

$$\begin{aligned}(h*T)\hat{}(\gamma)f&=\int_G(-x,\gamma)(h*T)R_{-x}f\ dx\\&=\int_G(-x,\gamma)\Big[\int_G(TR_zR_{-x}f)h(z)dz\Big]dx\\&=\int_G\Big[\int_G(-x,\gamma)TR_{z-x}fdx\Big]h(z)dz\\&=\int_G(-z,\gamma)\Big[\int_G(-w,\gamma)TR_{-w}f\ dw\Big]h(z)dz\\&=\int_G(-z,\gamma)h(z)dz\cdot\hat{T}(\gamma)f\\&=\hat{h}(\gamma)\hat{T}(\gamma)f\quad\text{for all } f\in B.\end{aligned}$$

Hence $(h*T)\hat{}=\hat{h}\cdot\hat{T}$ in $C_0(\hat{G},\mathscr{L}_s(B))$.

(3) Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a positive kernel of $L^1(G)$. From (1), for any $T\in\mathscr{L}^1_s(B)$, one has a net of operators $T_\alpha=e_\alpha*T$ in $\mathscr{L}^1_s(B)$ which converges to T. Since

$$\begin{aligned}\int_{\hat{G}}\hat{T}_\alpha(\gamma)fd\gamma&=\int_{\hat{G}}\hat{e}_\alpha(\gamma)\hat{T}(\gamma)f\,d\gamma\\&=\int_{\hat{G}}\hat{e}_\alpha(\gamma)\Big[\int_G(-x,\gamma)TR_{-x}f\,dx\Big]d\gamma\\&=\int_G\Big[\int_{\hat{G}}(-x,\gamma)\hat{e}_\alpha(\gamma)d\gamma\Big]TR_{-x}f\,dx\\&=\int_G e_\alpha(-x)TR_{-x}f\,dx\\&=\int_G e_\alpha(x)TR_xf\,dx\\&=(e_\alpha*T)f,\end{aligned}$$

it follows from (1) that $(e_\alpha*T)f\rightarrow Tf$. Consequently

$$\int_{\hat{G}} \hat{T}_\alpha(\gamma) f \, d\gamma \to Tf \text{ in } B \text{ for all } f \in B.$$

(4) Let $\hat{T}(\cdot)f \in L^1(\hat{G}, B)$ for any $f \in B$. Since the positive kernel $\{e_\alpha\}$ of $L^1(G)$ is an approximate identity for $L^1(G)$, thus for any $h \in L^1(G)$,

$$\begin{aligned} \|(e_\alpha * h - h)\hat{}\|_\infty &= \|\hat{e}_\alpha \hat{h} - \hat{h}\|_\infty \\ &\leq \|e_\alpha * h - h\|_1 \\ &\to 0, \end{aligned}$$

so $\hat{e}_\alpha(\gamma) \to 1$ for almost all $\gamma \in \hat{G}$. Now for $T \in \mathscr{L}^1_s(B)$ and $f \in B$, we have

$$\int_{\hat{G}} \|\hat{e}_\alpha(\gamma) \hat{T}(\gamma) f\|_B d\gamma \leq \int_{\hat{G}} \|\hat{T}(\gamma) f\|_B d\gamma < \infty.$$

Applying the dominated convergence theorem, we obtain

$$\lim_\alpha \int_{\hat{G}} \hat{e}_\alpha(\gamma) \hat{T}(\gamma) f \, d\gamma = \int_{\hat{G}} \hat{T}(\gamma) f \, d\gamma.$$

Consequently, by (3) we get

$$\int_{\hat{G}} \hat{T}(\gamma) f \, d\gamma = Tf.$$

(5) The result follows immediately from (4). □

4. Remark on multiplier property for $\mathscr{L}^1_s(B)$.

A *multiplier* for a topological algebra A is a continuous linear operator T on A which commutes with the ring multiplication, that is, for $a, b \in A$,

$$T(a \cdot b) = a \cdot Tb.$$

In 1952, Wendel proved that the space of multipliers for $L^1(G)$ is isometrically isomorphic to the measure algebra $M(G)$. For the general theory of multipliers one can refer to Larsen [6], while various characterization for multipliers one can consult Lai [8−12], Lai and Chang [14] and their cited references.

An equivalent definition of multiplier for $L^1(G)$ is that one calls a function φ on $\hat{G}$ a *multiplier* (function) for $L^1(G)$ if $\varphi \hat{f} \in L^1(G)\hat{}$ whenever $f \in L^1(G)$. It turns to discuss the multipliers for $\mathscr{L}^1_s(B)$. We call a function φ on $\hat{G}$ a *multiplier* for $\mathscr{L}^1_s(B)$ if $\varphi \hat{T} \in \mathscr{L}^1_s(B)\hat{} \subset C_0(\hat{G}, \mathscr{L}_s(B))$ whenever $T \in \mathscr{L}^1_s(B)$. From Theorem 3.2 (2), we can prove that for every $\mu \in M(G)$, one has

$$\hat{\mu} \hat{T} = (\mu * T)\hat{} \in \mathscr{L}^1_s(B)\hat{} \text{ for all } T \in \mathscr{L}^1_s(B)$$

where $\hat{\mu}(\gamma)=\int_G(-x,\gamma)d\mu(x)$ for $\gamma\in\hat{G}$ is the Fourier-Stieltjes transform of $\mu\in M(G)$. Hence each $\mu\in M(G)$ defines a linear map.

$$\phi_\mu:\mathscr{L}_s^1(B)\to\mathscr{L}_s^1(B) \text{ by } \phi_\mu(T)=\mu*T$$

and $$\phi_\mu(T)\hat{}=\hat{\mu}\hat{T} \text{ for all } T\in\mathscr{L}_s^1(B).$$

This operator ϕ_μ on $\mathscr{L}_s^1(B)$ is a multiplier (operator) for $\mathscr{L}_s^1(B)$. Actually the set $\mathscr{M}(\mathscr{L}_s^1(B))$ of all multipliers for $\mathscr{L}_s^1(B)$ is larger than $M(G)$. Thus we conclude that

PROPOSITION 4.1. *The measure algebra $M(G)$ is embedded as a subspace in the multiplier space $\mathscr{M}(\mathscr{L}_s^1(B))$ of $\mathscr{L}_s^1(B)$. That is, $M(G)\subset\mathscr{M}(\mathscr{L}_s^1(B))$. If G is a compact abelian group, then the identity operator $I\in\mathscr{L}_s^1(B)$ and*

$$[(\mu*I)\hat{}(\gamma)f](0)=\hat{\mu}(\gamma)\hat{f}(\gamma)$$

for all $\gamma\in G$ and $f\in B$.

REMARK. To characterize the multiplier space $\mathscr{M}(\mathscr{L}_s^1(B))$ as a function space is still open.

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