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A note on injective rings

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(Dedicated to Professor Hisao TOMINAGA on his 65th birthday)

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Introduction. Injective modules, specially self-injective rings, occupy a prominent position in ring theory and have drawn the attention of many authors since several years (cf. for example the bibliography of [1], [3], [4], [6]). Well-known examples of self-injective rings are self-injective regular rings, quasi-Frobeniusean rings and pseudo-Frobeniusean rings. The purpose of this note is to consider several nice conditions for rings to be self-injective. Test modules are given to ensure that rings are left self-injective regular with non-zero socle. Sufficient conditions for rings to be pseudo-Frobeniusean and quasi-Frobeniusean follow. Strongly regular rings with non-zero socle are characterized. The following are among the results proved for a ring A: (1) If A contains an injective maximal left ideal Y such that $r(Y)$ is a minimal right ideal, then A is left self-injective; (2) A is left self-injective if A is weakly right duo containing an injective maximal left ideal; (3) A is left pseudo-Frobeniusean if A is left p-injective left Kasch containing an injective maximal left ideal.

Throughout, A denotes an asscoiative ring with identity and A-modules are unital. J, Z will stand respectively for the Jacobson radical and the left singular ideal of A. A is called left non-singular iff $Z=0$. An ideal of A will always mean a two-sided ideal. Following E. H. FELLER, A is called a left duo ring if every left ideal of A is an ideal. A left (right) ideal of A is called reduced if it contains no non-zero nilpotient element. For any subset B of A, $r(B)$ (resp. $1(B)$) denotes the right (resp. left) annihilator of B.

§ 1. Self-injective rings.

PROPOSITION 1.1. *Let A have an injective maximal left ideal Y such that $r(Y)$ is a reduced ideal of A. Then A is left self-injective.*

PROOF. We have $A=Y\oplus U$, where $Y=Ae$, $e=e^2\in A$, $U=Au$, where $u=1-e$ isalso an idempotent and U is a minimal left ideal of A. Since $uA=r(Y)$ is an ideal of A, then $YAuA=YuA=0$ implies that $YA=1(u)=Y$, whence Y is a maximal right ideal of A. Now $uA=AuA$ and $eA\subseteq AeA=Ae$. If $uA\cap Aa\neq 0$, let $0\neq ua_0=a_1u$, where a_0, $a_1\in A$.

Then $ea_1e=eua_0=0$ implies that a_1e is a non-zero nilpotent element of $r(Y)$, which contradicts $r(Y)$ reduced. This proves that $uA\cap Ae=0$ and since $Y=Ae$ is a maximal right ideal of A, then $A=Y\oplus uA$. Therefore A/Y_A is projective which implies that ${}_AA/Y$ is injective by [10, Lemma 1]. Since ${}_AU\approx{}_AA/Y$, then $A=Y\oplus U$ is an injective left A-module which proves the proposition.

REMARK 1. If Y is a maximal left ideal of A which is a left annihilator such that $r(Y)$ is a minimal right ideal of A satisfying $r(Y)\cap Y=0$, then $r(Y)$ is reduced.

Proposition 1. 1. and Remark 1 may be used to prove the next theorem. The present independent proof is due to the referee. [8, Lemma 4(2)] and [13, Remark 9(2)] are strengthened.

THEOREM 1. 2. *Let A have an injective maximal left ideal Y such that $r(Y)$ is a minimal right ideal of A. Then A is left self-injective.*

PROOF. Since ${}_AY$ is injective and maximal, $A=Y\oplus V$ with a minimal left ideal V of A. We need only to show that ${}_AV$ is injective. Let $1=e+f$, where $e\in Y$, $f\in V$. Then e and $f(=1-e)$ are idempotents of A satisfying $Ae=Y$ and $Af=V$, and it follows that $r(Y)=fA$. First suppose that $YV=0$. Then Y is an ideal of A, since $YA=Y(Y+V)=YY+YV=YY\subseteq Y$. Therefore $r(Y)$ is also an ideal of A, and so $V=Af\subseteq r(Y)$ whence $A=Y+r(Y)$. Since $r(Y)$ is a minimal right ideal, we have that $Y\cap r(Y)=0$ (because otherwise $Y\cap r(Y)=r(Y)$ whence $f\in Y$, a contradiction !). Thus $A=Y\oplus r(Y)$. This implies that $(A/Y)_A$ is isomorphic to $r(Y)_A$ (and $V=AfA=r(Y)$) and hence $(A/Y)_A$ is simple and projective. Therefore, by [10, Lemma 1], ${}_A(A/Y)$ and hence ${}_AV$ is injective. Next suppose that $YV\neq 0$. Let $v\in V$ be such that $Yv\neq 0$. Then, since ${}_AV$ is simple, $Yv=V$, and indeed the mapping $y\rightarrow yv(y\in Y)$ gives an epimorphism ${}_AY\rightarrow{}_AV$. Since however ${}_AV$ is projective, the epimorphism splits and so ${}_AY$ has a direct summand isomorphic to ${}_AV$. Since ${}_AY$ is injective, its direct summand and hence ${}_AV$ must be injective too. This completes the proof.

COROLLARY 1. 3. *Let A be a left Kasch ring containing an injective maximal left ideal Y such that $r(Y)$ is a minimal right ideal. Then A is left pseudo-Frobeniusean. Consequently, a left duo left Kasch ring containing an injective maximal left ideal is left pseudo-Frobeniusean. In that case, the maximal right ideals of A coincide with the maximal left ideals of A.*

We are now in a position to give "test modules" for a ring to be left self-injective regular with non-zero socle.

THEOREM (MY). *The following conditions are equivalent*: (1) *A is left self-injective regular with non-zero socle*;

(2) *A has an injective maximal left ideal Y such that $r(Y)$ is a minimal right ideal and, for every $b \in J$, A/Yb is a flat left A-module*;

(3) *A has a non-singular injective maximal left ideal Y such that $r(Y)$ is a minimal right ideal of A.*

PROOF. Assume (1). Then A has a minimal left ideal V. Since V is cyclic, there exists an idempotent f such that $V = Af$. If we put $e = 1 - f$, then e is also an idempotent and we have a direct decomposition $A = Ae \oplus V$. Therefore Ae an injective maximal left ideal of A. Since A is regular, $J = Z = 0$ and $r(Ae) = fA$ is a minimal right ideal. Since the regularity of A implies that every left A-module is flat, we have (2), while since $Z = 0$, every left ideal of A is non-singular and so we have (3).

Now assume (2). Then A is left self-injective by Theorem 1.2. Let b be any element of J. Then ${}_A(A/Yb)$ is flat. It is known that for any left ideal I of A, ${}_A(A/I)$ is flat if, and only if, $a \in aI$ for every $a \in I$. Therefore, for every $y \in Y, yb = ybzb$ for some $z \in Y$. It follows that $yb(1 - zb) = 0$. But since b is, whence zb is, in J, $1 - zb$ is invertible in A and it follows that $yb = 0$. Thus we have $Yb = 0$, or equivalently, $b \in r(Y) \cap J$. On the other hand, since Y is an injective left ideal, Y is a direct summand of ${}_AA$ and hence $Y = Ae$ for an idempotent e of A. If we put $f = 1 - e$, then f is also an idempotent and we have $r(Y) = fA$. Thus we know that $b \in r(Y) \cap J = fJ$. But since $r(Y)_A$ is simple, it follows that $fJ = r(Y)J = 0$, whence $b = 0$. This shows that $J = 0$ and A is therefore regular by [3, Corollary 19.28]. Moreover, since $A = Y \oplus Af$ and Y is a maximal left ideal, Af is a minimal left ideal. This shows that (2) implies (1).

Assume finally (3). Then $A = Y \oplus Af$ for a minimal left ideal Af with idempotent f as seen above. But Af is and hence A is non-singular. So (1) follows from Theorem 1.2 and [3, Corollary 19.28].

[13, Remark 11] is extended to the non-commutative case by condition (3) in the above theorem.

COROLLARY 1.4. *A left duo ring containing a non-singular injective maximal left ideal is left and right self-injective strongly regular with non-zero socle.*

If A is prime, Y a maximal left ideal generated by an idempotent, then it is clear that $r(Y)$ is a minimal right ideal. If, further, A has non-zero socle, then $Z=0$. The following interesting result then follows immediately from Theorem (MY).

COROLLARY 1.5. *If A is a prime ring containing an injective maximal left ideal, then A is primitive left self-injective regular with non-zero socle. Consequently, A is simple Artinian iff A is a prime ring containing an injective maximal left and an injective maximal right ideals.*

Recall that a left A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to A. As in the definition of left self-injective rings, A is called left p-injective if ${}_AA$ is p-injective. A theorem of M. Ikeda-T. Nakayama guarantees that A is left p-injective iff every principal right ideal of A is a right annihilator [5, Theorem 1]. We now turn to von Neumann regular rings with non-zero socle. Note that a finitely generated p-injective left ideal of A is generated by an idempotent.

The proof of Theoren 1.2 yields an analogous p-injective result.

PROPOSITION 1.6. *Let A have a finitely generated p-injective maximal left ideal Y such that $r(Y)$ is a minimal right ideal. Then A is left p-injective.*

A characterization of regular rings with non-zero socle follows.

COROLLARY 1.7. *The following conditions are equivalent:*
(1) *A is regular with non-zero socle;*
(2) *Every principal left ideal of A is projective and A contains a finitely generated p-injective maximal left ideal Y such that $r(Y)$ is a minimal right ideal.*

At this point, let us give a sufficient condition for p-injective rings to be self-injective.

PROPOSITION 1.8. *Let A be a left p-injective ring containing an injective maximal left ideal Y. Then $r(Y)$ is a minimal right ideal and consequently, A is left self-injective.*

PROOF. Since ${}_AY$ is injective, $A=Y\oplus V$, where $V=Av$, $v=v^2\in A$, $Y=Ae$, $e=1-v$, and $r(Y)=vA$. For any $0\neq u\in vA$, since Y is a maximal left ideal, $Y=1(uA)$. In as much as $Y=1(vA)$, since A is left p-injective, by [5, Theorem 1], $vA=r(1(vA))=r(1(uA))=uA$ which proves that $r(Y)=vA$ is a minimal right ideal of A. The fact that A is left self-injective is a direct consequence of Theorem 1.2.

Applying [3, Corollary 24.22], we get

COROLLARY 1.9. *Let A have an injective maximal left ideal. Then*

A is quasi-Frobeniusean iff A is left p-injective satisfying either the maximum or the minimum condition on left annihilators.

Proposition 1.8. also yields

COROLLARY 1.10. *A left p-injective left Kasch ring containing an injective maximal left ideal is left pseudo-Frobeniusean.*

If A is left pseudo-Frobeniusean, then it is well-known that every left ideal of A is a left annihilator. Note that left pseudo-Frobeniusean rings need not be right pseudo-Frobeniusean [2].

However, the next result holds.

COROLLARY 1.11. *If A is left pseudo-Frobeniusean containing an injective maximal right ideal, then A is right pseudo-Frobeniusean.*

Left FP-injective rings are mentioned in [3, P. 108]. The next remark follows from Proposition 1.8.

REMARK 2. If A contains an injective maximal left ideal, then A is left self-injective iff A is left FP-injective.

§ 2. A generalization of duo rings.

Recall that A is WRD (weakly right duo) [9) if, for any $a \in A$, there exists a positive integer n such that $a^n A$ is an ideal of A. WRD rings generalize effectively right duo rings. For example, if

$$K = \mathbf{Z}/2\mathbf{Z}, R = \begin{bmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & K & 0 \end{bmatrix},$$

A the ring generated by R and identity, then A is WRD but not right duo.

PROPOSITION 2.1. *Let A be a WRD ring containing an injective maximal left ideal. Then A is left self-injective.*

PROOF. Let Y be an injective maximal left ideal. Then $A = Y \oplus U$, where $Y = Av$, $v = v^2 \in A$, $U = Au$, $u = 1 - v$. Suppose that Y is not an ideal of A: we have then $A = YA = AvA$ and since A is WRD, $vA = AvA$ which yields $A = vA$, whence $1 = vb$ for some $b \in A$. Therefore $u = uvb = 0$, which is impossible! This proves that Y must be an ideal of A. Now $Y = Av = AvA$ and since A is WRD, $Y = vA$ which implies that uA is a minimal right ideal of A. Therefore $A/Y_A \approx uA_A$ is projective which implies that ${}_AA/Y$ is injective by [10, Lemma 1]. Thus ${}_AU$ is injective and it follows that A is a left self-injective ring.

The proof of Proposition 2.1 shows the validity of the next proposition.

PROPOSITION 2.2. (1) *A is right self-injective if A is WRD containing an injective maximal right ideal;*
(2) *A is right self-injective if A contains an injective maximal right ideal and every complement right ideal of A is an ideal of A;*
(3) *A is left self-injective if A contains an injective maximal left ideal and every complement right ideal of A is an ideal of A.*

We know that A is a reduced ring if A contains a reduced maximal left ideal. Applying [14, Proppsition 7] and Lemma 1.2, we get

COROLLARY 2.3. *The following conditions are equivalent:*
(1) *A is left and right self-injective strongly regular with non-zero socle;*
(2) *A is a right duo ring containing a non-singular injective maximal left ideal;*
(3) *A is a WRD ring containing a non-singular injective maximal left ideal;*
(4) *A is a WRD ring containing a non-singular injective maximal right ideal;*
(5) *A contains a non-singular injective maximal right ideal and every complement right ideal of A is an ideal;*
(6) *A contains a non-singular injective maximal left ideal and every complement right ideal of A is an ideal;*
(7) *A contains a reduced injective maximal left ideal.*

If A is left p-injective, then $Z=J$ by [12, Proposition 3]. [14, Propositions 2 and 7] together with the proof of Proposition 2.1 yield the following p-injective analogue of Corollary 2.3.

PROPOSITION 2.4. *The following condiions are equivalent:*
(1) *A is strongly regular with non-zero socle;*
(2) *A is WRD containing a non-singular finitely generated p-injective maximal left ideal;*
(3) *A is WRD containing a non-singular finitely generated p-injective maximal right ideal;*
(4) *A contains a non-singular finitely generated p-injective maximal right ideal and every complement right ideal of A is an ideal;*
(5) *A contains a non-singular finitely generated p-injective maximal left ideal and every complement right ideal of A is an ideal;*
(6) *A contains a reduced finitely generated p-injective maximal left ideal.*

Following [7], a left A-module M is called semi-simple if the intersection of all maximal submodules of M is zero. Thus A is semi-simple iff $J=0$. A is called a left V-ring if every simple left A-module is injective. Then A is a left V-ring iff every left A module is semi-simple [7, Theorem 2.1]. Recall also that A is von Neumann regular iff every left A-module is p-injective.

Connecting semi-simplicity with p-injectivity, we have

REMARK 3. A is a regular left V-ring iff the cyclic semi-simple left A-modules coincide with the cyclic p-injective left A-modules.

Rings whose simple left modules are either p-injective or flat need not be regular (they need not be even simi-prime).

If $K=\boldsymbol{Z}/2\boldsymbol{Z}$, *set* $A=\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$. Then the simple left A-modules are either p-injective or flat but A is not semi-prime.

We finally consider a sufficient condition for A/J to be strongly regular.

PROPOSITION 2.5. *Let A be a ring whose cyclic semi-simple left modules are either p-injective or flat. If every maximal left ideal of A is an ideal, then A/J is strongly regular and every simple right A-module is either injective or flat.*

PROOF. Set $B=A/J$. Since ${}_AB$ is semi-simple, ${}_AB$ is either p-injective or flat. First suppose that ${}_AB$ is p-injective. Then B is a left p-injective ring. Since B is semi-simple and every maximal left ideal of B is an ideal, then B is a reduced ring [11, P. 27]. Now B, being a reduced left p-injective ring, is strongly regular.

Therefore every maximal right ideal of A is an ideal and every simple right A-modules is injective or flat by [10, Lemma 1]. Now suppose that ${}_AB$ is flat. For any $u\in J$, $u=uv$ for some $v\in J$. There exists $w\in A$ such that $(1-v)w=1$. Then $0=u(1-v)=u(1-v)w=u$ implies that $J=0$. Therefore A is reduced [11, P. 27] and every simple left A-module is flat or p-injective. Let Y be a maximal left ideal of A. If ${}_AA/Y$ is flat, for any $y\in Y$, $y=yz$ for some $z\in Y$. Then $1-z\in r(y)=1(y)$ (because A is reduced) which implies that A/Y_A is flat, whence ${}_AA/Y$ is p-injective [10, Lemma 1]. This proves that every simple left A-module is p-injective. Since every maximal left ideal of A is an ideal, then A is strongly regular. This proves the proposition.

COROLLARY 2.6. *Suppose that every cyclic semi-simple left A-module is either p-injective or flat. The following are then equivalent: (a) A is left Artinian and A/J is a finite direct sum of division rings; (b) A is left Noetherian, J is left T-nilpotent and every maximal left ideal of A is an ideal.*

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