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Blow-up for some equations with semilinear dynamical boundary conditions of parabolic and hyperbolic type

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1. Introdution

In recent years several results have been published concerning blow-up of solutions to semilinear parabolic and hyperbolic equations. We mention here the works [1]-[7] and their references.

A special attention was given not only to the description of the set of blow-up points but also to the description of the behaviour of the solutions near the blow-up points as time tends to the blow-up time [3]-[6].

All these papers dealt with semilinear parabolic and hyperbolic equations with classical boundary conditions, i. e, Dirichlet's, Neumann's and Robin's conditions.

In the present work blow-up results and characterization of the blow-up set (Bus) in the case of particular geometries are established for semilinear parabolic and hyperbolic equations of the following types:

$$\text{(P1)}\quad \begin{cases} \triangle u=0 & \text{in } D\times(0,\infty),\\ \partial u/\partial t+k\partial u/\partial\eta=h(x,t,u) & \text{on } S\times(0,\infty),\\ u(x,0)=u_0(x) & \text{on } S.\end{cases}$$

$$\text{(P2)}\quad \begin{cases} \triangle u=0 & \text{in } D\times(0,\infty),\\ \partial^2 u/\partial t^2+k\partial u/\partial\eta=f(u) & \text{on } S\times(0,\infty),\\ u(x,0)=u_0(x) & \text{on } S,\\ \partial u/\partial t(x,0)=u_1(x) & \text{on } S.\end{cases}$$

$$\text{(P3)}\quad \begin{cases} \partial u/\partial t-\triangle u=u^{1+\alpha} & \text{in } D\times(0,\infty),\\ \partial u/\partial t+k\partial u/\partial\eta=u^{1+\alpha} & \text{on } S\times(0,\infty),\\ u(x,0)=u_0(x) & \text{in } \bar{D}.\end{cases}$$

Here D is a bounded domain in R^N $(N\geq 1)$ with smooth boundary S and outer unit normal vector field η, $\triangle$ is the Laplace operator with respect to the space variables and $\partial/\partial\eta$ the outward normal derivative to S. The constants α and k are assumed to be positive.

The functions h and f are assumed to satisfy:

(H1) $h(x, t, u) \in C(D \times R^+ \times R, R)$

(H2) $h(x, t, u) \geq p(t)H(u)$ for $(x, t, u) \in D \times R^+ \times R$

where $H(u)$ is a continuous, convex and positive function on R^+ and satifying:

(H3) $\int_0^\infty du/H(u) < +\infty$,

and $p(t)$ is a continuous and positive function on R satisfying:

(P) $\int_0^t p(s)ds \geq C(t)$ for $t > 0$ and a positive function C such that $C(t)$ goes to infinity as t goes to infinity.

(F) f is a continuous, positive and convex function in R.

Problems of type (P 2) can be used as models to describe the motion of a fluid in a container or to describe the displacement of a fluid in a medium without gravity (artificial satellite).

Problems of type (P 3) occurs in describing the heat tranfer in a solid in contact with a fluid [9].

It is worth noting that Gröger [8] considered problems with dynamical boundary conditions from semiconductor device theory.

2. Known facts and statment of the problems

Throughout the paper, we will consider problems (P 1) and (P 2) in general bounded domains D and particulary in spherical domains $D := B_R(x) = (x \in R^N : |x| < R)$ with boundary $S := \partial D$.

The following results concerning existence of local (in time) solutions to problems (P 1), (P 2) and (P 3) are stated in [10].

A few words about notations. For $p \in [1, +\infty]$, we donote by $L_p(D)$ the space of measurable scalar functions on D for which

$$|u|_p = \left[\int_D |u(x)|^p dx\right]^{1/p} < +\infty, \text{ for } 1 \leq p < \infty.$$
$$|u|_\infty = \operatorname*{ess\,sup}_{x \in D} |u(x)| < +\infty, \text{ for } p = +\infty.$$

For $s \in R$ and $1 < p < \infty$, we denote by $H_p^s(D)$ and $B_{pp}^s(S)$ the local Bessel potential and Besov spaces with norms$\|\cdot\|_{s,p,D}$ and $\|\cdot\|_{s,p,S}$, respectively. They are defined by restrictions from the following Bessel potential and Besov spaces:

$$H_p^s(R^N)=\left[\left\{u\in S'\,|\,\mathcal{T}^s u\in L_p(R^N, C^n)\right\}, \|\cdot\|_{s,p}\right],\ \|u\|_{s,p}=|\mathcal{T}^s u|_p,$$

$$B_{pq}^s(R)^N=\begin{cases}(H_p^k, H_p^{k+1})_{s-k,q} & k<s<k+1,\ k\in Z,\\ (H_p^{k-1}, H_p^{k+1})_{s/2,q} & k=s\in Z\,;\end{cases}$$

Here $\mathcal{T}^s=\mathfrak{F}^{-1}\Lambda^s(1,\zeta)\,\mathfrak{F}\in\mathfrak{L}(S')$, $\Lambda^s(\eta,\zeta)=(|\eta|^2+|\zeta|^2)^{s/2}$ $\zeta\in R^N$, $s\in R$, $\eta\in C$ and S' denotes the space of C^n-valued tempered distributions on R^N and $\mathfrak{F}$ denotes the Fourier transform in S'. For further informations we refer to Triebel [13].

For simplicity, we put $H^s:=H_2^s:=B_{pp}^s$.

2.1 Facts

2.1.1 For each $u_0\in B_{pp}^{2-1/p}(S)$, problem (P 1) has a maximal solution

$$u\in C([0, T_{\max}), H_p^2(D))\cap C^1([0, T_{\max}), H_p^1(D))$$

If there exists a function $K\in C(R, R)$ such that

$$\|u(t)\|_{2,p,S}+\|u(t)\|_{2,p,D}\le K(t) \text{ for any } t\in[0, T_{\max})$$

then $T_{\max}=+\infty$,

2.1.2 For each $(u_0, u_1)\in H^{s+1/2}(S)\times H^s(S)$ $(s>1)$ there exists a maximal open interval $J=(T^-, T^+)$ with $0\in J$ and a unique weak solution to (P 2)

$$u\in C([0, T^+), H^{s+1}(S))$$

with trace

$$\gamma_0 u=u_{|S}\in C^1([0, T^+), H^{s+1}(S)).$$

If there exists a function $k\in C(R, R)$ such that

$\|\gamma_o u\|_{s+1/2,2,S}+\|d/dt((\gamma_o u(t)\|_{s,2,S}\le k(t)$, for any $t\in[0, T^+)$, then $T^+=+\infty$ (analogous result for T^-).

2.1.3 Let $p>N$. For each $u_0\in H^{2,p}(D)$, problem (P 3) has a unique maximal solution

$$u\in C([0, T_{\max}), H^{2,p}(D))\cap C^1((0, T_{\max}), L^p(D))$$

for $T_{\max}>0$.

Moreover, if $u(t)$, $t\in[0, T_{\max})$, is bounded in $H_p^{2-\varepsilon}(D)$ $(0<\varepsilon<<1)$, then the solution is global.

2.2 Definition: A point $x\in\overline{D}$ is a blow-up point if there exists

$((x_n, t_n))$ such that $t_n\to T_{\max}$, $x_n\to x$

and $\qquad u(x_n, t_n) \to +\infty$ as $n \to +\infty$.

In the sequel, we are going to answer the following questions:

1. When do the solutions blow up in finite time?
2. Where do the solutions blow up?

THEOREM 1: *Assume that* $u_0 \in B^{2-1/p}(S)$ *and let* u *be the solution to* (P 1). *If* h *satisfies* (H1) *and* (H2) *then* u *does not exist for all time.*

PROOF: Let $u(x, t)$ be the solution to problem (P 1), and consider the function:

$$U(t)=|S|^{-1}\int_S u(\sigma, t)d\sigma, \ t>0$$

where $|S|=\int_S 1, \ d\sigma.$

Integrating $(\mathrm{P}\,1)_2$ on S, we get:

$$U'(t)=-k|S|^{-1}\int_S \partial u/\partial\eta \ d\sigma+|S|^{-1}\int_S h(\sigma, t, u) \ d\sigma \tag{3.1}$$

Green's formula yields:

$$0=\int_D \triangle u dx=\int_S \partial u/\partial\eta \ d\sigma \tag{3.2}$$

From (H2) and Jensen's inequality we obtain:

$$\begin{aligned} |S|^{-1}\int_S h(\sigma, t, u) \ d\sigma &\geq |S|^{-1}\int_S p(t)H(u) \ d\sigma \\ &\geq |S|^{-1}p(t) \int_S H(u) \ d\sigma \geq p(t) \ H\left(|S|^{-1}\int_S u(\sigma, t) \ d\sigma\right) \\ &\geq p(t)H(U) \end{aligned} \tag{3.3}$$

then from (3.1), (3.2) and (3.3) it follows that:

$$U'(t)\geq p(t)H(U), \ t>0.$$

Since $H(U)>0$, this implies that:

$$\int_{U(0)}^{U(t)}\left[H(\sigma)\right]^{-1}d\sigma \geq \int_0^t p(\sigma)d\sigma \geq C(t),$$

hence

$$C(t)\leq \int_{U(0)}^{\infty}\left[H(\sigma)\right]^{-1}d\sigma<\infty. \tag{3.4}$$

Then if global existence of a solution to (P 1) is assumed, (3.4) leads to a contradiction because $C(t)$ goes to infinity as t goes to infinity.

REMARK 1: The result remains true even if H is not convex. In this case H has to be replaced by $H^{**}=\sup(H_i, i\in I)$, H_i convex, $H_i\leq H$ and , in this case, (*) reads $\int^{\infty} du/H^{**}(u)<\infty$.

The special case $D=B_R(0)$:

Now, assume that $D=B_R(0)\subset R^N$. By the mean value theorem we have:

$$u(0, t)=(1/N\omega_N R^{N-1})\int_S u(\sigma)d\sigma$$

where $\omega_N(=2\pi^{N/2}/N\Gamma(N/2))$ is the volume of the unit ball in R^N.

Hence $u(0, t)\to\infty$ as $t\to T_{\max}$.

On the other hand, as u is harmonic, u can not have a maximum in an interior point of D without being constant in a neighborhood of this point, thus:

$$u(x, t)\to+\infty \text{ as } t\to T_{\max} \text{ in all } D=B_R(0).$$

REMARK 2: If D included in R^2 ($\simeq C$) is a simply connected (no holes) domain whose boundary consists of more than one point then

$$u(x, t)\to+\infty \text{ as } t\to T_{\max} \text{ in } D \text{ all } D=B_R(0).$$

because, in this case, by Riemann's theorem D may be conformally mapped onto the interior of the unit circle $|W|<1$ of the W-plane [11, p. 256], the Laplace equation is preserved in a conformal mapping and the solution of the Dirichlet problem for the circle is obtained.

REMARK 3: The results remain true when:

$\triangle u=0$ is replaced by $\mathrm{div}(a(u)\mathrm{grad}u)=0(0<a_0\leq a(x)\leq a_1<+\infty)$ which can be rewritten by the Kirchhoff transform $v=\int_{u_0}^{u}a(s)ds$ into $\triangle v=0$.

Now for further reference, let's:

$$F(U)=\int_0^U f(s)ds \text{ and } d=(U'(0))^2-2\,F(U(0)).$$

THEOREM 2: *Assume that* $(u_0, u_1)\in H^{s+1/2}(S)\times H^s(S)$ $(s>1)$ $u_1\geq 0$, $u_1\not\equiv 0$ *and that* $\int_{U(0)}^{\infty}\left[d+2F(s)\right]^{-1/2}ds<+\infty$ *then the (local) week solution* $u\in C([0, T_{\max}), H^{s+1}(D))$ *to problem* (P 2) *blows up in a finite time.*

Moreover if $D=B_R(0)$ then $Bus=D$.

PROOF: Consider the function:

$$U(t)=|S|^{-1}\int_S u(\sigma, t)d\sigma.$$

By computation:

$$U''(t)=|S|^{-1}\int_S u_{tt}=-k|S|^{-1}\int_S \partial u/\partial\eta+|S|^{-1}\int_S f(u)$$

$$=|S|^{-1}\int_S f(u)\geq \text{ (by Jensen's inequality)} \geq f(U)\geq 0.$$

Now, as $U'(0)$ is positive and since U' is nondecreasing, $U'>0$ for any $t\in[0, T_{\max})$.
So, multiplying (3. 5) by U' yields:

$$(U'(t))^2\geq(U'(0))^2+2\int_0^t f(U(s))U'(s)ds.$$

Hence:

$$(U'(t))^2\geq(U'(0))^2+2F(U(t))-2F(U(0)) \text{ for } t\in[0, T_{\max})$$

so:

$$U'(t)\geq(d+2F(U(t))^{1/2} \text{ for } t\in[0, T_{\max}).$$

An integration on $(0, t)$ $(t<T_{\max})$ yields:

$$t\leq\int_{U(0)}^{U(t)}[d+2F(U(s))]^{-1/2}ds\leq T_{\max}<\infty,$$

it follows then that a global solution can not exist for all $t>0$.
Now, assume that $D=B_R(0)$ then, as for the parabolic case, we have:

$$u(0, t)=(1/N\omega_N R^{N-1})\int_S u(\sigma)d\sigma \to +\infty \text{ as } t \to T_{\max},$$

and proceeding as above we have:

$$u(x, t) \to +\infty \text{ as } t \to T_{\max} \text{ in } D \text{ all } D=B_R(0).$$

THEOREM 3: *Let $u_0\in H_p^2(D)$, $u_0\geq 0$, $u_0\not\equiv 0$ in D. Then the unique maximal solution:*

$$u\in C([0, T_{\max}), H_p^2(D))\cap C^1((0, T_{\max}), L_p(D)), \ T_{\max}>0,$$

to problem (P 3) blows up in a finite time.

For the proof of theorem 3 we need the following lemma.

LEMMA 3.1: *Assume that* $u_0 \in H_p^2(D)$, $u_0 \geq 0$, $u_0 \not\equiv 0$ *in* D, *then* $u \geq 0$ *in* $\bar{D}$ *and* $u > 0$ *in* D.

PROOF: We multiply the first equation of (P 3) by $u^- = \max(-u, 0) = -u^+$. Integrating over D and using Green's formula, we find:

$$\int_D u^- \partial u/\partial t = \int_D u^-(\partial u^+/\partial t - \partial u^-/\partial t) = -\int_D u^- \partial u^-/\partial t$$
$$= -d/dt\left[(1/2)\int_D |u^-|^2\right]$$
$$-\int_D \triangle u \cdot u^- = \int_D (\nabla u^+ - \nabla u^-)\cdot \nabla u^- - \int_S \partial u/\partial\eta \cdot u^- = -\int_D |\nabla u^-|^2$$

but:

$$-\int_S \partial u/\partial\eta \cdot u^- = -k^{-1}\int_S u^{1+\alpha}u^- + k^{-1}\int_S \partial u/\partial t \cdot u^-$$
$$= -k^{-1}\int_S u^{1+\alpha}u^- - (2k)^{-1} d/dt\left[\int_S |u^-|^2\right].$$

From the previous identities we infer:

$$d/dt\left[\int_D |u^-|^2 + \int_S |u^-|^2\right] \leq 2k^{-1}\int_S |u^{1+\alpha}u^-| + 2\int_D |u^{1+\alpha}u^-|.$$

For p large enough $|u(t)|_\infty$ is bounded for any interval $[0, T]$ with $T < T_{\max}$.

Hence, there exists a constant $C = C(T)$ such that:

$$\int_D |u^{1+\alpha}u^-| \leq C\int_D |u^-|^2 \text{ and } \int_S |u^{1+\alpha}u^-| \leq C\int_S |u^-|^2,$$

collecting all these estimates together, we get for $t \in (0, T)$:

$$d/dt\left[\int_D |u^-|^2 + \int_S |u^-|^2\right] \leq C(T)\left[\int_D |u^-|^2 + \int_S |u^-|^2\right].$$

Since $u^- \in C([0, T]; L_2)$ and $u^-(0, x) = 0$, Gronwall's lemma now implies:

$$\int_D |u^-|^2 + \int_S |u^-|^2 = 0 \text{ for all } t \in [0, T].$$

Since $T < T_{\max}$ is arbitrary, we see that $u^- = 0$ for all $t \in [0, T_{\max})$. Hence $u \geq 0$.

PROOF OF THEOREM 3:

Let $G:=\int_D u+\int_S u,$

then:

$$G'=d\int_D \triangle u+\int_D u^{1+\alpha}-d\int_S \partial u/\partial\eta+\int_S u^{1+\alpha}$$
$$=\int_D u^{1+\alpha}+\int_S u^{1+\alpha}.$$

as $u\geq 0$ in $\bar{D}$,

$$\left[\int_D u\right]^{1+\alpha}\leq|D|\int_D u^{1+\alpha},\quad \left[\int_S u\right]^{1+\alpha}\leq|S|\int_S u^{1+\alpha}$$

and $$(a+b)^{1+\alpha}\leq 2^{\alpha}(a^{1+\alpha}+b^{1+\alpha}),$$

we have:

$$G'\geq\beta\ 2^{-\alpha}\left[\int_D u+\int_S u\right]^{1+\alpha}=\beta 2^{-\alpha}G^{1+\alpha},$$

with $\beta=\min\ (|D|^{-\alpha},|S|^{-\alpha})$. (For simplicity let us $\beta\ 2^{-\alpha}=1$).
Or $G'\geq G^{1+\alpha}$.

A simple computation yields:

$$G(t)^{\alpha}\geq G(0)^{\alpha}/(1-\alpha G(0)^{\alpha}t)$$

which blows up at $t\rightarrow T^{\star}=\alpha^{-1}G(0)^{\alpha}$. ■

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