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Obstruction to circle group actions preserving symplectic structure

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

Kaoru ONO
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1. Introduction.

In a previous paper [O], we discussed non-existence of compact connected Lie group actions on some syplectic manifolds from a point of view of a symplectic analogue of Kähler geometry. For example, closed sympletic manifolds with " negative first Chern class " admit no compact connected Lie group actions preserving symplectic structure. In the proof we used moment maps and lifting of group actions on certain complex line bundles.

In this note, we give another sufficient condition for non existence compact Lie group actions by using generalised moment map [MD]. Our result is the following

THEOREM. *Let (M, ω) be a closed symplectic manifold.*
1) *If the second homotopy group $\pi_2(M)$ vanishes, there is no circle group action on M preserving ω with non empty fixed point set*

$$M^{S^1}=\{p\in M|t\cdot p=p \text{ for any } t\in S^1\}.$$

2) *Moreover if any abelian subgroup of the fundamental group $\pi_1(M)$ is cyclic, there is no circle group action on M preserving ω. Therefore there is no compact connected Lie group action preserving ω.*

These conditions are satisfied for closed negatively curved manifolds and it is known that they admit no circle group actions. The author is very grateful to Professor Akio Hattori for advice and encouragement. He also would like to thank Professor Dusa McDuff for sending him a preprint with helpful advice.

2. Preliminaries.

In this section we review the defintion of generalised moment map due to McDuff.

Let (M, ω) be a symplectic manifold admitting an S^1-action preserving ω. A function $\mu: M \to \boldsymbol{R}$ is called a moment map for the S^1-action if $d\mu(v)+i(v)\omega=0$ holds, where v is the vector field determined by the S^1-action. We give the definition of generalised moment map.

Let (M, ω) be a symplectic manifold satisfying the following condition

$$[\omega] \in \operatorname{Im}\{H^2(M; \boldsymbol{Z}) \to H^2(M; \boldsymbol{R})\}.$$

If a circle group S^1 acts on M symplectically, there is a map $\mu: M \to S^1$ such that $i(v)\omega+\mu^*\frac{d\theta}{2\pi}=0$ holds. We call μ a generalised moment map. (In [MD], μ is called a generalised moment map if it cannot be lifted to $M \to \boldsymbol{R}$.)

First of all, we recall the following

LEMMA 1 [MD]. *Let ω be an S^1-invariant symplectic form on M. Then there is an S^1-invariant symplectic from which admits a generalised moment map.*

LEMMA 2. *Assume that the generalised moment map $\mu: M \to S^1$ cannot be lifed to a continuous map $M \to \boldsymbol{R}$. Then given any point p is M, there exists a homologically non-trivial loop $\gamma: S^1 \to M$ passing through p such that $d\theta(\mu_* \dot{\gamma})>0$ everywhere except at fixed points.*

PROOF OF LEMMA2: Choose an S^1-invariant Riemannian metric and an S^1-invariant almost complex structure compatible with ω. Since a generalised moment map is locally a function, we can define the Hessian at critical points, their indices, i. e. the number of negative eigenvalues, and the gradient flow of μ. If there is a critical point at which the Hessian is positive or negative definite, it is easy to see that μ can be lifted to a continuous map $M \to \boldsymbol{R}$. Consider the quotient space X of M by the equivalence relation $\sim$ as follows.

$x \sim y$ if and only if there is $t \in S^1$ such that
x and y are contained in the same connected component of $\mu^{-1}(t)$.

Since the indices of the Hessian at critical points are even, X has no branch point. Since the Hessian at critical points are indefinite, X has no boundary point and is homeomorphic to a circle. Thus we can deform the trajectory of the gradient flow of μ passing through p to a loop as in

Lemma 2.

3. Proof of theorem.

First of all, we remark that there is no S^1-action admitting a moment map under the condition that the second homotopy group $\pi_2(M)=0$. In fact if there is an S^1-action on M admitting a moment map, the Hurewicz homomorphism $\pi_2(M) \to H_2(M ; \boldsymbol{Z})$ is non trivial. More precisely for generic point $p \in M$, we set

$$\overline{M(p)}=\overline{\bigcup_{t \in S^1} t(\gamma_p)}$$

where γ_p is the trajectory of the gradient flow of the moment map passing through p. It is easy to see that $\overline{M(p)}$ is homeomorphic to a 2-sphere S^2 and $\int_{\overline{M(p)}} \omega>0$, cf.[O]. This contradicts the condition $\pi_2(M)=0$.

Now we assume that S^1 acts symplectically on M and $\pi_2(M)=0$. By Lemma 1, we may assume that there is a generalised moment map $\mu: M \to S^1$. Above observation implies that this generalised moment map cannot be lifted to a continuous map $M \to \boldsymbol{R}$. If there is a fixed point p, there is a loop γ passing through p as in Lemma 2. Since p is a fixed point, $C=\bigcup_{t \in S^1} t \cdot \gamma$ is the image of a continuous map from 2-sphere and, by the condition $\pi_2(M)=0$, the homology class $[C]$ is zero in $H_2(M ; \boldsymbol{Z})$. On the other hand we can show that $\int_C \omega>0$, which contradicts $[C]=0$ in $H_2(M ; \boldsymbol{Z})$ and completes the proof of 1) in Theorem. The proof of the inequelity $\int_C \omega>0$ goes roughly as follows.

Let $\dot{\gamma}$ be the velocity vector field of γ. It is enough to show that $\omega(\dot{\gamma}, v)>0$. But $\omega(\dot{\gamma}, v)=\frac{1}{2\pi} d\theta(\mu_* \dot{\gamma})>0$.

We proceed to the proof of 2). As we have already proved that the fixed point set is empty, $C=\bigcup_{t \in S^1} t \cdot \gamma$, where γ is the loop in Lemma 2, is the image of a continuous map from torus T^2. Since any abelian subgroup of the fundamental group $\pi_1(M)$ is cyclic, there exists a homotopically non trivial loop on T^2 which in mapped to a null homotopic loop. Thus the homology class $[C]$ is represented by a continuous map from 2-sphere. From the assumption that $\pi_2(M)=0$, the homology class $[C]$ is zero in $H_2(M, \boldsymbol{Z})$. On the other hand we have the inequality $\int_C \omega > 0$. This is a contradition and implies that there could not be any circle action preserving ω.

References

[MD] D. McDuff, *The moment map for circle actions on symplectic manifolds*, *J. Geometry and Physics* 5 (1988), 149-160.

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Mathematical Institute, Tohoku Universty
Sendai 980, Japan

Current Address
Department of Mathematics, Faculty of Science,
Ochanomizu University
Otsuka Tokyo 112, Japan
and
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
5300 Bonn 3
Gremany

Added in proof.

R. Lashof combined the idea in this paper and the Pontrjagin product and obtained a related theorem.

R. Lashof: Circle actions on symplectic manifolds and the Pontrjagin product, MSRI preprint 06725-90.