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On levels of the distance function from the boundary of convex domain

Dedicated to Professor Haruo Suzuki on his 60th birhday

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1. Introduction

In this note we shall be concerned with the behaviour of levels of the distance function from the convex boundary of the 2-dimensional disc with real analytic riemannian metric of nonnegative curvature. First we explain the motivation. Let (S^2, g) be a riemannian metric of non-negative curvature on the 2-sphere. A. D. Alexandrov conjectured the following inequality with respect to the area and the diameter:

(1) $\quad \mathrm{Area}(S^2, g)/(\mathrm{Diam}(S^2, g))^2 \leq \pi/2,$

where the equality holds iff (S^2, g) is the double of the flat euclidean disc. For the partial results we refer to [Sa2], [Shi].

Now we consider the isoperimetric quantity $h:=\inf\{\text{length } \partial\Omega/\mathrm{Area}(S^2, g)$; Ω is a domain of S^2 with smooth boundary such that $\mathrm{Area}\Omega = \mathrm{Area}(S^2, g)/2\}$. Then in our case the infimum is realized by domain D whose boundary c is a connected regular simple closed curve of constant mean (i. e., geodesic) curvature (see e. g., [Ga]). Then $S^2 \backslash c$ is divided into the two discs $D_1 = D$, $D_2 = S^2 \backslash \bar{D}$ with the same area and the boundary c. Setting $d_i^* := \max\{d(p, c)\ ;\ p \in D_i\}$ (i=1, 2), we easily see that $d_1^* + d_2^* \leq \mathrm{Diam}(S^2, g)$. Then if we may estimate Area $D_i/(d_i^*)^2$, we may have estimate for (1). Since $d_1^* + d_2^*$ may smaller than $\mathrm{Diam}(S^2, g)$ this approach doesn't work very well for the original problem. Nevertheless it seems to be interesting to estimate Area $D_i/(d_i^*)^2$. For that purpose we consider the length l_t of level $d_c^{-1}(t)$, $0 \leq t \leq d_i^*$, where d_c denotes the distance function from the boundary c. In the present article we restrict ourself to the case when $D = D_i$ is 2-disc with real analytic riemannian metric of non-

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negative curvature and convex boundary.

Now in his nice paper F. Fiala ([F]) studied the behaviour of the length l_t of levels in general case (see also [Be], [Sal]). Under our assumption we have the following:

THEOREM. *Let D be the 2-disc with real analytic metric of non-negative curvature and convex boundary, namely geodesic curvature of the boudary curve c is positive. We denote by l_t the length of the level $d_c^{-1}(t)$, where d_c is the distance function from the boundary c.*

(1) *Set $d^* : =\max\{d_c(q) ;\ q\in D\}$. Then there exists the unique furthest point $p\in D$ from c which realizes d^*. The levels $d_c^{-1}(t)$, $0\leq t<d^*_;$ are connected simple closed curves and $\Omega_t : =d_c^{-1}(t, d^*]$ are discs.*

(2) *$t\to l_t$ is continuous and real analytic except for at most finitely many singular values $0<t_1<\ldots<t_k=d^*$ ([F]). Under our assumption we have furthermore*

$$d/dt\ l_t<0, \text{ and } \lim_{t\to t_i-0} d/dt\ l_t\geq \lim_{t\to t_i+0} d/dt\ l_t$$

(3) *For regular values t we have $d^2/dt^2\ l_t\leq 0$.*

As a corollary we get an estimate for Area $D/(d^*)^2$. Note that in genral we have no finite upper bound for Area $D/(d^*)^2$.

COROLLARY. *Under the assumption of the theorem we have the following.*

(1) *If there exist infinitely many minimal geodesics from c to the furthest point p, then we have* Area $D/(d^*)^2\leq\pi$.

(2) *If there exist only finitely many minimal geodesics from c to p, let $\alpha_1, \ldots, \alpha_k$ be the angles between tangent vectors at p to above minimal geodesics which are adjoining each other $(\alpha_1+\ldots+\alpha_k=2\pi)$. Then we have*

$$\text{Area } D/(d^*)^2\leq\pi+\textstyle\sum_i(\tan\alpha_i/2-\alpha_i/2).$$

2. Proof of the theorem and corollary.

Let the boundary curve $c(s)\,(0\leq s\leq l)$ be parametrized by arc length and $n(s)$ be the unit inward normal vector to c at $c(s)$. Then the geodesic curvature κ of c at $c(s)$ is given by $\langle n(s), \nabla_{\partial/\partial s}\dot{c}(s)\rangle$ where $\langle\ ,\ \rangle$ and ∇ denote the inner product and Levi-Civita covariant derivative respectively. Using normal exponential map exp we have a real analytic map

$$(2) \qquad x(t, s) : =\exp_{c(s)} t\, n(s)$$

Since $t \to x(t, s)$ is a geodesic γ_s parametrized by arc length and $\partial x/\partial s(0, s) = \dot{c}(s)$ is a unit vector perpendicular to $\partial x/\partial t(0, s) = n(s)$, we have $\langle \partial x/\partial t, \partial x/\partial s \rangle = 0$ everywhere. Note that the vector field $Y_s : t \to \partial x/\partial s(t, s)$ along γ_s is a c-Jacobi field.

LEMMA 1. *Up to the first focal value $t(s)$ of c along the c-Jacobi field Y_s, we have*

(3) $\quad \langle \nabla_{\partial/\partial t}\, \partial x/\partial s, \partial x/\partial s \rangle\ (t, s) < 0 \quad (0 < t < (s))$

PROOF. First we have

$$d/dt\{\langle \nabla_{\partial/\partial t}\, \partial x/\partial s, \partial x/\partial s\rangle / |\partial x/\partial s|\} =$$
$$\{\langle \nabla_{\partial/\partial t} \nabla_{\partial/\partial s}\, \partial x/\partial t, \partial x/\partial s\rangle + |\nabla_{\partial/\partial t}\, \partial x/\partial s|^2\} / |\partial x/\partial s| -$$
$$\langle \nabla_{\partial/\partial t}\, \partial x/\partial s, \partial/\partial s\rangle^2 / |\partial x/\partial s|^3 = \langle R(\partial x/\partial t, \partial x/\partial s)\partial x/\partial t, \partial x/\partial s\rangle \cdot |\partial x/\partial s|^{-1}$$
$$+ \{|\nabla_{\partial/\partial t}\, \partial x/\partial s|^2 |\partial x/\partial s|^2 - \langle \nabla_{\partial/\partial t}\, \partial x/\partial s, \partial x/\partial s\rangle^2\} \cdot |\partial x/\partial s|^{-3},$$

where R denotes the curvature tensor. Now the first term of the last equality is nonpositive because of the assumption on the curvature. Since Jacobi field $Y_s(t) = \partial x/\partial s(t, s)$ is perpendicular to γ_s for every value of t, $\nabla Y_s(t) = \nabla_{\partial/\partial t}\, \partial x/\partial s$ is also perpendicular to γ_s and linearly dependent on $Y_s(t)$. This implies that the second term vanishes. On the other hand for initial value we get

$$\langle \nabla_{\partial/\partial t}\, \partial x/\partial s, \partial x/\partial s\rangle(0, s) = \langle \nabla_{\partial/\partial s}\, \partial x/\partial t, \partial x/\partial s\rangle(0, s) =$$
$$-\langle n(s), \nabla_{\partial/\partial s} \dot{c}(s)\rangle < 0,$$

because c is convex. This completes the proof of the lemma.

Next we shall give key observation for our purpose.

LEMMA 2. *There is only one point at which d_c takes relative maximum. Thus we have the unique furthest point p from c with $d_c(p) = d^*$.*

PROOF. Let p be a point with $d_c(p) = d^*$ and suppose that d_c takes relative maximum at $p_1 \neq p$. Then from the convexity of D, the minimal geodesic τ joining p to p_1 lies in D. We may take a point r in the interior of τ at which $d_c|\tau$ takes the minimum. Take a minimal geodesic σ: $[0, a] \to \bar{D}$ from c to r parametrized by arc length which realizes the distance $d_c(r)$. By the first variation formula σ is orthogonal to c at $\sigma(0) = c(s)$ and to τ at $r = \sigma(a)$. Now consider the unit parallel vector field X along σ with $X(0) = \dot{c}(s)$. Since $X(a)$ is tangent to the geodesic τ, we have by the second variation formula (see e. g., [B-C])

(4) $$D^2L(X, X)=\int_0^a\{\langle\nabla X(t), \nabla X(t)\rangle-\langle R(X(t)\dot{\sigma}(t))\dot{\sigma}(t), X(t)\rangle\}dt$$
$$+\langle AX(0), X(0)\rangle,$$

where A denotes the shape operator of c with respect to the normal n. In our case we have $\nabla X(t)=0$ and

$$\langle AX(0), X(0)\rangle=\langle A\dot{c}(s), \dot{c}(s)\rangle=\langle\nabla_{\partial/\partial t}\,\partial x/\partial s, \partial x/\partial s\rangle(0, s)=$$
$$-\text{geodesic curvature of } c \text{ at } c(s)<0$$

because of convexity. Then we have $D^2L(X, X)<0$ which contradicts the fact that $d_c|\tau$ takes the minimum at r. q. e. d.

Now we recall the notion of the critical point of the distance function due to Gromov ([G]): $q\in D\backslash c$ is called a critical point of d_c if for any unit tangent vector $u\in T_qD$, there exists a minimal geodesic (parametrized by arc length) σ such that the angle $\sphericalangle(\dot{\sigma}(d_c(q)), u)\leq\pi/2$. It is known that the furthest point p from c is d_c-critical.

LEMMA 3. *p is the only one critical point of d_c. Namely for any point q of $D\backslash c$ different from p, the tangent vectors to minimal geodesics from c to q at q are contained in an open half plane of T_qD.*

PROOF. Let $q\neq p$ be a critical point of d_c. Take a minimal geodesic τ $(\subset D)$ from p to q parametrized by arc length and set $u:=\dot{\tau}(d(p, q))\in T_qD$, where $d(p, q)$ denotes the distance between p and q. Then there exists a minimal geodesic σ from c to q with $\sphericalangle(\dot{\sigma}(d_c(q)), u)\leq\pi/2$. If this angle is less than $\pi/2$, then from the first variation formula we may find points of τ whose distance from c is less than $d_c(q)$. In case where $\sphericalangle(\dot{\sigma}(d_c(q), u)=\pi/2$, the same argument as in the proof of Lemma 2 implies the same conclusion. Namely we see that $d_c|\tau$ takes the minimum at an interior point of τ. Again the same argument as in the proof of Lemma 2 derives a contradiction. q. e. d.

Note that for Lemma 1~3 we don't need real analycity of the metric. Now following Fiala ([F]) we investigate the behaviour of l_t by considering the cut locus of c in D (see also [B], [M], [Sal]). We list up some properties of cut locus which is necessary for later use. We mainly follow the notation of [Sal]. We denote by $N(c)$ the normal bundle of c. Let C (resp. $\tilde{C}$) be the (resp. tangent) cut locus of c. We may write as $\tilde{C}=\{(s, g_1(s)):=g_1(s)n(s)\in N(c), s\in[0, 1]/\{0, 1\}\}$. Then $g_1(s)\in(0, d^*]$ is continuous with respect to s. The normal exponential map exp is a diffeomorphism on the set $\tilde{\mathscr{I}} := \{(s, t) := tn(s) \in N(c)\,;\ s \in [0, 1]/\{0, 1\},\ 0 \leq t < g_1(s)\}$ and we get $\partial\tilde{\mathscr{I}} = \tilde{C}$.

CASE 1. If the first focal locus F of c reduces to one point, then $C = F = \{p\}$ and all unit speed geodesics emanating from c perpendicularly reach p at the same parameter value d^*. In this case we have $g_1(s) \equiv d^*$.

CASE 2. Otherwise we have the following;

1° There are only finitely many cut points which are also focal points of c along geodesics emanating from c perpendicularly.

2° The cut locus is a tree in the curve theory (i. e.,1-complex without closed curves). Its end points are the first focal points.

3° For $q \in C$, the number of minimal geodesics from c to q is finite and equal to the number of 1-cells of C which issue from q. This number will be called the order of the cut point q. In fact exactly one 1-cell issues from q between the two minimal geodesics from c to q adjoining each other. Note that end points are cut points of order 1.

4° Cut point $q \in C$ is called regular if q is of order 2 and is not a focal point. Otherwise $q \in C$ is called singular. The lift of regular (resp. singular) cut ponts to $\tilde{C} \subset N(c)$ via exp are called regular (resp. singular) tangent cut points. Then there are only finitely many singular (tangent) cut points. Singular cut points and the furthest point p from c form the set of vertices of the tree C.

5° There are only finitely many connected components of the set of regular cut points and each component, which is a 1-cell of C, is a regular analytic arc parametrized by analytic function $t = g_1(s)$. The number of critical points of $g_1(s)$ is at most finite in general. Moreover for regular cut point $q \in C$, two minimal geodesics from c to q make the equal angle at q with the real analytic curve $t = g_1(s)$ which is a 1-cell of the cut locus C (condition of bisection).

6° Now we consider the level $\Lambda_t := d_c^{-1}(t)$ and $\tilde{\Lambda}_t := \{(t, s) \in N(s)$, which lies in the closure of $\tilde{\mathscr{I}}\}$. Then $\tilde{\Lambda}_t \cap \tilde{C}$ consists of at most finitely many points. Now the value t_o $(0 < t_o < d^*)$ will be called regular if $\tilde{\Lambda}_{t_o} \cap \tilde{C}$ either is empty or consists only of regular tangent cut points. In the latter case for each tangent cut point $(g_1(\sigma_o), \sigma_o) \in \tilde{\Lambda}_{t_o} \cap \tilde{C}$, the equation $t = g_1(s)$ for $\tilde{C}$ is locally solvable in a neighbourhood of $t_o = g_1(\sigma_o)$ in the form $s = \sigma(t)$ with $\sigma_o = \sigma(t_o)$, where $\sigma(t)$ is real analytic. Note that the value t is singular iff Λ_t contains a singular cut point. Then for regular value t_o, by changing the origin of c if necessary, we have real analytic functions $s = \sigma_i^{\pm}(t)$ $(i = 1, \ldots, k)$ defined in a neighbourhood of t_o with $0 < \sigma_1^-(t) < \sigma_1^+(t) < \ldots \sigma_k^-(t) < \sigma_k^+(t) < l$ so that we have $\tilde{\Lambda}_t = \bigcup_{i=1}^k \{t\} \times [\sigma_i^-(t),$

$\sigma_i^+(t)]$ and $\tilde{\Lambda}_t \cap \tilde{C} = \{(t, \sigma_i^{\pm}(t))\}_{i=1}^k$. Then $\Lambda_t = \exp\tilde{\Lambda}_t$ is obtained from $\tilde{\Lambda}_t$ by identifying each $(t, \sigma_i^{\mp}(t))$ with exactly one $(t, \sigma_j^{\pm}(t))$ under exp. Note that $x|\{t\}\times(\sigma_i^-(t), \sigma_i^+(t))$ is a diffeomorphism. From this we see that for regular value t Λ_t consists of finitely many Jordan closed curves and we have

$$(5) \qquad l_t = \sum_{i=1}^k \int_{\sigma_i^-(t)}^{\sigma_i^+(t)} |\partial x/\partial s(t, s)| ds$$

Now we turn to our situation.

LEMMA 4. *Under the assumption of the theorem, for every* 1-*cell* e *of* C, *which is a real analytic curve consisting of regular cut points, there exists no critical points of real analytic function* $d_c|e$ (*i. e.*, $g_1(s)$).

PROOF. If $q \in e$ is a critical point of $d_c|e$, then by the first variation formula the two minimal geodesics γ_1, γ_2 from c to q intersect e perpendicularly at q. By parallel translating the unit tangent vector u to e at q along $\gamma_i^{-1}(i=1, 2)$, we see by the same argument as in Lemma 2 that d_c takes a local maximum at q along a geodesic $s \to \exp su$. From this we see that $d_c|_e$ also takes a local maximum at q. Since e is contained in the cut locus, $d_c: D \to \boldsymbol{R}$ takes a local maximum at q. This contradicts Lemma 2. q. e. d.

Now consider a 1-cell e of C issuing from an end point q of C. Since there is only one minimal geodesic from c to q, the condition of bisection, the first variation formula and Lemma 4 imply that $d_c|e$ is strictly increasing. Next we consider a vertex q of C different from p in general. Since q is not d_c-critical, unit tangent vectors at q to the minimal geodesics $\gamma_1, \ldots, \gamma_k$ from c to q adjoining each other are contained in an open half plane of

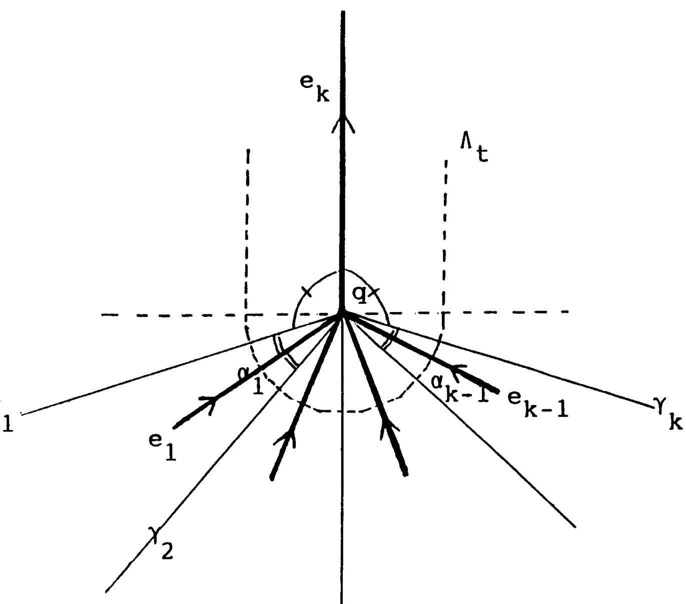

Figure 1

T_qD. We chose $\gamma_1,\ldots, \gamma_k$ so that the only one 1-cell e_k issuing from q, which lies in the above half plane, is adjoining to γ_1 and γ_k (see Figure 1). Then e_k makes an obtuse angle with γ_1^{-1} and γ_k^{-1} at q and $d_c|e_k$ is strictly increasing as above. Along other 1-cells $e_1, \ldots, e_{k-1}$ of C issuing from q, d_c is strictly decreasing. Thus for every cut point r, we can reach the furthest point p from r in the unique way along 1-cells of C so that d_c is strictly increasing.

LEMMA 5. *The level $d_c^{-1}(t)$ $(0\leq t<d^*)$ is a connected simple closed curve and $\Omega_t:=d_c^{-1}([t, d^*])$ is a disc.*

PROOF. First we consider the case when t is a regular value. Then from 6° $d_c^{-1}(t)$ consists of finitely many disjoint Jordan closed curves $\tau_i(i=1,\ldots, l)$. Now we show that Ω_t is connected. In fact for every point $q\in \Omega_t$ first proceed to a cut point q_1 along a minimal geodesic from c to q. Then we may reach p along cut locus as above. Thus we have a curve from q to p. By the same reason $d_c^{-1}((t, d^*])$ is connected. On the other hand $d_c^{-1}([0, t))$ is obviously connected. Now suppose that $l>1$. Then point r_1 of $d_c^{-1}((t, d^*])$ and point r_2 of $d_c^{-1}([0, t))$, which are close to τ_1, can be connected by a curve. In fact first take a curve from r_1 to a point of τ_2 in $d_c^{-1}([t, d^*])\backslash\tau_1$ and then join this point to r_2 by a curve in $d_c^{-1}([0, t])\backslash\tau_1$. Then we see that $D\backslash\tau_1$ is connected which is a contradiction. Then we see that $l=1$ and Ω_t is connected. By a limitting argument we have the same conclusion also for singular value t. q. e. d.

Now F. Fiala computed the first derivative $d/dt\ l_t$ for a regular value t in the following way: We denote by $\theta_i^{\pm}(t)$ the angle between $\mp(\partial x/\partial s)$ $(t, \sigma_i^{\pm}(t))$ and the tangent vector at $x(\sigma_i^{\pm}(t), t)$ to the 1-cell $t\to x(t, \sigma_i^{\pm}(t))$ of the cut locus $(i=1, \ldots, k)$. Then $0<\theta_i^{\pm}(t)\leq\pi/2$ and we get by setting $\Lambda_t:=d_c^{-1}(t)$

$$(6)\qquad d/dt\ l_t=-\int_{\Lambda_t}\langle\partial x/\partial t, \nabla_{\partial/\partial s}(\partial x/\partial s/|\partial x/\partial s|)\rangle\, ds-\sum\cot\theta_i^{\pm}(t)$$

(see [F], [Sal])

Note that $0<\theta_i^{\pm}(t)<\pi/2$ in our case.

REMARK. If Λ_t contains no cut points then the second term of right side of (6) vanishes. Next the geodesic curvature $\varkappa_t$ of the curve $s\to x(t, s)$, $\sigma_i^-(t)<s<\sigma_i(t)$ is given by

$$\varkappa_t d\sigma=\langle\partial x/\partial t, \nabla_{\partial/\partial s/|\partial x/\partial s|}(\partial x/\partial s/|\partial x/\partial s|\rangle)|\partial x/\partial s|ds,$$

where σ denotes arc length of $s \to x(t, s)$. Thus the integrand of the first term of right side is the geodesic curvature of Λ_t.

LEMMA 6. *Under the assumption of the theorem we have $d/dt\ l_t<0$ for regular value t.*

PROOF. This is clear from

$$\langle \partial x/\partial t,\ \nabla_{\partial/\partial s/|\partial x/\partial s|}(\partial x/\partial s/|\partial x/\partial s|)\rangle =$$

$$-|\partial x/\partial s|^{-1}<\nabla_{\partial/\partial s}\ \partial x/\partial t,\ \partial x/\partial s/|\partial x/\partial s|\rangle >0$$

by virtue of lemma 1. Note that this means that the geodesic curvature $\varkappa_t$ of the level is positive. q. e. d.

Now we apply Gauss-Bonnet to Ω_t. Since Ω_t is a disc we get by denoting K and ds Gauss curvature and area element respectively

$$(7) \qquad d/dt\ l_t=\int_{\Omega_t} K\ ds-2\pi-\Sigma\{\tan(\pi/2-\theta_i^{\pm}(t))-(\pi/2-\theta_i^{\pm}(t))\}$$

We set $\eta_i^{\pm}(t):=\pi/2-\theta_i{}^{\pm}(t)$.

LEMMA 7. *Let $T<d^*$ be a singular value. Then we have $\lim_{t\to T+0} d/dt\ l_t\leq \lim_{t\to T-0} d/dt\ l_t<0$*

PROOF. Let q be a singular cut point in $d_c^{-1}(t)$ of order k. Then from the argument given before Lemma 5, there exists only one 1-cell e_k of C issuing from q along which d_c is monotone increasing and other 1-cells e_i $(i=1, \dots, k\text{-}1)$ of C issuing from q are contained in an open half plane of T_qD (see Figure 1). Now for $t<T$, where $T\text{-}t$ is small, consider the contribution of $\eta_i^{\pm}(t)$ to (7) in a neighbourhood of q. Let $\alpha_1, \dots, \alpha_{k-1}$ be the angles at q between adjoing minimal geodesics $\gamma_1,\dots, \gamma_k$ from c to q contained in the open half plane. Then as $t\to T\text{-}0$, the above contribution to (7) converges to $-2\ \Sigma\ (\tan\ \alpha_i/2 - \alpha_i/2)$ by the condition of bisection. On the other hand for $t>T$, the 1-cell e_k of C consists only of regular cut points and as $t\to T+0$ the contribution of the angles $\eta^{\pm}(t)$ to (7) converges to

$$-2\{\tan((\alpha_1+\dots+\alpha_{k-1})/2)-(\alpha_1+\dots+\alpha_{k-1})/2\}.$$

Now since $(\alpha_1+\dots+\alpha_{k-1})/2<\pi/2$ by virtue of Lemma 3, we have

$$\tan((\alpha_1+\dots+\alpha_{k-1})/2)\ \geq\ \tan\ \alpha_1/2+\dots+\tan\ \alpha_{k-1}/2.$$

Then summing up the above contributions for all singular cut points in Λ_T we have easily the conclusion of the Lemma.

LEMMA 8. *Under the assumption of the theorem we have for regular value* t $d^2/dt^2\, l_t \leq 0$.

PROOF. we differentiate (7) for regular value t. Denoting $d\sigma$ the induced measure on Λ_{t_o} we get by Coarea formula (or directly by Fubini's theorem)

$$(8) \qquad d^2/dt^2\, l_t|_{t=t_o} = -\int_{\Lambda_{t_o}} K\, \mathrm{d}\sigma - \sum d\eta_i^{\pm}/dt(t_o) \cdot \{1/\cos^2 \eta_i^{\pm}(t_o) - 1\}$$

Thus to prove the lemma it suffices to show that $d\eta_i^{\pm}/dt(t_o)$ is nonnegative. Now recall that each $\eta_i^{\pm}(t)$ is equal to the half of the angle of the tangent vectors at cut point $q := x(t, \sigma_i^{\pm}(t))$ to two minimal geodesics from c to q by virtue of the condition of bisection. We parametrize the 1-cell e of the cut locus C containing q in the form $t \to x(t, \sigma_1(t)) = x(t, \sigma_2(t))$, where $\tau \to x(\tau, \sigma_i(t))$, $0 \leq \tau \leq t$ $(i=1, 2)$ are two minimal geodesics from c to the point of e. Here note that we parametrize e in a neighbourhood of $\sigma_i(t_o)$ so that $t \to s = \sigma_i(t)$ $(i=1, 2)$ are increasing (see Figure 2). We denote by $2\eta(t)$ the angle between the tangent vectors at cut point $x(t, \sigma_i(t))$ to two minimal geodesics from c to the cut point, namely we have

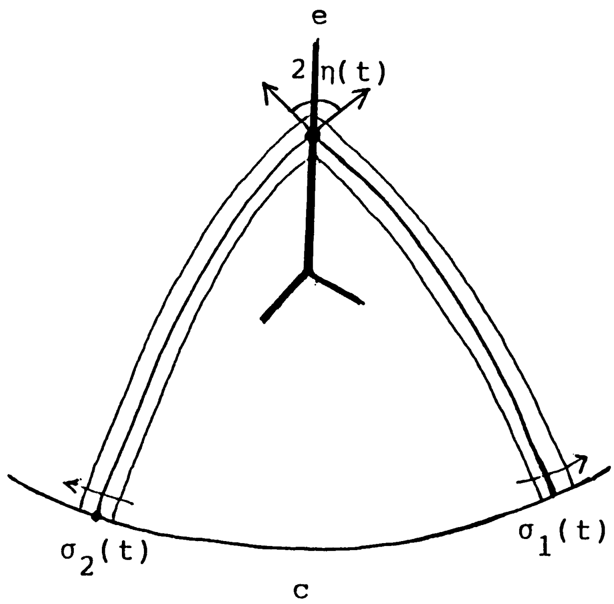

Figure 2

$$\cos 2\eta(t) = \langle \partial x/\partial t(t, \sigma_1(t)), \partial x/\partial t(t, \sigma_2(t))\rangle.$$

Note that each angle $\eta_i^{\pm}(t)$ may be written in this form $\eta(t)$. Now since $\nabla_{\partial/\partial t}\, \partial x/\partial t = 0$, we get

$$d/dt_{|t=t_o} \langle \partial x/\partial t(t, \sigma_1(t)), \partial x/\partial t(t, \sigma_2(t))\rangle$$

$$= \sigma_1'(t_o) \langle \nabla_{\partial/\partial s}\, \partial x/\partial t(t_o, \sigma_1(t_o)), \partial x/\partial t(t_o, \sigma_2(t_o))\rangle$$

$$+\sigma_2'(t_o)\langle \partial x/\partial t(t_o, \sigma_1(t_o)), \nabla_{\partial/\partial s}\, \partial x/\partial t(t_o, \sigma_2(t_o))\rangle$$

we consider the first term of the right side of the above equality. Since $t \to \partial x/\partial s(t, s)$ is a c-Jacobi field along γ_s which is perpendicular to γ_s everywhere we may write

$$\nabla_{\partial/\partial s}\, \partial x/\partial t(t, \sigma_1(t)) = \nabla_{\partial/\partial t}\, \partial x/\partial s\ (t, \sigma_1(t))$$

$$=\{\langle \nabla_{\partial/\partial t}\, \partial x/\partial s, \partial x/\partial s/|\partial x/\partial s|\rangle\, \partial x/\partial s/|\partial x/\partial s|\}(t, \sigma_1(t))$$

up to the first focal value. Thus the above first term is equal to

$$\sigma_1'(t_o)\ \langle \nabla_{\partial/\partial t}\, \partial x/\partial s, \partial x/\partial s/|\partial x/\partial s|\rangle(t_o, \sigma_1(t_o))\cdot\langle \partial x/\partial s/|\partial x/\partial s|\ (t_o, \sigma_1(t_o)),$$
$$\partial x/\partial t\ (t_o, \sigma_2(t_o))\rangle$$

Now $\sigma_1'(t_o) > 0$, and we see that from lemma 1

$$\langle \nabla_{\partial/\partial t}\, \partial x/\partial s, \partial x/\partial s/|\partial x/\partial s|\rangle < 0$$

Moreover from lemmas 3, 4 $\sphericalangle(\partial x/\partial t(t_o, \sigma_1(t_o)), \partial x/\partial t(t_o, \sigma_2(t_o)) < \pi$ and recalling the way of the parametrization of $\sigma_1(t)$, $\sigma_2(t)$ we have

$$\langle \partial x/\partial s/|\partial x/\partial s|(t_o, \sigma_1(t_o)), \partial x/\partial t(t_o, \sigma_2(t_o))\rangle < 0$$

Then the first term is negative and the same argument for the second term implies that $t \to \cos 2\eta(t)$ is decreasing and we have $d/dt\ \eta(t) \geq 0$. This completes the proof of the lemma. q. e. d.

REMARK. Consider the domain of revolution $(\tilde{D}, \tilde{g})$, $\tilde{D}=[0, d^*]\times S^1$, $\tilde{g}=dt^2+(l_t/2\pi)^2\ g_{s_1}$, where g_{s_1} denotes the canonical metric of unit circle S^1 and $\{d^*\}\times S^1$ reduces to one point $\tilde{p}$. Then the Gauss curvature $\tilde{K}$ of $(\tilde{D}, \tilde{g})$ is positive except singular values of t, because $\tilde{K}=-(d^2/dt^2\ l_t)/l_t$.

Now the theorem follows immediately from lemma 1~lemma 8. Finally we give a proof of the corollary: First consider the case (1). In this case the cut locus C consists of one point p. Then we have from (7)

$$\lim_{t\to d^*} d/dt\ l_t = \lim_{t\to d^*}\left(\int_{\Omega t} K\ ds - 2\pi\right) = -2\pi.$$

Now from lemma 8 we get $d/dt\ l_t \geq -2\pi$ and consequently $l_t \leq 2\pi(d^* - t)$. This implies that

$$\text{Area } D \leq 2\pi\int_0^{d^*}(d^*-t)dt = \pi(d^*)^2$$

We turn to the second case. Since by the same argument as in the proof of lemma 7 we have

$$\lim_{t\to d^*} d/dt\ l_t = -2\pi - 2\sum(\tan\alpha_i/2 - \alpha_i/2).$$

Then we get the desired inequality by lemmas 5, 6, 7 as above.

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Added in proof : d_c is a concave function (J. Cheeger-D. Gromoll, Ann. of Math., 96(1974), 413-443). *Using their argument it is possible to prove Theorem and Corollary under the weaker condition that the geodesic curvature κ of c is nonnegative.*