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Smooth $SL(n, C)$ actions on $(2n-1)$-manifolds

Dedicated to Professor Haruo Suzuki on his 60th birthday

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0. Introduction.

Smooth $SL(2, C)$ actions on closed connected 3-manifolds are classified by T. ASOH [1].

In this paper, we shall classify smooth $SL(n, C)$ actions on closed connected $(2n-1)$-manifolds for $n\geq 3$. We shall show that such a manifold is equivariantly diffeomorphic to the lens space $L^{2n-1}(p)$ or the product space $P_{n-1}(C)\times S^1$, with certain $SL(n, C)$ action. Our main result is stated in Theorem 3.

1. Certain subgroups of $SU(n)$.

Let K be a closed connected proper subgroup of $SU(n)$, and suppose $\dim SU(n)/K\leq 2n-1$, that is, $\dim K\geq n(n-2)$. Notice that the inclusion $i: K\rightarrow SU(n)$ gives a unitary representation of K.

Suppose first that the representation i is reducible, that is, there is a positive integer k such that $2k\leq n$ and K is contained in $S(U(k)\times U(n-k))$ up to an inner automorphism of $SU(n)$. If $k\geq 2$, then

$$2n-1<kn\leq 2k(n-k)=\dim SU(n)/S(U(k)\times U(n-k))$$
$$\leq\dim SU(n)/K.$$

Hence we obtain $k=1$. Moreover, we see that K coincides with $SU(n-1)$ or $S(U(1)\times U(n-1))$ up to an inner automorphism of $SU(n)$, by the fact that there is no closed subgroup of codimension 1 in $SU(n-1)$ for each $n\geq 3$.

Next we consider the case that the representation i is irreducible. We see that K is semi-simple, because K is contained in $SU(n)$.

Suppose that K is not simple. Then, there are closed normal subgroups H_1, H_2 of K and irreducible unitary representations $r_j: H_j\rightarrow U(n_j)$ such that the tensor product $r_1\otimes r_2$ is equivalent to $i\pi$, where $n=n_1n_2$, $n_j\geq 2$ and $\pi: H_1\times H_2\rightarrow K$ is a covering projection.

By our assupmtion, we obtain

$$2n-1 \geq \dim \boldsymbol{SU}(n)/K \geq n^2-(n_1{}^2+n_2{}^2)=(n_1{}^2-1)(n_2{}^2-1)-1,$$

and hence

$$(n_1-1)(n_2-1) \leq 2n_1n_2/(n_1+1)(n_2+1)<2.$$

This is a contradiction to $n_j \geq 2$. Therefore K is simple.

Now we consider the case that K is a simple and semi-simple Lie group, $i: K \rightarrow \boldsymbol{SU}(n)$ is an irreducible representation and dim $K \geq n(n-2)$. Denote by $m_1(K)$ (resp. $m_2(K)$) the smallest (resp. the second smallest) degree of non-trivial irreducible unitary representation of the universal covering group K^* of K. We obtain the following table by Weyl's dimension formula.

K^*	dim K	$m_1=m_1(K)$	$m_2=m_2(K)$
G_2	14	7	—
F_4	52	26	—
E_6	78	27	—
E_7	133	56	—
E_8	248	248	—
$\boldsymbol{Spin}(r)$, $r \geq 7$	$r(r-1)/2$	r	—
$\boldsymbol{Sp}(r)$, $r \geq 1$	$r(2r+1)$	$2r$	—
$\boldsymbol{SU}(r)$, $r \geq 4$	r^2-1	r	$r(r-1)/2$
$\boldsymbol{SU}(3)$	8	3	6

REMARK. $\boldsymbol{Spin}(6)=\boldsymbol{SU}(4)$, $\boldsymbol{Spin}(5)=\boldsymbol{Sp}(2)$, $\boldsymbol{Spin}(3)=\boldsymbol{SU}(2)=\boldsymbol{Sp}(1)$.

If K^* is an exceptional Lie group, $\boldsymbol{Spin}(r)$ $(r \geq 7)$ or $\boldsymbol{Sp}(r)$ $(r \geq 3)$, then $m_1(m_1-2)>$dim K. Therefore such a case does not happen. If $K^*=\boldsymbol{SU}(r)$ $(r \geq 3)$, then $m_2(m_2-2)>$dim K. Therefore we obtain $n=m_1=r$. This is a contradiction to the assumption $K \neq \boldsymbol{SU}(n)$. Therefore, the possibilities remain only when $K^*=\boldsymbol{Sp}(r)$ $(r=1,2)$. We see that K coincides with either $\boldsymbol{SO}(3)$ in $\boldsymbol{SU}(3)$ in $\boldsymbol{Sp}(2)$ in $\boldsymbol{SU}(4)$ up to an inner automorphism.

Summing up the above argument, we obtain the following:

LEMMA 1. *Suppose $n \geq 3$. Let K be a closed connected proper subgroup of $\boldsymbol{SU}(n)$ such that* dim $\boldsymbol{SU}(n)/K \leq 2n-1$. *Then K coincides with standardly embedded one of the following:*

$$\boldsymbol{SU}(n-1),\ \boldsymbol{S}(\boldsymbol{U}(1)\times \boldsymbol{U}(n-1)),\ \boldsymbol{SO}(3)\ (n=3)\ \text{or}\ \boldsymbol{Sp}(2)\ (n=4),$$

up to an inner automorphism of $\boldsymbol{SU}(n)$.

2. Certian subgroups of $SL(n,C)$

Let $L(n)$, $L^*(n)$, $N(n)$ and $N^*(n)$ denote the closed connected subgroups of $\boldsymbol{SL}(n, \boldsymbol{C})$ consisting of matrices in the form

$$\left(\begin{array}{c|c} 1 & * \cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right), \quad \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline * & \\ \vdots & * \\ * & \end{array}\right), \quad \left(\begin{array}{c|c} * & * \cdots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right) \text{ and } \left(\begin{array}{c|c} * & 0 \cdots 0 \\ \hline * & \\ \vdots & * \\ * & \end{array}\right)$$

respectively.

LEMMA 2.1. *Suppose $n \geq 3$. Let G be a closed connected proper subgroup of $\boldsymbol{SL}(n, \boldsymbol{C})$ such that G contains $\boldsymbol{SU}(n-1)$ and*

$$\dim \boldsymbol{SL}(n, \boldsymbol{C})/G \leq 2n-1.$$

Then, either $L(n) \subset G \subset N(n)$ or $L^*(n) \subset G \subset N^*(n)$.

PROOF. As usual, we regard $M_n(\boldsymbol{C})$ with the bracket operation $[A, B]=AB-BA$ as the Lie algebra of $\boldsymbol{GL}(n,\boldsymbol{C})$. Let $\mathfrak{sl}(n, \boldsymbol{C})$ and $\mathfrak{su}(n)$ denote the Lie subalgebras of $M_n(\boldsymbol{C})$ corresponding to the subgroups $\boldsymbol{SL}(n, \boldsymbol{C})$ and $\boldsymbol{SU}(n)$ respectively. Then

$$\mathfrak{sl}(n,\boldsymbol{C})=\{X \in M_n(\boldsymbol{C}) : \text{trace } X=0\},$$
$$\mathfrak{su}(n)=\{X \in M_n(\boldsymbol{C}) : X+X^*=0, \text{ trace } X=0\}.$$

Define certain real linear subspaces of $\mathfrak{sl}(n, \boldsymbol{C})$ as follows:

$$\mathfrak{sl}(n-1, \boldsymbol{C})=\{(a_{ij}) : a_{i1}=a_{1j}=0, \text{ for each } i, j\},$$
$$\mathfrak{su}(n-1)=\mathfrak{su}(n) \cap \mathfrak{sl}(n-1, \boldsymbol{C}),$$
$$\mathfrak{h}(n-1)=\{X \in \mathfrak{sl}(n-1, \boldsymbol{C}) : X=X^*\},$$
$$\mathfrak{a}=\{(a_{ij}) : a_{ij}=0 \text{ for } i \neq 1\},$$
$$\mathfrak{a}^*=\{(a_{ij}) : a_{ij}=0 \text{ for } j \neq 1\},$$
$$\mathfrak{b}=\{(a_{ij}) : a_{ij}=0 \text{ for } i \neq j,\ a_{22}=a_{33}= \cdots =a_{nn}\}.$$

Then

$$\mathfrak{sl}(n, \boldsymbol{C})=\mathfrak{sl}(n-1, \boldsymbol{C}) \oplus \mathfrak{a} \oplus \mathfrak{a}^* \oplus \mathfrak{b},$$
$$\mathfrak{sl}(n-1, \boldsymbol{C})=\mathfrak{su}(n-1) \oplus \mathfrak{h}(n-1)$$

as direct sums of real vector spaces. We have

(i) $\quad [\mathfrak{a}, \mathfrak{a}^*]=\mathfrak{sl}(n-1, \boldsymbol{C}) \oplus \mathfrak{b}.$

Let $Ad : SL(n, \boldsymbol{C}) \rightarrow \boldsymbol{GL}(\mathfrak{sl}(n, \boldsymbol{C}))$ be the adjoint representation defined by $Ad(A)X=AXA^{-1}$ for $A \in \boldsymbol{SL}(n, \boldsymbol{C})$, $X \in \mathfrak{sl}(n, \boldsymbol{C})$. Then the linear subspaces $\mathfrak{sl}(n-1, \boldsymbol{C})$, $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{b}$ are $Ad(\boldsymbol{SL}(n-1, \boldsymbol{C}))$ invariant, and the linear subspaces $\mathfrak{su}(n-1)$ and $\mathfrak{h}(n-1)$ are $Ad(\boldsymbol{SU}(n-1))$ invariant. Moreover, the linear subspaces $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{h}(n-1)$ are irreducible $Ad(\boldsymbol{SU}(n-1))$ spaces respectively.

Let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{sl}(n, \boldsymbol{C})$ corresponding to G. Since G contains $\boldsymbol{SU}(n-1)$, $\mathfrak{g}$ is $Ad(\boldsymbol{SU}(n-1))$ invariant, and

(ii) $\mathfrak{g}=\mathfrak{su}(n-1)\oplus(\mathfrak{g}\cap\mathfrak{h}(n-1))\oplus(\mathfrak{g}\cap(\mathfrak{a}\oplus\mathfrak{a}^*))\oplus(\mathfrak{g}\cap\mathfrak{b})$.

Here $\mathfrak{g}\cap\mathfrak{h}(n-1)=\{0\}$ or $\mathfrak{h}(n-1)$, because $\mathfrak{h}(n-1)$ is an irreducible $Ad(\boldsymbol{SU}(n-1))$ space. Notice that

(iii) $\mathfrak{g}\cap(\mathfrak{a}\oplus\mathfrak{a}^*)\neq\mathfrak{a}\oplus\mathfrak{a}^*$

by (i) and the assumption $\mathfrak{g}\neq\mathfrak{sl}(n, \boldsymbol{C})$.

Suppose $\mathfrak{g}\cap\mathfrak{h}(n-1)=\{0\}$. Then, by (i) and (ii), we obtain

$$\dim \mathfrak{sl}(n, \boldsymbol{C})-\dim \mathfrak{g}\geq\dim \mathfrak{h}(n-1)+\dim \mathfrak{a}=n^2-2.$$

But $n^2-2>2n-1$ for each $n\geq3$. This is a contradiction to the assumption $\dim \boldsymbol{SL}(n, \boldsymbol{C})/G\leq2n-1$. Therefore, $\mathfrak{g}$ contains $\mathfrak{h}(n-1)$, and hence $\mathfrak{g}$ contains $\mathfrak{sl}(n-1, \boldsymbol{C})$, that is, G contains $\boldsymbol{SL}(n-1, \boldsymbol{C})$. Then $\mathfrak{g}$ is $Ad((\boldsymbol{SL}(n-1, \boldsymbol{C}))$ invariant.

Suppose $n\geq4$. Then $\mathfrak{a}$ and $\mathfrak{a}^*$ are inequivalent as $Ad(\boldsymbol{SL}(n-1, \boldsymbol{C}))$ spaces. Therefore, by (iii), we see that

$$\mathfrak{g}=\mathfrak{sl}(n-1, \boldsymbol{C})\oplus\mathfrak{a}\oplus(\mathfrak{g}\cap b) \text{ or } \mathfrak{g}=\mathfrak{sl}(n-1, \boldsymbol{C})\oplus\mathfrak{a}^*\oplus(\mathfrak{g}\cap\mathfrak{b}).$$

Suppose $n=3$. Then $\mathfrak{a}$ and $\mathfrak{a}^*$ are equivalent as $Ad(\boldsymbol{SL}(2,\boldsymbol{C}))$ spaces. Put

$$\mathfrak{k}(u:v)=\left\{\left(\begin{array}{c|c} 0 & v^t(PX) \\ \hline uX & 0 \end{array}\right): X\in \boldsymbol{C}^2\right\}$$

for u, $v\in\boldsymbol{C}$ and $P=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We see that each $Ad(\boldsymbol{SL}(2, \boldsymbol{C}))$ invariant proper subspace of $\mathfrak{a}\oplus\mathfrak{a}^*$ coincides with certain $\mathfrak{k}(u:v)$. We have

$$[\mathfrak{k}(u:v), \mathfrak{k}(u:v)]=\mathfrak{b} \text{ for } uv\neq0,$$
$$[\mathfrak{b}, \mathfrak{k}(u:v)]=\mathfrak{k}(u:-v).$$

Hence $\mathfrak{g}\cap(\mathfrak{a}\oplus\mathfrak{a}^*)=\mathfrak{a}$ or $\mathfrak{a}^*$ for $n=3$.

Therefore, we see that

$$L(n)\subset G\subset N(n) \text{ or } L^*(n)\subset G\subset N^*(n)$$

for each $n\geq 3$. q. e. d.

LEMMA 2. 2. (a) *If G is a closed subgroup of $\boldsymbol{SL}(4, \boldsymbol{C})$ such that G contains $\boldsymbol{Sp}(2)$ and* $\dim \boldsymbol{SL}(4, \boldsymbol{C})/G\leq 7$, *then* $G=\boldsymbol{SL}(4, \boldsymbol{C})$.

(b) *If G is a closed subgroup of $\boldsymbol{SL}(3, \boldsymbol{C})$ such that G contains $\boldsymbol{SO}(3)$ and* $\dim \boldsymbol{SL}(3, \boldsymbol{C})/G\leq 5$, *Then* $G=\boldsymbol{SL}(3, \boldsymbol{C})$.

PROOF. Decompose $\mathfrak{sl}(4, \boldsymbol{C})$ (resp. $\mathfrak{sl}(3, \boldsymbol{C})$) into four irreducible $Ad(\boldsymbol{Sp}(2))$ (resp. $Ad(\boldsymbol{SO}(3))$) spaces, and consider the bracket operation. Then we obtain the desired result. We omit the detail. q. e. d.

3. $\boldsymbol{SL(n, C)}$ actions on $\boldsymbol{(2n-1)}$-manifolds.

Here we introduce two types of $\boldsymbol{SL}(n, \boldsymbol{C})$ actions on $(2n-1)$-manifolds. Denote by S^{2n-1} the unit sphere of $\boldsymbol{C}^n$, and let $L^{2n-1}(p)$ be the quotient space of S^{2n-1} by the equivalence relation: $z\sim\lambda z$ for $z\in S^{2n-1}$ and $\lambda^p=1$, where p is a positive integer. Denote by $[z]$ the equivalence class of $z\in S^{2n-1}$. For each real number c, define $\Phi_c, \Phi_c^*: \boldsymbol{SL}(n, \boldsymbol{C})\times L^{2n-1}(p)\to L^{2n-1}(p)$ by

$$\Phi_c(A,[z])=[\exp((ic-1)\log\|Az\|)Az],\quad \Phi_c^*(A,[z])=\Phi_c(A^{*-1},[z])$$

for $A\in\boldsymbol{SL}(n, \boldsymbol{C})$, $[z]\in L^{2n-1}(p)$, where $i=\sqrt{-1}$. Then we see that Φ_c and Φ_c^* are smooth $\boldsymbol{SL}(n, \boldsymbol{C})$ actions on $L^{2n-1}(p)$.

Let $P_{n-1}(\boldsymbol{C})$ be the projective space as the quotient space of $\boldsymbol{C}^n-\{0\}$ by the equivalence relation: $z\sim\lambda z$ for $z\in\boldsymbol{C}^n-\{0\}$ and a non-zero complex number λ. Denote by $[z]$ the equivalence class of $z\in\boldsymbol{C}^n-\{0\}$. Let $\phi:\boldsymbol{R}\times S^1\to S^1$ be a smooth $\boldsymbol{R}$ action on S^1, and define $\Psi_\phi, \Psi_\phi^*: \boldsymbol{SL}(n, \boldsymbol{C})\times P_{n-1}(\boldsymbol{C})\times S^1\to P_{n-1}(\boldsymbol{C})\times S^1$ by

$$\Psi_\phi(A,([z],x))=([Az],\phi(\log(\|z\|^{-1}\|Az\|),x)),$$
$$\Psi_\phi^*(A,([z],x))=\Psi_\phi(A^{*-1},([z],x))$$

for $A\in\boldsymbol{SL}(n, \boldsymbol{C})$, $[z]\in P_{n-1}(\boldsymbol{C})$ and $x\in S^1$. Then we see that Ψ_ϕ and Ψ_ϕ^* are smooth $\boldsymbol{SL}(n, \boldsymbol{C})$ actions on $P_{n-1}(\boldsymbol{C})\times S^1$.

THEOREM 3. *Suppose $n\geq 3$. Then any non-trivial smooth $\boldsymbol{SL}(n, \boldsymbol{C})$ action on a closed connected $(2n-1)$-manifold is equivariantly diffeomorphic to $\Phi_c, \Phi_c^*, \Psi_\phi$ or Ψ_ϕ^*.*

PROOF. Let $\Phi:\boldsymbol{SL}(n, \boldsymbol{C})\times M\to M$ be a smooth $\boldsymbol{SL}(n, \boldsymbol{C})$ action on a closed connected $(2n-1)$-manifold M, and denote by Φ_0 its restricted $\boldsymbol{SU}(n)$ action. Denote by $\boldsymbol{SL}(n, \boldsymbol{C})_x$ (resp. $\boldsymbol{SU}(n)_x$) the isotropy group at $x\in M$ with respect to the action Φ(resp. Φ_0). Then we see

(i) $\boldsymbol{SU}(n)_x = \boldsymbol{SL}(n, \boldsymbol{C})_x \cap \boldsymbol{SU}(n)$.

If the identity component of $\boldsymbol{SU}(n)_x$ is conjugate to $\boldsymbol{SO}(3)$ $(n=3)$ or $\boldsymbol{Sp}(2)$ $(n=4)$, then $\boldsymbol{SL}(n, \boldsymbol{C})_x = \boldsymbol{SL}(n, \boldsymbol{C})$ by Lemma 2.2. This is a contradiction to (i).

Therefore, by Lemma 1, we see that the identity component of $\boldsymbol{SU}(n)_x$ is conjugate to $\boldsymbol{SU}(n-1)$, $\boldsymbol{S}(U(1)\times \boldsymbol{U}(n-1))$ or $\boldsymbol{SU}(n)$ for each $x \in M$.

3.1. Suppose first that the identity component of $\boldsymbol{SU}(n)_x$ is conjugate to $\boldsymbol{SU}(n-1)$ for some $x \in M$. Then the action Φ_0 is transitive, and we see that the $\boldsymbol{SU}(n)$ manifold M is equivariantly diffeomorphic to $L^{2n-1}(p)$ with the natural $\boldsymbol{SU}(n)$ action given by $[z] \to [Kz]$ for $K \in \boldsymbol{SU}(n)$, where p is the number of connected components of $\boldsymbol{SU}(n)_x$. In the following, we can assume that $M = L^{2n-1}(p)$ and the action Φ satisfies the condition:

(ii) $\Phi(K, [z]) = [Kz] : K \in \boldsymbol{SU}(n),\ [z] \in L^{2n-1}(p)$.

Then $F(\boldsymbol{SU}(n-1)) = \{[w\boldsymbol{e}_1] : w \in \boldsymbol{U}(1)\}$, where $\boldsymbol{e}_1 = {}^t(1, 0, \cdots, 0) \in \boldsymbol{C}^n$, and $F(H)$ denotes the fixed point set of the restricted H action on M. By Lemma 2.1, we obtain

$$F(L(n)) \cup F(L^*(n)) = F(\boldsymbol{SU}(n-1)).$$

Since $F(L(n)) \cap F(L^*(n)) = F(\boldsymbol{SL}(n, \boldsymbol{C}))$, we can show that $F(L(n))$ and $F(L^*(n))$ are disjoint, from the condition (ii). Then we obtain

$$F(\boldsymbol{SU}(n-1)) = F(L(n)) \text{ or } F(\boldsymbol{SU}(n-1)) = F(L^*(n)),$$

because $F(\boldsymbol{SU}(n-1))$ is connected.

Now we assume $F(\boldsymbol{SU}(n-1)) = F(L(n))$. Since $F(L(n))$ is $N(n)$ invariant, the action Φ induces naturally a $\boldsymbol{C}^\times$ action ξ on $F(L(n))$ given by

$$\Phi(T, [z]) = \xi(t_{11}, [z]) \text{ for } T = (t_{ij}) \in N(n).$$

Here $\boldsymbol{C}^\times$ denotes the multiplicative group of non-zero complex numbers. By the condition (ii), we see that the $\boldsymbol{C}^\times$ action ξ satisfies

$$\xi(u, [z]) = [uz] \text{ for } u \in \boldsymbol{U}(1),\ [z] \in F(L(n)).$$

On the other hand, we obtain a smooth mapping $f : \boldsymbol{R} \to \boldsymbol{U}(1)$ determined by $\xi(e^t, [\boldsymbol{e}_1]) = [f(t)\boldsymbol{e}_1]$, and we see that f is a homomorphism. Hence, there exists a real number c such that $f(t) = \exp(ict)$. Therefore,

$$\xi(u, [z])=[\exp((ic-1)\log|u|)uz]$$

for $u\in \boldsymbol{C}^{\times}$, $[z]\in F(L(n))$. Since $T\boldsymbol{e}_1=t_{11}\boldsymbol{e}_1$ for $T=(t_{ij})\in N(n)$, we obtain

$$\text{(iii)}\quad \Phi(T, [z])=[\exp((ic-1)\log\|Tz\|)Tz]$$

for $T\in N(n)$, $[z]\in F(L(n))$. Consequently, by the conditions (ii) and (iii), we obtain $\Phi=\Phi_c$, because there is a decomposition $A=KT: K\in \boldsymbol{SU}(n)$ and $T\in N(n)$ for each $A\in \boldsymbol{SL}(n, \boldsymbol{C})$, and the restricted $\boldsymbol{SU}(n)$ action Φ_0 on $L^{2n-1}(p)$ is transitive. Similarly, we obtain $\Phi=\Phi_c{}^*$, for the case $F(\boldsymbol{SU}(n-1))=F(L^*(n))$.

3. 2. Suppose next that the identity component of $\boldsymbol{SU}(n)_x$ is conjugate to $\boldsymbol{S}(\boldsymbol{U}(1)\times \boldsymbol{U}(n-1))$ or $\boldsymbol{SU}(n)$ for each $x\in M$. Then the action Φ_0 has codimension one principal orbits. If Φ_0 has a nonprincipal orbit, then it is a fixed point. Considering the slice representation at the fixed point, we see that Φ_0 has no nonprincipal orbit, because $P_{n-1}(C)$ is not homeomorphic to the $(2n-2)$-sphere. Then we see that the $\boldsymbol{SU}(n)$ manifold M is equivariantly diffeomorphic to $P_{n-1}(\boldsymbol{C})\times S^1$, where $\boldsymbol{SU}(n)$ acts on $P_{n-1}(C)$ by $[z]\to[Kz]$ for $K\in \boldsymbol{SU}(n)$ and trivially on $\boldsymbol{S}^1$. In the following, we can assume that $M=P_{n-1}(\boldsymbol{C})\times S^1$ and the action Φ satisfies

$$\text{(iv)}\quad \Phi(K, ([z], x))=([Kz], x)$$

for $K\in \boldsymbol{SU}(n)$, $[z]\in P_{n-1}(\boldsymbol{C})$ and $x\in S^1$. Then

$$F(\boldsymbol{SU}(n-1))=[\boldsymbol{e}_1]\times S^1,$$

and $F(\boldsymbol{SU}(n-1))=F(L(n))$ or $F(\boldsymbol{SU}(n-1))=F(L^*(n))$ as above. Now we assume $F(\boldsymbol{SU}(n-1))=F(L(n))$. Then the action Φ induces naturally a $\boldsymbol{C}^{\times}$ action ξ on S^1 given by

$$\Phi(T, ([\boldsymbol{e}_1], x))=([\boldsymbol{e}_1], \xi(t_{11}, x))$$

for $T=(t_{ij})\in N(n)$. If $T\in N(n)\cap \boldsymbol{SU}(n)$, then each point of $F(\boldsymbol{SU}(n-1))$ leaves fixed by T under the action Φ. Therefore, the $\boldsymbol{C}^{\times}$ action ξ satisfies $\xi(u, x)=x$ for $|u|=1$. On the other hand, we obtain a smooth $\boldsymbol{R}$ action $\phi: \boldsymbol{R}\times S^1\to S^1$ given by $\phi(t, x)=\xi(e^t, x)$. Then we see that

$$\text{(v)}\quad \Phi(T, ([\boldsymbol{e}_1], x))=([\boldsymbol{e}_1], \phi(\log\|T\boldsymbol{e}_1\|, x))$$

for $T\in N(n)$ and $x\in S^1$. Consequently, by the conditions (iv) and (v), we obtain $\Phi=\Psi_\phi$, because the $\boldsymbol{SU}(n)$ action on $P_{n-1}(\boldsymbol{C})$ is transitive. Similarly we obtain $\Phi=\Psi_\phi{}^*$, for the case $F(\boldsymbol{SU}(n-1))=F(L^*(n))$.

This completes the proof of Theorem 3. Similar argument is used in [2].

References

[1] T. ASOH: On smooth $\boldsymbol{SL}(2, \boldsymbol{C})$ actions on 3-manifolds, Osaka J. Math. **24** (1987), 271-298.

[2] F. UCHIDA: Actions of special linear groups on a product manifold, Bull. of Yamagata Univ., Nat. Sci. **10-3** (1982), 227-233.

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