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Twisted linear actions on complex Grassmannians

Dedicated to Professor Haruo Suzuki on his 60th birthday

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0. Introduction

In this paper, we shall study twisted linear actions of noncompact Lie groups on complex Grassmannians as the sequel to [3]. The first example of twisted linear actions on spheres was given by F. Uchida (cf. [5], [6]) and later the author (cf. [3]) gave such an example over complex (or quaternionic) projective spaces. It seems interesting to examine twisted linear actions on simply connected compact irreducible symmetric spaces of rank greater than one as well. The paper is organized as follows; some preliminary facts are collected to describe complex Grassmannians for our use in Section 1, the twisted linear actions are dealt with in Section 2 and 3.

One of the main results is that any twisted linear actions of compact Lie groups on complex Grassmannians are equivalent to the linear actions (cf. Theorem 2.2). On the contrary we emphasize that, as well as on the complex projective spaces, there are uncountably many topologically inequivalent twisted linear C^{ω}-actions of the noncompact Lie group $SL(n, \boldsymbol{C})$ on the complex Grassmannian $\boldsymbol{G}_{nk,m}$ of all m-dimensional linear subspaces in the nk-dimensional complex Enclidean space $\boldsymbol{C}^{nk}$, where $n>mk$ and $k>1$ (cf. Theorem 3.3). For complex Grassmannians, the author could not obtain the results corresponding to Theorem 3.3 and 3.5 of [3]. For quaternionic Grassmannians, our methods can not be used, since the quaternion field is noncommutative. The author does not know how twisted linear actions of Lie groups on quaternionic Grassmannians are defined.

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1. A description of complex Grassmannian

1.1. Let $M(n, m; \boldsymbol{C})$ be the set of all complex matrices of type $n\times m$ and put $M_n(\boldsymbol{C})=M(n, n; \boldsymbol{C})$. For $X, Y\in M(n, m; \boldsymbol{C})$, we define their hermitian inner product by $\langle X, Y\rangle=\mathrm{trace}(X^* Y)$ and the norm of X by

$\|X\|=\sqrt{\langle X, X\rangle}$. Then $\boldsymbol{C}^n=M(n,1;\ \boldsymbol{C})$ is the n-dimensional complex Euclidean space. Set $\boldsymbol{C}_0^n=\boldsymbol{C}^n-\{0_n\}$, $\boldsymbol{C}_0=\boldsymbol{C}-\{0\}=GL(1, \boldsymbol{C})$, where 0_n is the zero vector of $\boldsymbol{C}^n$. We say that $X\in M_n(\boldsymbol{C})$ satisfies the *condition* (T) if $\frac{1}{2}(X+X^*)$ is a positive definite hermitian matrix. It is easy to see that the condition (T) is equivalent to the following:

$$(T')\quad \frac{d}{dt}\|\exp(tX)z\|>0 \text{ for each } z\in \boldsymbol{C}_0^n,\ t\in\boldsymbol{R}.$$

If X satisfies (T'), then

$$\lim_{t\to+\infty}\|\exp(tX)z\|=+\infty \text{ and } \lim_{t\to-\infty}\|\exp(tX)z\|=0$$

for each $z\in \boldsymbol{C}_0^n$ and hence there exists a unique real valued C^{ω}-function τ on $\boldsymbol{C}_0^n$ such that

$$\|\exp(\tau(z)X)z\|=1 \text{ for } z\in \boldsymbol{C}_0^n.$$

The following lemma is proved in [5, Lemma 2.2].

LEMMA 1.1. *For $X \in M_n(\boldsymbol{F})$, assume that all the eigenvalues of X have positive real parts, where $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$. Then there exists $P \in GL(n, \boldsymbol{F})$ such that $P^{-1}XP$ satisfies the condition (T).*

1.2. For positive integers n. k such that $n>k$, set

$$\Lambda(n, k)=\{\alpha=(\alpha_1, \cdots, \alpha_k)\in\boldsymbol{Z}^k|\ 1\leqq\alpha_1<\cdots<\alpha_k\leqq n\}.$$

By introducing the lexicographic order in $\Lambda(n, k)$, identify $\Lambda(n, k)$ with the orderd set $\{1,\cdots, N\}$ of positive integers $1\leqq m\leqq N$, where

$$N=\binom{n}{k}$$

throughout Section 1 and 2. We define complex analytic mappings λ^k : $GL(n, \boldsymbol{C})\longrightarrow GL(N, \boldsymbol{C})$, μ_k : $M(n, k;\ \boldsymbol{C})\longrightarrow \boldsymbol{C}^N$ by

$$(\lambda^k A)_{\alpha,\beta}=\det(A(\alpha, \beta)),\qquad (\mu_k Z)_{\alpha}=\det(Z(\alpha)),$$

where $A=(a_{p,q})\in GL(n, \boldsymbol{C})$, $Z=(z_{p,j})\in M(n, k;\ \boldsymbol{C})$; $\alpha=(\alpha_1,\cdots, \alpha_k)$, $\beta=(\beta_1, \cdots, \beta_k)\in \Lambda(n, k)$; $(\lambda^k A)_{\alpha,\beta}$, $(\mu_k Z)_{\alpha}$ are the (α, β)-component of $\lambda^k A\in GL(N, \boldsymbol{C})$, the α-th component of $\mu_k Z\in \boldsymbol{C}^N$, respectively, and $A(\alpha, \beta)=(a_{\alpha_i, \beta_j})$, $Z(\alpha)=(Z_{\alpha_i,j})\in M_k(\boldsymbol{C})$ are square submatrices of A, Z, respectively. Moreover we define a $\boldsymbol{C}$-linear mapping λ_k : $M_n(\boldsymbol{C})\longrightarrow M_N(\boldsymbol{C})$ by

$$(\lambda_k X)_{\alpha,\beta}=\begin{cases}\sum_{i=1}^{k} x_{\alpha_i,\beta_i} & (\alpha=\beta)\\ (-1)^{i+j}x_{\alpha_i,\beta_j} & \begin{pmatrix}\alpha_i\neq\beta_j\\ (\alpha_1,\cdots,\alpha_{i-1},\alpha_{i+1},\cdots,\alpha_k)\\ =(\beta_1,\cdots,\beta_{j-1},\beta_{j+1},\cdots,\beta_k)\end{pmatrix}\\ 0 & \text{otherwise,}\end{cases}$$

where $X=(x_{p,q})\in M_n(\boldsymbol{C})$; $\alpha=(\alpha_1,\cdots,\alpha_k)$, $\beta=(\beta_1,\cdots,\beta_k)\in\Lambda(n,k)$ and $(\lambda_k X)_{\alpha,\beta}$ is the (α,β)-component of $\lambda_k X\in M_N(\boldsymbol{C})$. There is the following lemma.

LEMMA 1.2. i) *λ^k is a matrix representation of the Lie group $GL(n,\boldsymbol{C})$.*

ii) *$\mu_k(AZg)=(\lambda^k A)(\mu_k Z)\det g\in\boldsymbol{C}^N$ for $A\in GL(n,\boldsymbol{C})$, $Z\in M(n,k;\boldsymbol{C})$ and $g\in GL(k,\boldsymbol{C})$.*

iii) *For Z, $W\in M(n,k;\boldsymbol{C})$ such that Z^*Z, $W^*W\in GL(k,\boldsymbol{C})$, if there is a certain element $\zeta\in\boldsymbol{C}_0$ such that $\mu_k W=(\mu_k Z)\zeta$, then there exists some matrix $g\in GL(k,\boldsymbol{C})$ such that $W=Zg$ and $\zeta=\det g$.*

iv) *$\langle\mu_k Z,\mu_k W\rangle=\det(Z^*W)$ for Z, $W\in M(n,k;\boldsymbol{C})$.*

v) *λ_k is the differential representation of λ^k, i.e.,*

$$\exp(\theta\lambda_k X)=\lambda^k(\exp(\theta X))\in GL(N,\boldsymbol{C})$$

for $\theta\in\boldsymbol{R}$, $X\in M_n(\boldsymbol{C})$.

vi) *$\lambda_k(P^{-1}XP)=(\lambda^k P)^{-1}(\lambda_k X)\lambda^k P\in M_N(\boldsymbol{C})$ for $X\in M_n(\boldsymbol{C})$, $P\in GL(n,\boldsymbol{C})$.*

vii) *$\lambda^k(A^*)=(\lambda^k A)^*\in GL(N,\boldsymbol{C})$, $\lambda_k(X^*)=(\lambda_k X)^*\in M_N(\boldsymbol{C})$ for $A\in GL(n,\boldsymbol{C})$, $X\in M_n(\boldsymbol{C})$.*

viii) *If $X\in M_n(\boldsymbol{C})$ is a positive definite hermitian matrix, then $\lambda_k X\in M_N(\boldsymbol{C})$ is also a positive definite hermitian matrix.*

PROOF. i) and ii) are proved in [4, pp. 97-98]. iii) and iv) are proved in [4, pp. 258-259 and pp. 78-80], respectively. Now consider v). We have only to show

$$\lambda_k X=\left(\frac{d}{d\theta}\lambda^k(\exp\theta X)\right)_{\theta=0}=\left(\frac{d}{d\theta}\lambda^k(I_n+\theta X)\right)_{\theta=0}.$$

Hence it needs to be proved that

$$(\lambda_k X)_{\alpha,\beta}=\left(\frac{d}{d\theta}\det((I_n+\theta X)(\alpha,\beta))\right)_{\theta=0}$$

for $\alpha=(\alpha_1,\cdots,\alpha_k)$, $\beta=(\beta_1,\cdots,\beta_k)\in\Lambda(n,k)$, where $X=(x_{p,q})\in M_n(\boldsymbol{C})$,

$(\lambda_k X)_{\alpha,\beta}$ is the (α, β)-component of $\lambda_k X \in M_N(\boldsymbol{C})$ and $(I_n+\theta X)(\alpha, \beta)=(\delta_{\alpha_i,\beta_j}+\theta x_{\alpha_i,\beta_j}) \in M_k(\boldsymbol{C})$. The following equality holds;

$$\left(\frac{d}{d\theta}\det((I_n+\theta X)(\alpha, \beta))\right)_{\theta=0}=\mathrm{tr}({}^t(\Delta(I_n(\alpha, \beta)))X(\alpha, \beta)),$$

where the (i, j)-component of $\Delta(I_n(\alpha, \beta)) \in M_k(\boldsymbol{C})$ is the (i, j)-cofactor of $I_n(\alpha, \beta)=(\delta_{\alpha_i,\beta_j}) \in M_k(\boldsymbol{C})(1 \leqq i, j \leqq k)$ and $X(\alpha, \beta)=(x_{\alpha_i,\beta_j}) \in M_k(\boldsymbol{C})(1 \leqq i, j \leqq k)$. Then it is easily shown that

$$\mathrm{tr}({}^t(\Delta(I_n(\alpha, \beta)))X(\alpha, \beta))=(\lambda_k X)_{\alpha,\beta}$$

for $\alpha, \beta \in \Lambda(n, k)$. Thus v) has been proved. vi) follows immediately from i) and v). vii) follows directly from the definitions of λ^k and λ_k. viii) is easily shown by the definition of λ_k, vi) and vii). q. e. d.

1. 3. Set

$$V'_{n,k}=\{Z \in M(n, k;\ \boldsymbol{C}) | Z^* Z \in GL(k, \boldsymbol{C})\},$$

$$V_{n,k}=\{Z \in M(n, k;\ \boldsymbol{C}) | Z^* Z=I_k\},$$

$$W'_{n,k}=\mu_k(V'_{n,k}),\quad W_{n,k}=\mu_k(V_{n,k}).$$

Then

$$\mu_k: V'_{n,k} \longrightarrow W'_{n,k},\quad \mu_k: V_{n,k} \longrightarrow W_{n,k}$$

are principal fibrations whose structure groups are $SL(k, \boldsymbol{C})$, $SU(k)$, respectively. It follows from iv) of Lemma 1. 2 that

(1. 1) $\qquad W'_{n,k} \subset \boldsymbol{C}_0^N,\quad W_{n,k}=S^{2N-1} \cap W'_{n,k}$

where $S^{2N-1}=\{z \in \boldsymbol{C}^N |\ \|z\|=1\}$. For $X \in M_n(\boldsymbol{C})$, we define a real analytic right $\boldsymbol{C}_0$-action $\alpha_X: W'_{n,k} \times \boldsymbol{C}_0 \longrightarrow W'_{n,k}$ by

$$\alpha_X(\mu_k Z, \zeta)=\exp((\log|\zeta|)(\lambda_k X))(\mu_k Z)\frac{\zeta}{|\zeta|},$$

where $Z \in V_{n,k}'$. It needs to be checked that $\alpha_X(\mu_k Z, \zeta) \in W'_{n,k}$. For some matrix $g \in GL(k, \boldsymbol{C})$ such that $\det g=\zeta/|\zeta|$, it follows from ii) and v) of Lemma 1. 2 that $\alpha_X(\mu_k Z, \zeta)=\mu_k(\exp((\log|\zeta|)X)Zg) \in W'_{n,k}$. Now, for the above matrix $X \in M_n(\boldsymbol{C})$, assume that all the eigenvalues of X have positive real parts. Then by Lemma 1. 1, there exists $P \in GL(n, \boldsymbol{C})$ such that $X_0=P^{-1}XP$ satisfies the condition (T). By viii) of Lemma 1. 2, one sees easily that $\lambda_k X_0$ satisfies also the condition (T). For this matrix X_0, we define C^ω-diffeomorphisms $\Phi_{\lambda_k X_0}$, $\Psi_{\lambda_k X_0}: \boldsymbol{C}_0^N \longrightarrow \boldsymbol{C}_0^N$ and a real analytic mapping $\Phi_{X_0}: V'_{n,k} \longrightarrow V'_{n,k}$ (unless $k=1$, this is not a homeomorphism) by

$$\Phi_{\lambda_k X_0}(z)=(\exp((\log\|z\|)(\lambda_k X_0))z/\|z\|,$$
$$\Psi_{\lambda_k X_0}(w)=(\exp(\tau(w)\lambda_k X_0))we^{-\tau(w)},$$
$$\Phi_{X_0}(Z)=(\exp((\frac{1}{2}\log(\det(Z^*Z)))X_0))Z(Z^*Z)^{-1/2}.$$

Then it is verified directly from their definitions that $\Phi_{\lambda_k X_0}^{-1}=\Psi_{\lambda_k X_0}$. Moreover, it is noted that

$$\mu_k\circ\Phi_{X_0}=\Phi_{\lambda_k X_0}\circ\mu_k. \tag{1.2}$$

This follows from ii), iv) and v) of Lemma 1. 2. Hence one has that $\Phi_{\lambda_k X_0}(\boldsymbol{W}'_{n,k})\subset\boldsymbol{W}'_{n,k}$. It is easily proved that $\boldsymbol{W}'_{n,k}\supset\Psi_{\lambda_k X_0}(\boldsymbol{W}'_{n,k})$. Therefore $\Phi_{\lambda_k X_0}$, $\Psi_{\lambda_k X_0}$ are C^{ω}-diffeomorphisms of $\boldsymbol{W}'_{n,k}$ and it is also true that $\Phi_{\lambda_k X_0}^{-1}=\Psi_{\lambda_k X_0}$ on $\boldsymbol{W}'_{n,k}$.

For the above matrices P and X_0, let us define a C^{ω}-diffeomorphism $F_{\lambda_k X}$ of $\boldsymbol{C}_0^N$ and a real analytic mapping F_X of $\boldsymbol{V}'_{n,k}$ by

$$F_{\lambda_k X}=L_{\lambda^k P}\circ\Phi_{\lambda_k X_0},\quad F_X=L_P\circ\Phi_{X_0},$$

where $L_{\lambda^k P}(z)=(\lambda^k P)z$ for $z\in\boldsymbol{C}_0^N$ and $L_P(Z)=PZ$ for $Z\in\boldsymbol{V}'_{n,k}$. The mappings $F_{\lambda_k X}$ and F_X depend on a choice of P. By (1. 2) and that $\mu_k\circ L_P=L_{\lambda^k P}\circ\mu_k$, it is also true that

$$\mu_k\circ F_X=F_{\lambda_k X}\circ\mu_k. \tag{1.3}$$

Hence $F_{\lambda_k X}$ is a C^{ω}-diffeomorphism of $\boldsymbol{W}'_{n,k}$ and it holds that $F_{\lambda_k X}^{-1}=\Psi_{\lambda_k X_0}\circ L_{(\lambda^k P)^{-1}}$ on $\boldsymbol{W}'_{n,k}$. We have the commutative diagram:

$$\begin{array}{ccc} \boldsymbol{W}'_{n,k}\times\boldsymbol{C}_0 & \xrightarrow{\alpha_{I_n}} & \boldsymbol{W}'_{n,k} \\ \downarrow{\scriptstyle F_{\lambda_k X}\times 1} & & \downarrow{\scriptstyle F_{\lambda_k X}} \\ \boldsymbol{W}'_{n,k}\times\boldsymbol{C}_0 & \xrightarrow{\alpha_X} & \boldsymbol{W}'_{n,k}. \end{array} \tag{1.4}$$

We denote the orbit space of the action α_X by $\boldsymbol{G}_{n,k}^X$.

If we choose the identity matrix as X, then the orbit $\boldsymbol{G}_{n,k}^{I_n}$ is the usual complex Grassmannian $\boldsymbol{G}_{n,k}$ of all k-dimensional linear subspaces of $\boldsymbol{C}^n$. In fact, the $\boldsymbol{C}_0$-action α_X: $\boldsymbol{W}'_{n,k}\times\boldsymbol{C}_0\longrightarrow\boldsymbol{W}'_{n,k}$ is extended to a real analytic right $\boldsymbol{C}_0$-action α_X: $\boldsymbol{C}_0^N\times\boldsymbol{C}_0\longrightarrow\boldsymbol{C}_0^N$. If the matrix X is the identity matrix I_n, then $\alpha_X=\alpha_{I_n}$ is the usual right $\boldsymbol{C}_0$-action on $\boldsymbol{C}_0^N$. Hence its orbit space is the usual $(N-1)$-dimensional complex projective space $P_{N-1}(\boldsymbol{C})$. Then the orbit space $\boldsymbol{G}_{n,k}^{I_n}$ of α_{I_n}: $\boldsymbol{W}'_{n,k}\times\boldsymbol{C}_0\longrightarrow\boldsymbol{W}'_{n,k}$ is an image of the Plücker embedding of the usual complex Grassmannian $\boldsymbol{G}_{n,k}$ into $P_{N-1}(\boldsymbol{C})$ (cf. [1, pp. 209-211]). Thus $\boldsymbol{G}_{n,k}^{I_n}$ may be identified with $\boldsymbol{G}_{n,k}$.

Let $[z]_X=[\mu_k Z]_X$ denote the α_X-orbit through $z=\mu_k Z\in\boldsymbol{W}'_{n,k}$, where $Z\in\boldsymbol{V}'_{n,k}$ and π_X denote the canonical projection of $\boldsymbol{W}'_{n,k}$ onto $\boldsymbol{G}_{n,k}^X$. Then there is the commutative diagram:

(1.5)
$$\begin{array}{ccc} \boldsymbol{W}'_{n,k} & \xrightarrow{F_{\lambda_k X}} & \boldsymbol{W}'_{n,k} \\ \downarrow \pi & & \downarrow \pi_X \\ \boldsymbol{G}_{n,k} & \xrightarrow{\widetilde{F}_{\lambda_k X}} & \boldsymbol{G}'_{n,k}, \end{array}$$

where $\pi=\pi_{I_n}$, $\widetilde{F}_{\lambda_k X}$ is a homeomorphism defined by $\widetilde{F}_{\lambda_k X}([\mu_k Z])=\widetilde{F}_{\lambda_k X}([_{\lambda_k X}(\mu_k Z)]_X$ and $[\mu_k Z]=[\mu_k Z]_{I_n}$ for $Z\in V'_{n,k}$. Now we introduce a C^ω-manifold structure to $\boldsymbol{G}_{n,k}^X$ induced from the usual C^ω-manifold structure of $\boldsymbol{G}_{n,k}$ by the homeomorphism $\widetilde{F}_{\lambda_k X}$: $\boldsymbol{G}_{n,k}\longrightarrow\boldsymbol{G}_{n,k}^X$ and regard $\boldsymbol{G}_{n,k}^X$ as a C^ω-manifold with this structure. Then local expressions of $\widetilde{F}_{\lambda_k X}$: $\boldsymbol{G}_{n,k}\longrightarrow\boldsymbol{G}_{n,k}^X$ are identity mappings of open sets in the $k(n-k)$-dimensional complex Euclidean space $\boldsymbol{C}^{k(n-k)}$. Hence the homeomorphism $\widetilde{F}_{\lambda_k X}$: $\boldsymbol{G}_{n,k}\longrightarrow\boldsymbol{G}_{n,k}^X$ is a C^ω-diffeomorphism. It follows from the commutative diagram (1.5) that $\pi_X=\widetilde{F}_{\lambda_k X}\circ\pi\circ F_{\lambda_k X}^{-1}$. Thus π_X is a C^ω-mapping. Since π: $\boldsymbol{W}'_{n,k}\longrightarrow\boldsymbol{G}_{n,k}$ is a principal fibration induced from the Hopf fibration π: $\boldsymbol{C}_0^N\longrightarrow P_{N-1}(\boldsymbol{C})$ by the Plücker embedding, π_X: $\boldsymbol{W}'_{n,k}\longrightarrow\boldsymbol{G}_{n,k}^X$ is also a principal fibration whose structure group is $\boldsymbol{C}_0=GL(1,\boldsymbol{C})$.

2. Twisted linear actions on complex Grassmannians

2.1. Let G be a Lie group, ρ: $G\longrightarrow GL(n,\boldsymbol{C})$ a matrix representation and X a square $\boldsymbol{C}$-matrix of degree n whose all eigenvalues have positive real parts. We call (ρ, X) a *TC-pair* of *degree n*, if $\rho(g)X=X\rho(g)$ for each $g\in G$. For TC-pair (ρ, X) of degree n, define a C^ω-mapping ξ: $G\times\boldsymbol{G}_{n,k}^X\longrightarrow\boldsymbol{G}_{n,k}^X$ by

$$\xi(g,\ [\mu_k Z]_X)=[\mu_k(\rho(g)Z)]_X,$$

where $Z\in \boldsymbol{V}'_{n,k}$. It is easily seen that ξ is a real analytic G-action on $\boldsymbol{G}^X_{n,k}$. We call $\xi=\xi_{(\rho,X)}$ a *twisted linear action* G on $\boldsymbol{G}^X_{n,k}$ determined by TC-pair (ρ, X) and we say that ξ *is associated* to the matrix representation ρ. Moreover a real analytic G-action ξ^0: $G\times \boldsymbol{G}_{n,k}\longrightarrow \boldsymbol{G}_{n,k}$ is defined by

$$\begin{aligned}\xi^0(g, [\mu_k Z])&=[F^{-1}_{\lambda_k X}(\lambda^k(\rho(g))F_{\lambda_k X}(\mu_k Z))]\\&=[\Psi_{\lambda_k X_0}(\lambda^k(P^{-1}\rho(g)P)\Phi_{\lambda_k X_0}(\mu_k Z))]\\&=[\Psi_{\lambda_k X_0}(\mu_k(P^{-1}\rho(g)P\Phi_{X_0}(Z)))],\end{aligned}$$

where $Z\in \boldsymbol{V}'_{n,k}$, $P\in GL(n, \boldsymbol{C})$ and $X_0=P^{-1}XP$ satisfies the condition (T). The the following diagram is commutative:

$$(2.1)\qquad \begin{array}{ccc} G\times \boldsymbol{G}_{n,k} & \xrightarrow{\ \xi^0\ } & \boldsymbol{G}_{n,k}\\ \Big\downarrow{\scriptstyle 1\times \tilde{F}_{\lambda_k X}} & & \Big\downarrow{\scriptstyle \tilde{F}_{\lambda_k X}}\\ G\times \boldsymbol{G}^X_{n,k} & \xrightarrow{\ \xi\ } & \boldsymbol{G}^X_{n,k}. \end{array}$$

We call also $\xi^0=\xi^0_{(\rho,X)}$ *a twisted linear action* of G on $\boldsymbol{G}_{n,k}$ determined by the TC-pair (ρ, X) and we say that ξ^0 *is associated* to the matrix representation ρ.

2.2. For a given Lie group G, we introduce an equivalence relation on TC-pairs. Let (ρ, X) and (σ, Y) be TC-pairs of degree n. Note that ρ, σ: $G\longrightarrow GL(n, \boldsymbol{C})$ are matrix representations and X, Y are square $\boldsymbol{C}$-matrices of degree n whose all eigenvalues have positive real parts. We say that (ρ, X) *is algebraically equivalent* to (σ, Y), if there exist $A\in GL(n, \boldsymbol{C})$, a positive real number c and a real number d satisfying

$$(2.2)\qquad Y=cAXA^{-1}+\sqrt{-1}dI_n,\ \sigma(g)=A\rho(g)A^{-1}$$

for each element $g\in G$. We say that (ρ, X) is C^r-*equivalent* to (σ, Y), if there exists a C^r-diffeomorphism f: $\boldsymbol{G}^X_{n,k}\longrightarrow \boldsymbol{G}^Y_{n,k}$ $(r=0, 1, 2, \cdots, \infty, \omega)$ such that the following diagram is commutative:

$$\begin{array}{ccc} G\times \boldsymbol{G}^X_{n,k} & \xrightarrow{\xi_{(\rho,X)}} & \boldsymbol{G}^X_{n,k} \\ \Big\downarrow 1\times f & & \Big\downarrow f \\ G\times \boldsymbol{G}^Y_{n,k} & \xrightarrow{\xi_{(\sigma,Y)}} & \boldsymbol{G}^Y_{n,k}. \end{array} \tag{2.3}$$

We call f a *G-equivariant C^r-diffeomorphism.* The following results are proved similarly as in [3, Lemma 1.2, Theorem 1.4].

LEMMA 2.1. *If (ρ, X) is algebraically equivalent to (σ, Y), then (ρ, X) is C^ω-equivalent to (σ, Y).*

THEOREM 2.2. *Let G be a compact Lie group and $\rho: G\longrightarrow GL(n, \boldsymbol{C})$ a matrix representation. Then any TC-pair (ρ, X) is C^ω-equivalent to (ρ, I_n). In other words, any twisted linear action of G on $\boldsymbol{G}_{n,k}$ associated to ρ is equivariantly C^ω-diffeomorphic to the linear action of G on $\boldsymbol{G}_{n,k}$ associated to ρ.*

3. Example

In this section, We shall study twisted linear actions of $G=SL(n, \boldsymbol{C})$ on the complex Grassmannian $\boldsymbol{G}_{nk,m}$ ($n>mk$ and $k>1$) associated to a representation $\rho=\rho_n\otimes I_k$, that is, $\rho(A)=A\otimes I_k$ for each element $A\in G$.

3.1. For $A\in M_n(\boldsymbol{C})$, $B\in M_k(\boldsymbol{C})$, let $A\otimes B$ stand for the Kronecker product which has the form

$$A\otimes B=\begin{pmatrix} b_{11}A & \cdots & b_{1k}A \\ \vdots & & \vdots \\ b_{k1}A & \cdots & b_{kk}A \end{pmatrix}\in M_{nk}(\boldsymbol{C}).$$

We obtain the following lemma.

LEMMA 3.1. *Let $\hat{K}$ be a square $\boldsymbol{C}$-matrix of degree nk. Then the commutativity $\hat{K}(A\otimes I_k)=(A\otimes I_k)\hat{K}$ holds for each $A\in SL(n, \boldsymbol{C})$ if and* *only if* $\hat{K}=I_n\otimes K$, *where* K *is a certain square* $\boldsymbol{C}$-*matrix of degree* k. *It is noted that all the eigenvalues of* $I_n\otimes K$ *have positive real parts if and only if all the eigenvalues of* K *have positive real parts.*

By this lemma, for $\hat{K}\in M_{nk}(\boldsymbol{C})$ whose all eigenvalues have positive real parts, $(\rho_n\otimes I_k, \hat{K})$ is a TC-pair if and only if $\hat{K}=I_n\otimes K$ for $K\in M_k(\boldsymbol{C})$ whose all eigenvalues have positive real parts. Furthermore TC-pairs $(\rho_n\otimes I_k, I_n\otimes K)$ and $(\rho_n\otimes I_k, I_n\otimes L)$ are algebraically equivalent if and only if there exist $X\in GL(k, \boldsymbol{C})$, a positive real number c and a real number d satisfying $L=cXKX^{-1}+\sqrt{-1}dI_k$.

3. 2 Let K be a square $\boldsymbol{C}$-matrix of degree k whose all eigenvalues have positive real parts. Denote by ζ_K the twisted linear $SL(n, \boldsymbol{C})$-action on the complex Grassmannian $\boldsymbol{G}^{\hat{K}}_{nk,m}$ determined by the TC-pair $(\rho_n\otimes I_k, I_n\otimes K)$, where $\hat{K}=I_n\otimes K$. From now on, assume that $n>mk$ and $k>1$. We define a matrix $J^{(i)}_{n,m}\in M(n, m\,;\, \boldsymbol{C})$ by

$$J^{(i)}_{n,m}=\begin{pmatrix} O_i \\ I_m \\ O'_i \end{pmatrix} \quad (i=1,\cdots,k)$$

where $O_i\in M(m(i-1), m\,;\, \boldsymbol{C})$ and $O'_i\in M(n-mi, m\,;\, \boldsymbol{C})$ are zero matrices. Moreover we set

$$Z_0=\begin{pmatrix} J^{(1)}_{n,m} \\ \vdots \\ J^{(k)}_{n,m} \end{pmatrix}\in \boldsymbol{V}'_{nk,m}.$$

With respect to the twisted linear action ζ_K, let $I(K)$ denote the isotropy group at the point

$$[\mu_m Z_0]_{\hat{K}}\in \boldsymbol{G}^{\hat{K}}_{nk,m}$$

and $O(K)$ denote the orbit through this point. Define an injective homomorphism $\phi_K: GL(m, \boldsymbol{C})\longrightarrow GL(mk, \boldsymbol{C})$ by

$$\phi_K(g)=h\otimes\exp(\theta\, {}^tK),$$

where $h=|\det g|^{-1/m}g$, $\theta=\dfrac{1}{m}\log|\det g|$. Then the following lemma is obtained.

LEMMA 3. 2. *Suppose that* $n>mk>m$. *Then*

i) *the isotropy group* $I(K)$ *is written in the form*

$$I(K)=\left\{\begin{pmatrix} \phi_K(g) & \vdots & * \\ \cdots\cdots & \vdots & \cdots \\ 0 & \vdots & * \end{pmatrix} \in SL(n, \boldsymbol{C})\,;\ g\in GL(m, \boldsymbol{C})\right\};$$

ii) *the orbit* $O(K)$ *is equal to*

$$\{[\mu_m((A\otimes I_k)Z_0)]_{\hat{K}}\in \boldsymbol{G}^{\hat{K}}_{nk,m} | A\in SL(n, \boldsymbol{C})\}\,;$$

iii) *the orbit* $O(K)$ *is an open dense subset of* $\boldsymbol{G}^{\hat{K}}_{nk,m}$.

3. 3. The purpose of this section is to prove the following theorem.

THEOREM 3. 3. *Assume that* $n>mk>m$. *Then two of TC-pairs in the form* $(\rho_n\otimes I_k, I_n\otimes K)$ *algebraically equivalent if and only if they are* C^0*-equivalent.*

REMARK. This theorem implies that if, for any positive real number c and any real number d, the matrix $cK+\sqrt{-1}dI_k$ is not similar to L, then ζ_K is not C^0-equivalent to ζ_L, where $K, L\in M_k(\boldsymbol{C})$ whose all eigenvalues have positive real parts. Therefore there are uncountably many topologically inequivalent C^ω-actions of the noncompact Lie group $SL(n, \boldsymbol{C})$ on the complex Grassmannian $\boldsymbol{G}_{nk,m}$ $(n>mk>m)$.

First we prepare two lemmas for the proof.

LEMMA 3. 4. *For* $K\in M_k(\boldsymbol{C})$ *whose all eigenvalues have positive real parts, the homomorphism* ϕ_K ; $GL(m, \boldsymbol{C})\longrightarrow GL(mk, \boldsymbol{C})$ *defined in Subsection* 3. 2 *is an into-homeomorphism.*

PROOF. Set

$$H(m)=\{g\in GL(m, \boldsymbol{C})\,;\ |\det g|=1\}.$$

Then $H(m)$ is a closed subgroup of $GL(m, \boldsymbol{C})$. Define an isomorphism ψ_m of a Lie group $H(m)\times \boldsymbol{R}$ onto $GL(m, \boldsymbol{C})$ by $\psi_m(h, \theta)=e^\theta h$. Moreover define an injective homomorphism $\tilde{\phi}_K$; $H(m)\times\boldsymbol{R}\longrightarrow H(mk)\times\boldsymbol{R}$ by

$$\tilde{\phi}_K(h, \theta)=(h\otimes\exp(\theta\,{}^tM),\ a\theta),$$

where $a=\dfrac{1}{k}(\mathrm{Re}(\mathrm{tr}K))$, $M=K-aI_k$. Then it is easy to see that $\tilde{\phi}_K$ is an into-homeomorphism and the following diagram is commutative;

$$\begin{array}{ccc} H(m)\times \boldsymbol{R} & \xrightarrow{\quad \tilde{\phi}_K \quad} & H(mk)\times \boldsymbol{R} \\ \Big\downarrow \psi_m & & \Big\downarrow \psi_{mk} \\ GL(m, \boldsymbol{C}) & \xrightarrow{\quad \phi_K \quad} & GL(mk, \boldsymbol{C}), \end{array}$$

Hence ϕ_K is an into-homeomorphism. q. e. d.

Set

$$\mathfrak{sl}(m, \boldsymbol{C})=\{X\in M_m(\boldsymbol{C})|\mathrm{tr}X=0\}.$$

We regard $\mathfrak{sl}(m, \boldsymbol{C})$ as a real Lie algebra. For each element $A\in\mathfrak{sl}(m, \boldsymbol{C})$, define an automorphism τ_A of $\mathfrak{sl}(m, \boldsymbol{C})$ by $\tau_A(X)=AXA^{-1}$, where $X\in\mathfrak{sl}(m, \boldsymbol{C})$. Moreover define two automorphisms γ, χ of $\mathfrak{sl}(m, \boldsymbol{C})$ by $\gamma(X)=-{}^tX$, $\chi(X)=\bar{X}$, where $X\in\mathfrak{sl}(m, \boldsymbol{C})$. Then it is easy to see that these automorphisms satisfy the following relations:

$$\gamma\circ\iota_A=\iota_{{}^tA^{-1}}\circ\gamma,\ \chi\circ\iota_A=\iota_{\bar{A}}\circ\chi,\ \gamma\circ\chi=\chi\circ\gamma,\ \gamma^2=\chi^2=1,$$

where $A\in SL(m, \boldsymbol{C})$. Denote by $A_{\boldsymbol{R}}(\mathfrak{sl}(m, \boldsymbol{C}))$ the group of all automorphisms of $\mathfrak{sl}(m, \boldsymbol{C})$. Shin-ichi Watanabe pointed out the following fact to the author.

LEMMA 3.5. *Each element of* $A_{\boldsymbol{R}}(\mathfrak{sl}(m, \boldsymbol{C}))$ $(m\geqq 2)$ *is equal to one of*

$$\iota_A,\ \gamma\circ\iota_A,\ \chi\circ\iota_A,\ \gamma\circ\chi\circ\iota_A$$

for some element $A\in SL(m, \boldsymbol{C})$.

The author learned its proof from him. We refer to [2] for detail.

PROOF OF THEOREM 3.3. Since the only-if part follows immediately from Lemma 2.1, we have only to show the if part. For K, $L\in M_k(\boldsymbol{C})$ whose all eigenvalues have positive real parts, assume that f: $\boldsymbol{G}^{\hat{K}}_{nk,m}\longrightarrow \boldsymbol{G}^{\hat{L}}_{nk,m}$ is an $SL(n, \boldsymbol{C})$-equivariant homeomorphism between twisted linear actions ζ_K and ζ_L, where $n>mk>m$. Since f maps the orbit $O(K)$ onto the orbit $O(L)$, the isotropy subgroup $I(K)$ is conjugate to the isotropy subgroup $I(L)$ by some element T, namely $I(L)=TI(K)T^{-1}$ in $SL(n, \boldsymbol{C})$. Then it is easily shown that

$$T=\begin{pmatrix} C & E \\ O & D \end{pmatrix}$$

for some matrices $C\in GL(mk, \boldsymbol{C})$, $D\in GL(n-mk, \boldsymbol{C})$ and $E\in M(mk, n-mk; \boldsymbol{C})$. Now we assign to each matrix g ($\in GL(m, \boldsymbol{C})$) a matrix g' ($\in GL(m, \boldsymbol{C})$) such that $C\phi_K(g)C^{-1}=\Phi_L(g')$, where C is the above one. This correspondence defines a homomorphism q of $GL(m, \boldsymbol{C})$ onto itself. Moreover q must be a homeomorphism of $GL(m, \boldsymbol{C})$ be Lemma 3. 4. Therefore q is an automorphism of the Lie group $GL(m, \boldsymbol{C})$.

Each element $g\in GL(m, \boldsymbol{C})$ is written in the form $g=\exp(X+zI_m)$, where $X\in\mathfrak{sl}(m, \boldsymbol{C})$, $z=x+\sqrt{-1}y\in\boldsymbol{C}$ ($x, y\in\boldsymbol{R}$). It holds that $q(\exp Y)=\exp(dq(Y))$ for each matrix $Y\in\mathfrak{gl}(m, \boldsymbol{C})=M_m(\boldsymbol{C})$, where dq is the differential of the automorphism q and an automorphism of the real Lie algebra $\mathfrak{gl}(m, \boldsymbol{C})=M_m(\boldsymbol{C})$. It is well known that $\mathfrak{sl}(m, \boldsymbol{C})=[\mathfrak{sl}(m, \boldsymbol{C}), \mathfrak{sl}(m, \boldsymbol{C})]$. Thus it is easy to see that $dq(\mathfrak{sl}(m, \boldsymbol{C}))\subset\mathfrak{sl}(m, \boldsymbol{C})$. On the other hand, it is easily shown that dq maps the center of $\mathfrak{gl}(m, \boldsymbol{C})=M_m(\boldsymbol{C})$ into itself. Hence there exist some element $\eta\in A_{\boldsymbol{R}}(\mathfrak{sl}(m, \boldsymbol{C}))$ and a certain automorphism $\tilde{\omega}$ of the real Lie algebra $\boldsymbol{C}$ such that $(dq)(X+zI_m)=\eta(X)+\tilde{\omega}(z)I_m$ for each matrix $X\in\mathfrak{sl}(\mathrm{m}, \boldsymbol{C})$, each element $z\in\boldsymbol{C}$. Therefore it is obtained that

$$q(\exp(X+zI_m))=\exp(\eta(X)+\tilde{\omega}(z)I_m)$$

for each matrix $X\in\mathfrak{sl}(m, \boldsymbol{C})$, each element $z\in\boldsymbol{C}$. Hence it follows from the definition of q that

$$C(\phi_K(\exp(X+zI_m))C^{-1}=\phi_L(\exp(\eta(X)+\tilde{\omega}(z)I_m))$$

for each matrix $X\in\mathfrak{sl}(m, \boldsymbol{C})$, each element $z\in\boldsymbol{C}$.

Since $\tilde{\omega}$ is a linear bijection of the 2-dimensional real vector space $\boldsymbol{C}$, it is given by the form

$$\tilde{\omega}(z)=ax+cy+\sqrt{-1}(bx+dy)$$

($z=x+\sqrt{-1}y\in\boldsymbol{C}$; x, $y\in\boldsymbol{R}$), where a, b, c, $d\in\boldsymbol{R}$ satisfy that $ad-bc\neq 0$. Hence we obtain that

$$\begin{aligned} &C(\exp(X+\sqrt{-1}yI_m)\otimes\exp(x^tK)) \\ &\qquad=(\exp(\eta(X)+\sqrt{-1}(bx+dy)I_m)\otimes\exp((ax+cy)^tL))C \end{aligned}$$

for each matrix $X\in\mathfrak{sl}(m, \boldsymbol{C})$, arbitrary element x, $y\in\boldsymbol{R}$. By differentiating both sides of this equality, it is obtained that

$$\begin{aligned} &C(X\otimes I_k)+xC(I_m\otimes{}^tK)+\sqrt{-1}yC=(\eta(X)\otimes I_k)C \\ &\qquad+x(I_m\otimes(a^tL+\sqrt{-1}bI_k))C+y(I_m\otimes(c^tL+\sqrt{-1}dI_k))C \end{aligned}$$

for each matrix $X\in\mathfrak{sl}(m, \boldsymbol{C})$, arbitrary element x, $y\in\boldsymbol{R}$.
Thus the equality:

$$(3.1) \qquad C(I_m\otimes{}^tK)=(I_m\otimes(a{}^tL+\sqrt{-1}bI_k))C$$

holds and it follows that

$$(3.2) \qquad C(X\otimes I_k)=(\eta(X)\otimes I_k)C$$

for each matrix $X\in\mathfrak{sl}(m, \boldsymbol{C})$. Moreover it is also seen that $c=0$ and $d=1$.

By Lemma 3.5, the equality (3.2) holds for each matrix $X\in\mathfrak{sl}(m, \boldsymbol{C})$ if and only if there exist matrices $A\in SL(m, \boldsymbol{C})$, $B\in GL(k, \boldsymbol{C})$ such that $\eta=\iota_A$ and $C=A\otimes B$. Hence we obtain

$$L=\left(\frac{1}{a}\right){}^tB^{-1}K{}^tB+\sqrt{-1}\left(-\frac{b}{a}\right)I_k$$

from (3.1). Therefore TC-pairs $(\rho_n\otimes I_k, I_n\otimes K)$ and $\rho_n\otimes I_k, I_n\otimes L)$ are algebraically equivalent. q. e. d.

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