Hokkaido Mathematical Journal Vol. 21 (1992) p. 151-158

An analogy of the theorem of Hector and Duminy

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(Received April 16, 1991, Revised July 30, 1991)

1. Introduction

In this paper, we intend to study qualitative properties of foliations of higher codimensions. For the first step, we consider some analogies of the theory of codimension one foliations.

Let $\mathscr{F}$ be a transversely orientable, codimension one foliation of class C^2 on a closed smooth manifold M. A leaf F of $\mathscr{F}$ is *semiproper* if it is asymptotic to itself from at most one side. In particular, F is called a *nonproper semiproper* leaf if it is asymptotic from exactly one side and this side is called the *nonproper side*.

For nonproper semiproper leaves of $\mathscr{F}$, the following theorems are important.

THEOREM (Sacksteder [6]). *Let M, $\mathscr{F}$ be as above and F a nonproper semiproper leaf of $\mathscr{F}$. Then the closure $\bar{F}$ of F contains a leaf with a linearly contracting holonomy.*

THEOREM (Hector [4], Duminy (unpublished, but see Cantwell-Conlon [3])). *Let M, $\mathscr{F}$ be as above and F a nonproper semiproper leaf of $\mathscr{F}$. Then F has a germinal contracting holonomy on the nonproper side of F.*

For the proof of these theorems, we use the holonomy pseudogroup of $\mathscr{F}$ acting on the real line $\boldsymbol{R}$.

Let $\mathscr{G}$ be a codimension q foliation on a manifold. A foliation $\mathscr{G}$ is called *transversely similar* if all holonomy transition functions of $\mathscr{G}$ are local similarity transformations of $\boldsymbol{R}^q$. Therefore, if $\mathscr{G}$ is a transversely similar foliation of codimension q, we obtain the holonomy pseudogroup of $\mathscr{G}$ which consists of local similarity transformations of $\boldsymbol{R}^q$. So we treat a pseudogroup of local similarity transformations of $\boldsymbol{R}^q$.

Recently, for such pseudogroups, an analogy of Sacksteder's theorem is obtained by Nishimori (see [5], or Section 2). Now we consider an analogy of the theorem of Hector-Duminy.

2. Similarity pseudogroups and statement of the result

In this section, we review about pseudogroups and state out result. For more informations about pseudogroups of our sense, see Nishimori [5].

Let $\Gamma_{q,+}^{r,*}$ ($r=0, 1, \dots, \infty, \omega$) be the set of orientation preserving local C^r diffeomorphisms $h: U \to V$ of $\boldsymbol{R}^q$ satisfying that the domain U and the range V of h are both non-empty, bounded, convex open subsets of $\boldsymbol{R}^q$. We denote $D(h)=U$ and $R(h)=V$. Let $\Gamma_{q,+}^{r}=\Gamma_{q,+}^{r,*}\cup\{\mathrm{id}_{\boldsymbol{R}^q}, \mathrm{id}_{\emptyset}\}$, where $\mathrm{id}_{\emptyset}$ is the unique transformation on the empty set $\emptyset$.

DEFINITION 2.1. A subset Γ of $\Gamma_{q,+}^{r}$ is called a *pseudogroup* if it satisfies the following three conditions:

(1) $\mathrm{id}_{\boldsymbol{R}^q}\in\Gamma$.
(2) If $f, g\in\Gamma$, then $f\circ g\in\Gamma$.
(3) If $f\in\Gamma$, then $f^{-1}\in\Gamma$.

For example, $\Gamma_{q,+}^{r}$ itself is a pseudogroup, but $\Gamma_{q,+}^{r,*}$ is not.

DEFINITION 2.2. Let Γ_0 be a subset of $\Gamma_{q,+}^{r,*}$.

(1) Γ_0 is called *symmetric* if $h\in\Gamma_0$ implies $h^{-1}\in\Gamma_0$.
(2) Denote $\langle\Gamma_0\rangle$ the intersection of all the pseudogroups $\Gamma\subset\Gamma_{q,+}^{r}$ which contain Γ_0. Then $\langle\Gamma_0\rangle$ is also a pseudogroup, which is called the *pseudogroup generated by* Γ_0.

Let Γ_0 be a symmetric subset of $\Gamma_{q,+}^{r,*}$ and $\Gamma=\langle\Gamma_0\rangle$. Denote $W(\Gamma_0)$ the set of all words with Γ_0 as alphabet, that is, $W(\Gamma_0)=\coprod_{n=0}^{\infty}(\Gamma_0)^n$, where $(\Gamma_0)^n$ means n-direct product of Γ_0 and $(\Gamma_0)^0$ the singleton which consists of the empty word ( ). This set $W(\Gamma_0)$ is useful to treat the pseudogroup $\langle\Gamma_0\rangle$, because

PROPOSITION 2.3 ([5], Proposition 2.6). *Define a map $\Phi: W(\Gamma_0)\to\Gamma=\langle\Gamma_0\rangle$ by $\Phi(( ))=\mathrm{id}_{\boldsymbol{R}^q}$ for the empty word ( ) and $\Phi(w)=h_m\circ\cdots\circ h_1$ for a word $w=(h_m, \cdots, h_1)$. Then this map Φ is surjective.*

For a word $w=(h_m, \cdots, h_1)\in W(\Gamma_0)$, we put $g_w=\Phi(w)=h_m\circ\cdots\circ h_1$. Note that for the *inverse word* $w^{-1}=(h_1^{-1}, \cdots, h_m^{-1})$ of w, $g_w^{-1}=g_{w^{-1}}=\Phi(w^{-1})=h_1^{-1}\circ\cdots\circ h_m^{-1}$.

DEFINITION 2.4. Let $x_0\in\boldsymbol{R}^q$. The Γ-*orbit of* x_0 is the set $\Gamma(x_0)=\{g(x_0)\mid g\in\Gamma, x_0\in D(g)\}$.

Af first, we consider in the case of $q=1$ and $r=2$, which is related to holonomy pseudogroups of codimension one foliations of class C^2.

Let $\Gamma\subset\Gamma^2_{1,+}$ be a finitely generated pseudogroup, that is, a pseudogroup of local C^2 diffeomorphisms of $\boldsymbol{R}$.

DEFINITION 2.5. (1) The Γ-orbit $\Gamma(x_0)\subset\boldsymbol{R}$ of $x_0\in\boldsymbol{R}$ is called *proper* if for every $x\in\Gamma(x_0)$, the closure $\overline{\Gamma(x_0)\backslash\{x\}}$ does not contain x, that is, $\Gamma(x_0)$ is discrete. Otherwise, $\Gamma(x_0)$ is called *nonproper*.

(2) $\Gamma(x_0)$ is called *exceptional* if the closure $\overline{\Gamma(x_0)}$ is a perfect set with empty interior. Clearly, each exceptional orbit is nonproper.

To investigate the structure of the closure of a Γ-orbit is an important problem.

DEFINITION 2.6. The Γ-orbit $\Gamma(x_0)$ of $x_0\in\boldsymbol{R}$ is called *semiproper* if for every $x\in\Gamma(x_0)$, there exists an open interval $J\subset\boldsymbol{R}$ such that x is a boundary point of J and $J\cap\Gamma(x_0)=\emptyset$. Therefore a semiproper orbit is either proper or exceptional.

PROPOSITION 2.7. *The Γ-orbit $\Gamma(x_0)$ of $x_0\in\boldsymbol{R}$ is exceptional if and only if for some (and thus any) $x\in\Gamma(x_0)$, there exists a compact neighborhood I_x of x in $\boldsymbol{R}$ such that $\overline{\Gamma(x_0)}\cap I_x$ is a Cantor set. Furthermore, if $\Gamma(x_0)$ is a semiproper orbit of exceptional type, then for every $x\in\Gamma(x_0)$, x is a semi-isolated point of the Cantor set $\overline{\Gamma(x_0)}\cap I_x$.*

For semiproper orbits, the following theorems are important (compare with theorems in Introduction):

THEOREM 2.8 (Sacksteder [6]). *Suppose that $\Gamma\subset\Gamma^2_{1,+}$ is a pseudogroup generated by a finite, symmetric subset $\Gamma_0\subset\Gamma^{2;*}_{1,+}$ and $x_0\in\boldsymbol{R}$ satisfying that $\Gamma(x_0)$ is a nonproper, semiproper orbit and there exists a constant $\varepsilon>0$ such that the distance* $\mathrm{dist}(\Gamma(x_0), \cup_{h\in\Gamma_0}\partial D(h))$ *is greater than ε. Then there exists $x\in\overline{\Gamma(x_0)}$ and $g\in\Gamma$ such that $x\in D(g)$, $g(x)=x$ and g is a (hyperbolic) contraction to x, that is, the derivative $g'(x)$ at x is less than* 1.

THEOREM 2.9 (Hector [4], Duminy (unpublished, but see Cantwell-Conlon [3])). *On the same assumptions of Theorem 2.8, there exists $g\in\Gamma$ such that $x_0\in D(g)$, $g(x_0)=x_0$ and g is a contraction to x_0 on the nonproper side.*

To consider analogies of these theorems for $q\geq 2$, we work in a restricted category of pseudogroups, namely, pseudogroups of local *similarity transformations* of $\boldsymbol{R}^q$.

Let $\Gamma^{\mathrm{sim};*}_{q,+}$ be the subset of $\Gamma^{\omega;*}_{q,+}$ so that for every $h\in\Gamma^{\omega;*}_{q,+}$, there exists an orientation preserving similarity transformation $\bar{h}: \boldsymbol{R}^q \to \boldsymbol{R}^q$ such that $\bar{h}(D(h))=R(h)$ and the restriction $\bar{h}|_{D(h)}=h$. Such $\bar{h}$ is determined uniquely by h, and is called the *extension* of h. Put $\Gamma_{q,+}^{\mathrm{sim}}=\Gamma_{q,+}^{\mathrm{sim},*}\cup\{\mathrm{id}_{\boldsymbol{R}}, \mathrm{id}_{\emptyset}\}$.

Let $\Gamma\subset\Gamma_{q,+}^{\mathrm{sim}}$ be a finitely generated pseudogroup.

DEFINITION 2.10. The Γ-orbit $\Gamma(x_0)$ of $x_0\in\boldsymbol{R}^q$ is called *proper* if for every $x\in\Gamma(x_0)$, $\overline{\Gamma(x_0)\setminus\{x\}}$ does not contain x. Otherwise, $\Gamma(x_0)$ is called *nonproper*.

In order to consider analogies of the theorems of Sacksteder and Hector-Duminy, we have to introduce a substitute concept of "semiproper Γ-orbits". As one attempt, Nishimori introduced the concept of "Γ-orbits with bubbles".

DEFINITION 2.11 ([5], Definition 3.2). Let $x_0\in\boldsymbol{R}^q$. We say that the Γ-orbit $\Gamma(x_0)$ of x_0 is *with bubbles* if for each $x\in\Gamma(x_0)$, there exists a non-empty, bounded, convex open subset B_x (called a *bubble at* x) of $\boldsymbol{R}^q$ satisfying the following three properties:

(a) $x\in\partial B_x$, where ∂B_x denotes the boundary of B_x.

(b) If $x_1, x_2\in\Gamma(x_0)$ and $x_1\neq x_2$, then $B_{x_1}\cap B_{x_2}=\emptyset$.

(c) If $h\in\Gamma_0$ and $x\in D(h)\cap\Gamma(x_0)$ satisfying $h(x)\neq x$, then $\bar{h}(B_x)=B_{h(x)}$, where $\bar{h}$ is the extension of h.

EXAMPLE. Let D^q be the unit disk in $\boldsymbol{R}^q$, $x_0\in\partial D^q=S^{q-1}$ and $D_1,\cdots,D_n$ $(n\geq 2)$ mutually disjoint disks contained in D^q and $\partial D_1\ni x_0$. Let $\bar{h}_i(i=1,\cdots,n)$ be a similarity transformation which maps the unit disk D^q to the disk D_i and $\bar{h}_1(x_o)=x_0$. Let h_i be a suitable restriction of $\bar{h}_i$ such that the domains of h_i are bounded, convex open neighborhoods of D^q and the ranges of h_i are mutually disjoint. (Clearly each h_i is a contraction.) Now we obtain a pseudogroup $\Gamma=\langle\Gamma_0\rangle\subset\Gamma_{q,+}^{\mathrm{sim}}$, where $\Gamma_0=\{h_1,\ldots,h_n,h_1^{-1},\ldots,h_n^{-1}\}$. Then the Γ-orbit $\Gamma(x_0)$ is with bubbles and the closure $\overline{\Gamma(x_0)}$ is a Cantor set in $\boldsymbol{R}^q$. Furthermore h_1 is a contraction to $x_0\in\Gamma(x_0)$. This construction is closely related to that of exceptional minimal sets of Markov type for $q=1$ (see Cantwell-Conlon [2]).

Hereafter, we consider the following situation.

Let $\Gamma_0\subset\Gamma_{q,+}^{\mathrm{sim},*}$ be a finite, symmetric subset, $\Gamma=\langle\Gamma_0\rangle$ and $x_0\in\boldsymbol{R}^q$ satisfying the following two properties:

(S1) There exists a constant $\epsilon>0$ such that the distance $\mathrm{dist}(\Gamma(x_0), \bigcup_{h\in\Gamma_0}\partial D(h))$ is greater than ϵ.

(S2) The Γ-orbit $\Gamma(x_0)$ of x_0 is nonproper and with bubbles $\{B_x\}_{x\in\Gamma(x_0)}$.

Remark that if $x \in \Gamma(x_0) \cap D(h)$ for some $h \in \Gamma_0$, then by (S1), $U(x;\epsilon) \subset D(h)$, where $U(x;\epsilon)$ denotes the ϵ-neighborhood of x.

Then an analogy of Sacksteder's theorem is as follows.

THEOREM 2.12 (Nishimori [5], Theorem 3.3). *Let Γ be the pseudogroup generated by a finite, symmetric subset Γ_0 of $\Gamma_{q,+}^{\mathrm{sim},*}$ and $x_0 \in \boldsymbol{R}^q$ satisfying the assumptions* (S1) *and* (S2). *Then there exist $g \in \Gamma$ and $z \in \overline{\Gamma(x_0)}$ such that $z \in D(g)$, $g(z)=z$ and g is a contraction, that is, the similitude ratio of g is less than 1.*

We prove, in the rest of this paper, the following result which is a weak version of an analogy of the theorem of Hector-Duminy.

THEOREM 2.13. *Let Γ be the pseudogroup generated by a finite, symmetric subset Γ_0 of $\Gamma_{q,+}^{\mathrm{sim},*}$ and $x_0 \in \boldsymbol{R}^q$ satisfying the assumptions* (S1) *and* (S2). *Then there exists $g \in \Gamma$ such that $x_0 \in D(g)$, $g(x_0)=x_0$ and g is not the identity of $D(g)$.*

REMARK. Therefore, such g is possibly a rotation at x_0. We do not know whether there exists an example that all elements of Γ which fix x_0 are rotations at x_0.

3. The proof of Theorem 2.13

Let Γ be the pseudogroup generated by a finite, symmetric subset Γ_0 of $\Gamma_{q,+}^{\mathrm{sim},*}$ and $x_0 \in \boldsymbol{R}^q$ satisfying the assumptions (S1) and (S2). Let $\{B_x\}_{x \in \Gamma(x_0)}$ be bubbles of $\Gamma(x_0)$.

At first, we prepare some notions which play an important role in the proof of Theorem 2.13.

DEFINITION 3.1. (1) For a word $w \in W(\Gamma_0)$, $|w|$ denotes the word length of w, that is, $|w|=0$ for the empty word $w=(\ )$ and $|w|=m$ for $w=(h_m, \ldots, h_1)$.

(2) For x, $y \in \boldsymbol{R}^q$ with $y \in \Gamma(x)$, put

$$d_{\Gamma_0}(x, y)=\min\{|w| \mid w \in W(\Gamma_0),\ x \in D(g_w) \text{ and } g_w(x)=y\}.$$

Then d_{Γ_0} is a natural distance on the orbit $\Gamma(x)$.

DEFINITION 3.2. Let x, $y \in \boldsymbol{R}^q$. A word $w \in W(\Gamma_0)$ is called a *short-cut at x to y* if $x \in D(g_w)$, $g_w(x)=y$ and $|w|=d_{\Gamma_0}(x, y)$.

Remark that if $w=(h_m, \ldots, h_1) \in W(\Gamma_0)$ is a short-cut at x to y, then the inverse word $w^{-1}=(h_1^{-1}, \ldots, h_m^{-1})$ of w is a short-cut at y to x and for every $k=1, \ldots, m-1$, the word $w_k=(h_k, \ldots, h_1)$ is a short-cut at x to $g_{w_k}(x)$

$=h_k \circ \cdots \circ h_1(x)$.

Following three lemmas are fundamental and for the proofs, see Nishimori [**5**].

LEMMA 3.3 ([**5**], Lemma 4.3). *Let* $x \in \Gamma(x_0)$ *and* $w=(h_m, \ldots, h_1) \in W(\Gamma_0)$ *be a short-cut at* x. *Then* $\bar{g}_w(B_x)=B_{g_w(x)}$, *where* $g_w=h_m \circ \cdots \circ h_1$ *and* $\bar{g}_w$ *is the extension of* g_w (*in the sense of Section 2*). *Therefore the similitude ratio of* g_w *is the ratio of the diameters of bubbles,* $\operatorname{diam}(B_{g_w(x)})/\operatorname{diam}(B_x)$. *In particular, if* $D(g_w) \supset U(x\,;r)$, *then*

$$g_w(U(x\,;r))=U\left(g_w(x)\,;r\cdot\frac{\operatorname{diam}(B_{g_w(x)})}{\operatorname{diam}(B_x)}\right).$$

LEMMA 3.4 ([**5**], Lemma 4.4, 4.5). (1) *The union* $\bigcup_{x \in \Gamma(x_0)} B_x$ *of bubbles is a bounded subset of* $\boldsymbol{R}^q$.
(2) *The total volume* $\sum_{x \in \Gamma(x_0)} \operatorname{vol}(B_x)$ *of bubbles is bounded. So* $\sum_{x \in \Gamma(x_0)} (\operatorname{diam}(B_x))^q$ *is also bounded.*

LEMMA 3.5 (The short-cut theorem. [**5**], Lemma 4.7). *Let* $w \in W(\Gamma_0)$ *be a short-cut at* x_0. *Then*

$$U\left(x_0\,;\epsilon\cdot\frac{\operatorname{diam}(B_{x_0})}{\delta}\right) \subset D(g_w),$$

where $\delta=\sup\{\operatorname{diam}(B_y) \mid y \in \Gamma(x_0)\}$.

For the proof of our theorem, the following argument is essentially due to Hector [**4**, Théorème CIII 1] in the case of $q=1$.

Put $\Delta=\{y \in \Gamma(x_0) \mid \operatorname{diam}(B_y) \geqq \operatorname{diam}(B_{x_0})\}$, then by Lemma 3.4, it is a non-empty, finite subset of $\Gamma(x_0)$ which contains x_0. Since the pseudo-group Γ is finitely generated and Δ is finite, so there exists a non-negative integer $N=\sup\{d_{\Gamma_0}(x, y) \mid x, y \in \Delta\}$.

LEMMA 3.6. *There exists* $\epsilon'>0$ *such that*

(1) $\epsilon/3 \geqq \epsilon'>0$,

(2) $d_{\Gamma_0}(x_0, z)>N$ *for each* $z \in U(x_0\,;\epsilon'\cdot\operatorname{diam}(B_{x_0})/\delta)$ *with* $z \in \Gamma(x_0)\backslash\{x_0\}$. *Therefore* $z \notin \Delta$.

PROOF. Since Γ is finitely generated, the set $\{y \in \Gamma(x_0) | d_{\Gamma_0}(x_0, y) \leqq N\}$ is finite. By assumption, the orbit $\Gamma(x_0)$ is nonproper, so we can take $\epsilon'>0$ satisfying (1) and (2). □

Hereafter we assume that

(#) for each $g \in \Gamma$ which fixes x_0, g is the identity on $D(g)$

and deduce a contradiction.

LEMMA 3.7. *Let $\epsilon'>0$ be a constant as in Lemma 3.6 and $z\in U(x_0;\epsilon'\cdot\mathrm{diam}(B_{x_0})/\delta)$ with $z\in\Gamma(x_0)\setminus\{x_0\}$. Let $w\in W(\Gamma_0)$ be a short-cut at x_0 to z. Then $x_0\in D(g_w^{-1})$ and w^{-1} is a short-cut at x_0 to $g_w^{-1}(x_0)$.*

PROOF. Note that the word length $|w|=d_{\Gamma_0}(z, x_0)>N$. By assumption, $w^{-1}\in W(\Gamma_0)$ is a short-cut at z to x_0.

We write $w^{-1}=(h_m, \dots, h_1)$ $(m\geqq 1,\ h_i\in\Gamma_0)$, and put $w_k^{-1}=(h_k, \dots, h_1)$ and $g_k=g_{w_k}^{-1}=g_{w_k^{-1}}=h_k\circ\cdots\circ h_1$ for $k=1, 2, \dots, m$. And, for convention, $w_0^{-1}=(\ )$ (the empty word) and $g_0=g_{w_0^{-1}}=\mathrm{id}_{R^q}$. Then w_k^{-1} is a short-cut at z to $g_k(z)$ for $k=0, 1, \dots, m$.

We prove the following assertions by induction on $k=0, 1, \dots, m$:

$(\mathrm{A})_k$: $\quad U\left(x_0\,;\,\epsilon'\cdot\dfrac{\mathrm{diam}(B_{x_0})}{\delta}\right)\subset D(g_k)$.

$(\mathrm{B})_k$: $\quad$ The word w_k^{-1} is a short-cut at x_0 to $g_k(x_0)$.

For $k=0$, all assertions are trivial.

Assume that the assertions $(\mathrm{A})_k$ and $(\mathrm{B})_k$ hold true for $k\geqq 0$. By the choice of $z\in U(x_0;\epsilon'\cdot\mathrm{diam}(B_{x_0})/\delta)$ and $(\mathrm{A})_k$,

$$\begin{aligned} g_k(z)&\in g_k\left(U\left(x_0\,;\,\epsilon'\cdot\frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\right)\\ &=U\left(g_k(x_0)\,;\,\epsilon'\cdot\left(\frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\cdot\left(\frac{\mathrm{diam}(B_{g_k(x_0)})}{\mathrm{diam}(B_{x_0})}\right)\right)\\ &=U\left(g_k(x_0)\,;\,\epsilon'\cdot\frac{\mathrm{diam}(B_{g_k(x_0)})}{\delta}\right)\\ &\subset U(g_k(x_0)\,;\,\epsilon'). \end{aligned}$$

Since $g_k(x)\in D(h_{k+1})\cap\Gamma(x_0)$, $U(g_k(z)\,;\,\epsilon)\subset D(h_{k+1})$ by (S1). Therefore

$$\begin{aligned} g_k\left(U\left(x_0\,;\,\epsilon'\cdot\frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\right)&\subset U(g_k(x_0)\,;\,\epsilon')\\ &\subset U(g_k(z)\,;\,\epsilon)\\ &\subset D(h_{k+1}). \end{aligned}$$

Then $U(x_0\,;\,\epsilon'\cdot\mathrm{diam}(B_{x_0})/\delta)\subset D(h_{k+1}\circ g_k)=D(g_{k+1})$. This establishes the assertion $(\mathrm{A})_{k+1}$.

In order to prove the assertion $(\mathrm{B})_{k+1}$, we take a short-cut $\zeta\in W(\Gamma_0)$ at x_0 to $g_{k+1}(x_0)$. Then $g_\zeta^{-1}\circ g_{k+1}(x_0)=x_0$, so $g_\zeta=g_{k+1}$ on $D(g_\zeta)\cap D(g_{k+1})$ by assumption (#).

Since w_{k+1}^{-1} is a short-cut at z, then $z\in D(g_{k+1})$ and by Lemma 3.5 and the choice of ϵ', $z\in U(x_0\,;\,\epsilon'\cdot\mathrm{diam}(B_{x_0})/\delta)\subset D(g_\zeta)$. Therefore

$z\in D(g_{\zeta})\cap D(g_{k+1})$.

By the definition of a short-cut,

$$|w_{k+1}^{-1}|=d_{\Gamma_0}(z,\ g_{k+1}(z))\leqq|\zeta|=d_{\Gamma_0}(x_0,\ g_{k+1}(x_0))\leqq|w_{k+1}^{-1}|,$$

so $|w_{k+1}^{-1}|=d_{\Gamma_0}(x_0,\ g_{k+1}(x_0))$, that is, w_{k+1}^{-1} is a short-cut at x_0 to $g_{k+1}(x_0)$. This establishes the assertion $(\mathrm{B})_{k+1}$, and completes the induction.

Now $(\mathrm{B})_m$ is the desired result. □

Remark that $g_w^{-1}(x_0)\notin\Delta$. This is because $d_{\Gamma_0}(x_0,\ g_w^{-1}(x_0))=|w^{-1}|=d_{\Gamma_0}(x_0,\ z)>N$.

By Lemma 3.7, the word w^{-1} is a short-cut at $z\notin\Delta$ to $x_0\in\Delta$ and is also a short-cut at $x_0\in\Delta$ to $g_w^{-1}(x_0)\notin\Delta$. Then, by Lemma 3.3, the similitude ratio of g_w^{-1} is

$$\frac{\mathrm{diam}(B_{x_0})}{\mathrm{diam}(B_z)}=\frac{\mathrm{diam}(B_{g_w^{-1}(x_0)})}{\mathrm{diam}(B_{x_0})}.$$

But the definition of the set Δ yields

$$1<\frac{\mathrm{diam}(B_{x_0})}{\mathrm{diam}(B_z)}=\frac{\mathrm{diam}(B_{g_w^{-1}(x_0)})}{\mathrm{diam}(B_{x_0})}<1,$$

a contradiction. This completes the proof of Theorem 2.13.

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