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A qualitative theory of similarity pseudogroups and an analogy of Sacksteder's theorem

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1. Introducion

The qualitative theory of foliations has been developed for foliations of codimension one (see Sacksteder [7], Cantwell-Conlon [1] and Hector [4] for example). Now we intend to study qualitative properties of foliations of higher codimensions. Note that all the non-singular dynamical systems can be considered as foliations and there are numberless researches on the qualitative theory of dynamical systems. Such researches are not our intention. So we must make our purpose more concrete. The most typical result in the qualitative theory of codimension one foliations is the following theorem.

THEOREM (Sacksteder's Theorem, see Sacksteder [7]). *Let $\mathscr{F}$ be a codimension one C^2 foliation of a closed manifold M, and $\mathscr{M}\subset M$ an exceptional minimal set with respect to $\mathscr{F}$. Then there exists a leaf F of $\mathscr{F}$ contained in $\mathscr{M}$ such that F has a contracting element in its linear holonomy group* LHol(F).

We demand that our intended study should contain an analogy of the above theorem, and look for an appropriate and simple category of foliations on which we should work. A natural idea is to consider foliations with transverse geometric structure (see Godbillon [3] for example). The automorphism groups of the appropriate geometric structures are requested to contain contracting elements for an expected analogy of Sacksteder's theorem. These considerations guide us to investigate foliations with transverse similarity structure (see Ghys [2] and Nishimori [6]).

In this paper, we are going to treat similarity pseudogroups Γ on $\boldsymbol{R}^q$ in place of codimension q foliations $\mathscr{F}$ with tranverse similarity structure. As is well known, there exist natural correspondences between the terms in the qualitative theories of these objects. For example, one considers Γ-orbits in place of leaves of $\mathscr{F}$, and the stabilizer at a point in a Γ-orbit in place of the holonomy group of a leaf of $\mathscr{F}$. It is easy to translate results on pseudogroups to those on foliations. The reason why we treat pseudogroups is to avoid the ambiguities completely and to make the skeleton of our arguments simple and apparent.

The plan of this paper is as follows. In §2 we give our formulation for similarity pseudogroups. In §3, we introduce a concept " Γ-orbits with bubbles " and state our main theorem (Theorem 3.3), which is an analogy of Sacksteder's theorem. In §4, we prove this theorem.

2. Similarity pseudogroups and the qualitative theory

In this section, we give a convenient formulation of similarity pseudogroups for our purpose. This formulation makes the arguments simple and avoids the ambiguities (for example, those on the domains of elements of pseudogroups) but does not lose the generality of phenomena in the view point of the qualitative theory.

DEFINITION 2.1. (1) Denote by $\Gamma_{q,+}^{\mathrm{sim},*}$ the set of homeomorphisms h: $U \to V$ satisfying the following conditions:

(a) The domain U and the range V of h are non-empty, bounded, convex, open subsets of $\boldsymbol{R}^q$. (We denote $D(h)=U$ and $R(h)=V$.)

(b) There exists an orientation preserving similarity transformation $\bar{h}: \boldsymbol{R}^q \to \boldsymbol{R}^q$ such that $\bar{h}(U)=V$ and $\bar{h}|_U=h$. (Such $\bar{h}$ is determined uniquely by h and we call $\bar{h}$ the *extension* of h.)

(2) Let $\Gamma_{q,+}^{\mathrm{sim}}=\Gamma_{q,+}^{\mathrm{sim},*}\cup\{\mathrm{id}_{\boldsymbol{R}^q}, \mathrm{id}_\emptyset\}$, where $\mathrm{id}_\emptyset$ is the unique transformation on the empty set $\emptyset$. (We bring in the transformation $\mathrm{id}_\emptyset$ in order that we can cosider the composition for any paris of elements of pseudogroups and make the description simple.)

DEFINITION 2.2. (1) For $f, g\in\Gamma_{q,+}^{\mathrm{sim}}$, let $U=g^{-1}(R(g)\cap D(f))$ and $V=f(R(g)\cap D(f))$, and difine the *composition* $f\circ g: U \to V$ by

$$(f\circ g)(x)=f(g(x)) \quad \text{for all } x\in U.$$

(2) For $f\in\Gamma_{q,+}^{\mathrm{sim}}$, let $U=R(f)$ and $V=D(f)$, and define the *inverse* $f^{-1}: U \to V$ by

$$f^{-1}(f(x))=x \quad \text{for all } x\in V.$$

(Note that if f, $g\in\Gamma_{q,+}^{\mathrm{sim}}$ then $f\circ g$, $f^{-1}\in\Gamma_{q,+}^{\mathrm{sim}}$.)

DEFINITION 2.3. A subset Γ of $\Gamma_{q,+}^{\mathrm{sim}}$ is called a *pseudogroup* if it satisfies the following conditions:

(a) $\mathrm{id}_{\boldsymbol{R}^q} \in \Gamma$.
(b) If $f, g \in \Gamma$, then $f \circ g \in \Gamma$.
(c) If $f \in \Gamma$, then $f^{-1} \in \Gamma$.

(Noe that $\Gamma_{q,+}^{\mathrm{sim}}$ is itself a pseudogroup.)

DEFINITION 2.4. Let Γ_0 be a subset of $\Gamma_{q,+}^{\mathrm{sim},*}$.
(1) Γ_0 is called *symmetric* if $h \in \Gamma_0$ implies $h^{-1} \in \Gamma_0$.
(2) Denote by $\langle \Gamma_0 \rangle$ the intersection of all the pseudogroups $\Gamma \subset \Gamma_{q,+}^{\mathrm{sim}}$ which contain Γ_0. (Cleary $\langle \Gamma_0 \rangle$ is a pseudogroup.) We call $\langle \Gamma_0 \rangle$ the *pseudo-group generated by* Γ_0.

Hereafter let Γ_0 be a symmetric subset of $\Gamma_{q,+}^{\mathrm{sim},*}$, and $\Gamma = \langle \Gamma_0 \rangle$.

DEFINITION 2.5. (1) Denote by $W(\Gamma_0)$ the set of words with Γ_0 as the alphabet. In order to distinguish a word from a composition, we prefer to write a word $w \in W(\Gamma_0)$ in such a way as $w = (h_m, \cdots, h_1)$ rather than $w = h_m \cdots h_1$. In this way, we identify $W(\Gamma_0)$ with the disjoint union $\coprod_{n=0}^{\infty} (\Gamma_0)^m$, where $(\Gamma_0)^m$ denotes the product of m-copies of Γ_0 and $(\Gamma_0)^0$ is the singleton consisting of the empty word ( ).
(2) For $w = (h_m, \cdots, h_1) \in (\Gamma_0)^m (m \geqq 1)$, let $g_w = h_m \circ \cdots \circ h_1$. For the empty word ( ), let $g_{(\,)} = \mathrm{id}_{\boldsymbol{R}^q}$.

The following proposition gives a description of elements of the pseudogroup Γ generated by the symmetric subset $\Gamma_0 \subset \Gamma_{q,+}^{\mathrm{sim},*}$.

PROPOSITION 2.6. (1) *For each $w \in W(\Gamma_0)$, $g_w \in \Gamma = \langle \Gamma_0 \rangle$.*
(2) *The map $\Phi: W(\Gamma_0) \to \Gamma$ defined by*

$$\Phi(w) = g_w \quad \textit{for all } w \in W(\Gamma_0)$$

is surjective.

PROOF: (1) is clear. (2) follows from the assumption that Γ_0 is symmetric. □

The terms in the qualitative theory are defined as follows.

DEFINITION 2.7. For $x \in \boldsymbol{R}^q$, we call

$$\Gamma(x) = \{g(x) : g \in \Gamma,\ x \in D(g)\}$$

the Γ-*orbit* of x. (Note that $x \in \Gamma(x)$.)
(2) A subset E of $\boldsymbol{R}^q$ is called a Γ-*orbit* if there exists $x \in \boldsymbol{R}^q$ with $E = \Gamma(x)$.

DEFINITION 2.8. A subset $A \subset \boldsymbol{R}^q$ is called Γ-*invariant* if, for any $x \in A$, the Γ-orbit $\Gamma(x)$ is contained in A.

DEFINITION 2.9. A subset $\mathscr{M} \subset \boldsymbol{R}^q$ is called a Γ-*minimal set* if $\mathscr{M}$ is a minimal element of the set of closed, non-empty, Γ-invariant subsets of $\boldsymbol{R}^q$ partially ordered by the inclusions.

The concept for a Γ-orbit corresponding to the limit set of a leaf of a foliation is the derived set in the following.

DEFINITION 2.10. For a subset A of $\boldsymbol{R}^q$, denote by $\mathrm{Der}(A)$ the set of the points $y \in \boldsymbol{R}^q$ such that there exists a sequence $x_1, x_2, \cdots \in A - \{y\}$ with $y = \lim_{n\to\infty} x_n$. We call $\mathrm{Der}(A)$ the *derived set* of A.

DEFINITION 2.11. A Γ-orbit E is called *infinite* if $\#(E) = \infty$, *bounded* if E is bounded as a subset of $\boldsymbol{R}^q$, and *proper* if $E \cap \mathrm{Der}(E) = \emptyset$.

We give the following propositions as typical examples of the propositions in the qualitative theory of similarity pseudogroups, and omit the other natural propositions in it.

PROPOSITION 2.12. *If a subset A of $\boldsymbol{R}^q$ is Γ-invariant, then so are the interior* $\mathrm{Int}(A)$, *the closure $\bar{A}$ and the derived set* $\mathrm{Der}(A)$.

PROOF: This follows from the standard arguments. □

PROPOSITION 2.13. *If a Γ-orbit E is infinite and bounded, then the derived set* $\mathrm{Der}(E)$ *contains a compact Γ-minimal set.*

PROOF: The assumption implies that the derived set $\mathrm{Der}(E)$ is non-empty, compact and Γ-invariant. Hence the proposition follows from Zorn's lemma. □

3. Statement of the main theorem

The purpose of this section is to describe briefly how we reach the concept "Γ-orbits with bubbles" and to state our main result.

We are going to find an object corresponding to an exceptional mimimal set $\mathscr{M}$ of a codimension one foliation. Note that a boundary leaf of such $\mathscr{M}$ is non-compact, non-proper and semi-proper. We begin by describing a Γ-orbit which may be considered as an analogy of such a leaf.

Hereafter let Γ be the pseudogroup generated by a finite symmetric subset Γ_0 of $\Gamma_{q,+}^{\mathrm{sim},*}$ and x_0 a point in the bounded Γ-invariant open subset $\Omega := \bigcup_{h \in \Gamma_0} D(h)$ of $\boldsymbol{R}^q$ such that the Γ-orbit $\Gamma(x_0)$ is infinite and non-proper. Since $\Gamma(x_0) \subset \Omega$, it follows that $\Gamma(x_0)$ is bounded.

An observation on the holonomy pseudogroup of an exceptional minimal set of a codimension one foliation leads us to make the following natural assumption, which can always be satisfied for such a holonomy pseudogroup.

ASSUMPTION (S). *There exists a constant* $\varepsilon>0$ *such that the distance* $\mathrm{dist}(\Gamma(x_0), \Delta)$ *is greater than* ε, *where* $\Delta=\bigcup_{h\in\Gamma_0}\partial D(h_0)$.

Note that Δ is a compact subset of $\boldsymbol{R}^q$. This assumption (S) implies that the closure $\overline{\Gamma(x_0)}$ is compact and contained in the open subset $\Omega-\Delta$ of $\boldsymbol{R}^q$.

In order to obtain a result analogous to Sacksteder's theorem on codimension one foliations, we must look for a point x in the closure $\overline{\Gamma(x_0)}$ such that there exists a contracting element in the stabilizer $\Gamma_x:=\{g\in\Gamma : x\in D(g), g(x)=x\}$, which is the concept for the Γ-orbit $\Gamma(x)$ corresponding to the holonomy group of a leaf of a codimension one foliation. Here we investigate the following two examples.

EXAMPLE 3.1. Consider the case $q=2$.
(1) Let $U=]-\varepsilon, 1+\varepsilon[\times]-\varepsilon, 1+\varepsilon[$ for some $\varepsilon\in]0, 1/100[$. Take four points

$$x_0=(0,0),\ x_1=(1,0),\ x_2=(1,1),\ x_3=(0,1)\in\boldsymbol{R}^2$$

and define similarity transformations $\bar{h}_0, \bar{h}_1, \bar{h}_2, \bar{h}_3 : \boldsymbol{R}^2\to\boldsymbol{R}^2$ by

$$\bar{h}_i(x)=\frac{1}{3}(x-x_i)+x_i \quad \text{for all } x\in\boldsymbol{R}^2,$$

and let $h_i=\bar{h}_i|_U : U\to\bar{h}_i(U)$. Denote by Γ the pseudogroup generated by the finite symmetric subset

$$\Gamma_0:=\{h_0,\cdots,h_3,h_0^{-1},\cdots,h_3^{-1}\}\subset\Gamma_{2,+}^{\mathrm{sim},*}.$$

It is easy to see that $\overline{\Gamma(x_0)}=\overline{\Gamma(x_1)}=\overline{\Gamma(x_2)}=\overline{\Gamma(x_3)}=C\times C$, where C is the standard Cantor set. Note that the stabilizer Γ_{x_0} contains the contracting element h_0.
(2) Let $U=\{x\in\boldsymbol{R}^2 : \|x\|<1+\varepsilon\}$ for some $\varepsilon\in]0, 1/100[$. Take an irrational rotation $\bar{h}:\boldsymbol{R}^2\to\boldsymbol{R}^2$ fixing the origin $0\in\boldsymbol{R}^2$ and let $h=\bar{h}|_U : U\to U$. Denote by Γ the pseudogroup generated by $\Gamma_0:=\{h, h^{-1}\}$ and put $x_0=(1,0)\in\boldsymbol{R}^2$. Then we see that

$$\overline{\Gamma(x_0)}=S^1:=\{x\in\boldsymbol{R}^2 : \|x\|=1\}.$$

Clearly the stabilizer Γ_{x_0} contains no contracting element.

The first example is affirmative for our problem but the second one is not. By watching carefully these examples (and also boundary leaves of exceptional minimal sets of codimension one foliations), we find the difference: the first example admits "bubbles" defined below and the second one does not.

DEFINITION 3.2. A Γ-orbit $E\subset \boldsymbol{R}^q$ is called *with bubbles* if, for each $x\in E$, there exists a non-empty, bounded, convex, open subset B_x (called a *bubble* at x) of $\boldsymbol{R}^q$ satisfying the following conditions:

(a) $x\in\partial B_x$, where $\partial B_x := \overline{B_x} - B_x$.
(b) $B_x\cap B_y=\emptyset$ if $x\neq y$.
(c) If $h\in\Gamma_0$ and $x\in D(h)\cap E$ satisfy $h(x)\neq x$, then $\bar{h}(Bx)=B_{h(x)}$, where $\bar{h}$ is the extension of h.

One can easily find "bubbles" for the Γ-orbit $\Gamma(x_0)$ in Example 3.1 (1) and cannot in the case of Example 3.1 (2). Our main result is the following theorem.

THEOREM 3.3 (An Analogy of Sacksteder's Theorem). *Let Γ be the pseudogroup generated by a finite symmetric subset Γ_0 of $\Gamma_{q,+}^{\mathrm{sim},*}$ and x_0 a point in the union $\Omega := \bigcup_{h\in\Gamma_0} D(h)$ such that the Γ-orbit $\Gamma(x_0)$ is infinite and non-proper. Suppose that the condition* (S) *is satisfied and the Γ-orbit $\Gamma(x_0)$ is with bubbles. Then there exists a point x in the closure $\overline{\Gamma(x_0)}$ such that the stabilizer Γ_x contains a contracting element.*

We consider this result as a starting point for the qualitative theory of foliations of higher codimension. Now we have two immediate ways to proceed. One is to prove this theorem in the more general situation. The other is to prove an analogy of another theorem in the qualitative theory of codimension one foliations. An attempt in this way is done by Matsuda [4].

4. The proof of Theorem 3.3.

Let Γ be the pseudogroup generated by a finite symmetric subset Γ_0 of $\Gamma_{q,+}^{\mathrm{sim},*}$ and x_0 a point in $\Omega=\bigcup_{h\in\Gamma_0} D(h)$ such that the orbit $\Gamma(x_0)$ is infinite and non-proper. Suppose that the condition (S) is satisfied and the Γ-orbit $\Gamma(x_0)$ is with bubbles $\{B_x\}_{x\in\Gamma(x_0)}$.

We begin by some definitions.

DEFINITION 4.1. (1) For a word $w\in W(\Gamma_0)$, denote by $|w|$ the word length of w; that is, $|w|=m$ if $w=(h_m,\cdots,h_1)$, and $|w|=0$ if w is the empty word ( ).

(2) For $x, y \in \boldsymbol{R}^q$ with $y \in \Gamma(x)$, put

$$d_{\Gamma_0}(x, y) = \min\{|w| : w \in W(\Gamma_0),\ x \in D(g_w) \text{ and } g_w(x) = y\}.$$

(Distinguish $d_{\Gamma_0}(x, y)$ from the Euclidean distance $\|x - y\|$.)

DEFINITION 4.2. Let $x, y \in \boldsymbol{R}^q$. A word $w \in W(\Gamma_0)$ is called a *short-cut* at x to y if $x \in D(g_w)$, $g_w(x) = y$ and $|w| = d_{\Gamma_0}(x, y)$.

The bubbles $\{B_x\}_{x \in \Gamma(x_0)}$ are preserved by short-cuts as follows.

LEMMA 4.3. *Let $x, y \in \Gamma(x_0)$ be distinct points and $w = (h_m, \cdots, h_1)$ a short-cut at x to y. Then $\bar{g}_w(B_x) = B_y$, where $\bar{g}_w$ is the extension of $g_w = h_m \circ \cdots \circ h_1$.*

PROOF: For $i = 1, \cdots, m$, let $y_i = h_i \circ \cdots \circ h_1(x) \in \Gamma(x_0)$ and put $y_0 = x \in \Gamma(x_0)$. Since w is a short-cut at x, the points $x = y_0, \cdots, y_m = y$ are pairwise distinct. Clearly $h_{i+1} \in \Gamma_0$ and $y_i \in D(h_{i+1}) \cap \Gamma(x_0)$ for $i = 0, \cdots, m-1$. Therefore

$$\begin{aligned} \bar{g}_w(B_x) &= \bar{h}_m \circ \cdots \circ \bar{h}_2 \circ \bar{h}_1(B_{y_0}) \\ &= \bar{h}_m \circ \cdots \circ \bar{h}_2(B_{y_1}) \\ &= \cdots = \bar{h}_m(B_{y_{m-1}}) = B_{y_m} = B_y. \end{aligned}$$

This completes the proof of Lemma 4.3. □

The following is a key observation.

LEMMA 4.4. *The union $B := \bigcup_{x \in \Gamma(x_0)} B_x$ is bounded.*

PROOF. Take a cone C_{x_0} with x_0 as vertex such that $\mathrm{Int}(C_{x_0}) \subset B_{x_0}$. For each $x \in \Gamma(x_0)$, choose a short-cut $w \in W(\Gamma_0)$ at x_0 to x, and put $C_x = \bar{g}_w(C_{x_0})$. Then C_x is a cone with x as vertex and $\mathrm{Int}(C_x) \subset B_x$. Note that C_x is simirlar to C_{x_0} and the similarity ratio of C_x to C_{x_0} coincides with that of B_x to B_{x_0}. We proceed by intuitive arguments. Take a very large sphere S with x_0 as center. We may suppose that S is suffiiently large in such a way that the union $\Omega = \bigcup_{h \in \Gamma_0} D(h)$ can be almost identified with x_0. There exists a large sphere S' with x_0 as center such that if a bubble B_x intersects S, then the correspoding cone C_x intersects S'. By taking S' of an appropriate size, we may suppose that, for all the points $x \in \Gamma(x_0)$ with $B_x \cap S \neq \emptyset$, the intersections $C_x \cap S'$ are almost congruent and so their q-1 dimensional volumes have almost the same positive value v. The number of such x's is finite since it is almost overestimated by the ratio of the volume of S' to the value v. Hence the union $B = \bigcup_{x \in \Gamma(x_0)} B_x$

is contained in the union of the disk surrounded by S and a finite number of bubbles, which implies that B is bounded. □

As an application of Lemma 4.4, we have the following.

LEMMA 4.5. (1) $\sum_{x\in\Gamma(x_0)} \mathrm{vol}(B_x)<\infty$.
(2) $\sum_{x\in\Gamma(x_0)}(\mathrm{diam}(B_x))^q<\infty$.
(3) There exists a sequence $\{\mu_n\}_{n=1}^\infty$ of positive numbers such that

(a) $\mu_n>\mu_{n+1}$ *for all* $n\in\boldsymbol{N}$,
(b) $\lim_{n\to\infty}\mu_n=0$,
(c) *if* $w\in W(\Gamma_0)$ *is a short-cut at* x_0, *then* $\mathrm{diam}(B_{g_w(x_0)})<\mu_n$, *when* $n=d_{\Gamma_0}(x_0, g_w(x_0))=|w|$.

PROOF. (1) This follows directly from Lemma 4.4.
(2) Since the bubbles $\{B_x\}_{x\in\Gamma(x_0)}$ are similar, the volumes $\{\mathrm{vol}(B_x)\}_{x\in\Gamma(x_0)}$ are directly proportional to the numbers $\{(\mathrm{diam}(B_x))^q\}_{x\in\Gamma(x_0)}$. Hence (1) implies (2).
(3) For $n\in\boldsymbol{N}$, let

$$\delta_n=\sup\{\mathrm{diam}(B_x):\ x\in\Gamma(x_0),\ d_{\Gamma_0}(x, x_0)\geqq n\},$$

which is not infinity because

$$(\delta_n)^q\leqq\sum_{x\in\Gamma(x_0)}(\mathrm{diam}(B_x))^q<\infty.$$

Since the sequence $\delta_1, \delta_2, \delta_3, \cdots$ is weakly decreasing and has a lower bound 0, there exists the limit $\delta_\infty:=\lim_{n\to\infty}\delta_n\geqq 0$. If $\delta_\infty>0$, then there exists an infinite number of $x\in\Gamma(x_0)$ with $\mathrm{diam}(B_x)\geqq\delta_\infty/2$, which contradicts the inequality in (2). Hence $\delta_\infty=0$. Now put $\mu_n=\delta_n+1/n$ for each $n\in\boldsymbol{N}$. It is easy to see that the sequence $\{\mu_n\}_{n=1}^\infty$ satisfies the conditions (a), (b) and (c). □

In contrast with the action of a group of diffeomorphisms on a manifold, we must always worry about the domains of elements in the pseudogroup Γ, which occupies an important part in our arguments. Here we give a lemma which follows immediately from the assumption (S). For $x\in\boldsymbol{R}^q$ and $r>0$, put $U(x;r)=\{y\in\boldsymbol{R}^q:\ \|y-x\|<r\}$.

LEMMA 4.6. *If* $h\in\Gamma_0$ *and* $x\in\Gamma(x_0)\cap D(h)$, *then* $U(x;\varepsilon)\subset D(h)$.

PROOF: Suppose that $U(x;\varepsilon)\not\subset D(h)$ and take a point $y\in U(x;\varepsilon)-D(h)$. Since $U(x;\varepsilon)$ is a convex subset of $\boldsymbol{R}^q$ containing the points x and y, the line segment L connecting x and y lies in $U(x;\varepsilon)$. Since $x\in D(h)$ and $y\notin D(h)$, the line segment L must intersect $\partial D(h)$.

Hence $U(x\,;\varepsilon)\cap\partial D(h)\neq\emptyset$. This contradicts the inequality $\mathrm{dist}(\Gamma(x_0),\Delta)>\varepsilon$ in the condition (S). □

The following lemma is an analogy of a useful lemma in Sacksteder [7]. Let $\delta=\sup\{\mathrm{diam}(B_x):x\in\Gamma(x_0)\}$.

LEMMA 4.7. (The Short-cut Theorem). *If $w\in W(\Gamma_0)$ is a short-cut at x_0, then*

$$U\left(x_0\,;\varepsilon\cdot\frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\subset D(g_w).$$

PROOF: We proceed by an induction on $m=|w|$.

(I) If $m=1$, then $h:=g_w$ is an element of the generating set Γ_0. Since $\mathrm{diam}(B_{x_0})/\delta\leqq 1$ and $x_0\in\Gamma(x_0)\cap D(h)$, Lemma 4.6 implies that

$$U\left(x_0\,;\varepsilon\cdot\frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\subset U(x_0\,;\varepsilon)\subset D(h)=D(g_w).$$

(II) Suppose that Lemma 4.7 is satisfied for short-cuts of word length less than m. For a short-cut $w=(h_m,\cdots,h_1)\in W(\Gamma_0)$ at x_0, let $w'=(h_{m-1},\cdots,h_1)$ and $g'=g_{w'}$. Note that w' is also a short-cut at x_0 and that $g'(x_0)\in\Gamma(x_0)\cap D(h_m)$. By the induction hypothesis, it follows that $U(x_0\,;\varepsilon\cdot\mathrm{diam}(B_{x_0})/\delta)\subset D(g')$ and the following computation has the meaning:

$$\begin{aligned} g'\left(U\left(x_0\,;\varepsilon\cdot\frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\right)&=U\left(g'(x_0)\,;\varepsilon\cdot\frac{\mathrm{diam}(B_{x_0})}{\delta}\cdot\frac{\mathrm{diam}(B_{g'(x_0)})}{\mathrm{diam}(B_{x_0})}\right)\\ &\subset U(g'(x_0)\,;\varepsilon)\\ &\subset D(h_m). \end{aligned}$$

This implies that

$$U\left(x_0\,;\varepsilon\cdot\frac{\mathrm{diam}(B_{x_0})}{\delta}\right)\subset D(h_m\circ g')=D(g_w).\quad\square$$

Now we are in the final stage of the proof of Theorem 3.3. Let $\varepsilon_0=\varepsilon\cdot\mathrm{diam}(B_{x_0})/\delta$ and take $n\in\boldsymbol{N}$ with $\mu_n<\mathrm{diam}(B_{x_0})/3$. Since the Γ-orbit $\Gamma(x_0)$ is non-proper, there exists a point $x\in(\Gamma(x_0)-\{x_0\})\cap U(x_0\,;\varepsilon_0/3)$ at a short-cut $w=(h_m,\cdots,h_1)\in W(\Gamma_0)$ at x_0 such that $g_w(x_0)=x$ and $m\geqq n$. Then

$$\mathrm{diam}(B_x)=\mathrm{diam}(B_{g_w(x_0)})<\mu_m\leqq\mu_n<\frac{\mathrm{diam}(B_{x_0})}{3}.$$

By Lemma 4.7, the domain $D(g_w)$ contains $U(x_0\,;\varepsilon_0)$. It follows that

$$g_w(U(x_0\,;\,\varepsilon_0))=U\left(g_w(x_0)\,;\,\varepsilon_0\cdot\frac{\operatorname{diam}(B_{g_w(x_0)})}{\operatorname{diam}(B_{x_0})}\right)$$
$$\subset U\left(x\,;\,\frac{\varepsilon_0}{3}\right)$$
$$\subset U(x_0\,;\,\frac{2}{3}\cdot\varepsilon_0).$$

Hence, according to the Brouwer fixed point theorem, there exists a point $z\in U(x_0\,;2\varepsilon_0/3)$ fixed by g_w. Furthermore we see that the similitude ratio of g_w is smaller than one. Therefore g_w is a contracting element of Γ and

$$z=\lim_{k\to\infty}(g_w)^k(x_0)\in\overline{\Gamma(x_0)}.$$

This completes the proof of Theorem 3.3.

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