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A new algorithm derived from the view-point of the fluctuation-dissipation principle in the theory of KM_2O-Langevin equations

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§ 1. Introduction and statements of results

We have constructed in [4] a theory of KM_2O-Langevin equations for multi-dimensional weakly stationary processes with discrete time, and from the view-point of the so-called fluctuation-dissipation theorem in irreversible statistical physics ([2]), we have established *a fluctuation-dissipation theorem* which gives a relation between the fluctuant and deter -ministic terms in the KM_2O-Langevin equation. Such *a fluctuation-dissipation theorem* had already been found as *the Levinson-Whittle-Wiggins-Robinson algorithm* for *the fitting problem of AR-models* in the field of system, control and information ([3], [1], [10], [11]). Sublimating a certain philosophical structure behind our fluctuation-dissipation theorem to form *the fluctuation-dissipation principle*, we have applied the theory of KM_2O-Langevin equations to data analysis and developped *a stationary analysis* as well as *a causal analysis* ([7], [6]). Furthermore, on these lines, we have solved the non-linear prediction problem for one-dimensional strictly stationary processes with discrete time and developped *a prediction analysis* as our third project in data analysis ([5], [9], [8]).

Let $\mathbf{X}=(X(n); n\in\mathbf{Z})$ be an $\mathbf{R}^d$-valued weakly stationary process on a probability space $(\Omega, \mathscr{B}, P)$ with expectation vector zero and covariance matrix function R:

$$(1.1) \qquad R(m-n)\equiv E(X(m)\,{}^tX(n)) \qquad (m, n\in\mathbf{Z}),$$

where d is any fixed natural number.

For each $n\in\mathbf{N}$, a block Toeplitz matrix $S_n\in M(nd; \mathbf{R})$ is defined by

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$$(1.2)\qquad S_n\equiv\begin{pmatrix} R(0) & R(1) & \cdots & R(n-1)\\ R(-1) & R(0) & \cdots & R(n-2)\\ \vdots & \vdots & \ddots & \vdots\\ R(-(n-1)) & R(-(n-2)) & \cdots & R(0)\end{pmatrix}.$$

In this paper we shall assume the Toeplitz condition:

$$(1.3)\qquad S_n\in GL(nd\,;\,\mathbf{R})\quad \text{for any}\quad n\in\mathbf{N}.$$

It then follows from the theory of KM_2O-Langevin equations that the time evolution *in the future* (resp. *past*) of the process $\mathbf{X}$ is governed by *the forward* (resp, *backward*) *KM_2O-Langevin equation* (1. 5_+) (resp. (1. 5_-)) with (1. 4):

$$(1.4)\qquad X(0)=\nu_+(0)=\nu_-(0)$$

$$(1.5_+)\qquad X(n)=-\sum_{k=1}^{n-1}\gamma_+(n,k)X(k)-\delta_+(n)X(0)+\nu_+(n)\qquad (n\in\mathbf{N})$$

$$(1.5_-)\qquad X(-n)=-\sum_{k=1}^{n-1}\gamma_-(n,k)X(-k)-\delta_-(n)X(0)+\nu_-(-n)\quad (n\in\mathbf{N}).$$

Here the random force $\boldsymbol{\nu}_+=(\nu_+(l)\,;\,l\in\mathbf{N}^*)$ (resp. $\boldsymbol{\nu}_-=(\nu_-(l)\,;\,l\in-\mathbf{N}^*)$ is said to be *the forward* (resp. *backward*) *KM_2O-Langevin force* associated with $\mathbf{X}$. We call the system $\{\gamma_\pm(n,k),\delta_\pm(m),V_\pm(l)\,;\,l\in\mathbf{N}^*,k,m,n\in\mathbf{N},n>k\}$, whose elements belong to $M(d\,;\,\mathbf{R})$, *the KM_2O-Langevin data* associated with the covariance matrix function R of $\mathbf{X}$, where $V_\pm(l)$ are the covariance matrices of KM_2O-Langevin forces $\nu_\pm(\pm l)$ $(l\in\mathbf{N}^*)$:

$$(1.6)\qquad V_+(l)\equiv E(\nu_+(l)\,{}^t\nu_+(l))\quad\text{and}\quad V_-(l)\equiv E(\nu_-(-l)\,{}^t\nu_-(-l)).$$

In particular, the subsystem $\{\delta_\pm(n)\,;\,n\in\mathbf{N}\}$ is called *the partial autocorrelation coefficient* in the field of system, control and information.

We are now ready to formulte the fluctuation-dissipation theorem mentioned above:

Dissipation-Dissipation Theorem ([3], [1], [10], [11], [4]). *For any $n,k\in\mathbf{N}$, $n>k$,*

$$(1.7_\pm)\qquad \gamma_\pm(n,k)=\gamma_\pm(n-1,k-1)+\delta_\pm(n)\gamma_\mp(n-1,n-k-1),$$

where

$$(1.8)\qquad \gamma_+(n,0)\equiv\delta_+(n)\quad\textit{and}\quad \gamma_-(n,0)\equiv\delta_-(n).$$

Fluctuation-Dissipation Theorem ([3], [1], [10], [11], [4]). *For any $n\in\mathbf{N}$,*

$$(1.9_\pm)\qquad V_\pm(n)=(I-\delta_\pm(n)\delta_\mp(n))\,V_\pm(n-1)$$

(1.10) $\delta_-(n)V_+(n-1)=V_-(n-1)^t\delta_+(n)$
(1.11) $\delta_-(n)V_+(n)=V_-(n)^t\delta_+(n)$.

Recalling the theory of KM_2O-Langevin equations, we should note that the relations $(1.9_\pm)$–(1.11) can be derived from the following *Burg's relation*:

Burg's relation ([3], [1], [10], [11], [4]). *For any* $n\in\mathbf{N}$,

$$(1.12)\qquad \sum_{k=0}^{n-1}\gamma_+(n,k)R(k+1)=\sum_{k=0}^{n-1}R(k+1)^t\gamma_-(n,k).$$

As will be shown in § 2, we can paraphrase Burg's relation in terms of the KM_2O-Langevin forces $\boldsymbol{\nu}_\pm$:

$$(1.13)\qquad E(\nu_+(n)^t\nu_-(-1))=E(\nu_+(1)^t\nu_-(-n))\quad (n\in\mathbf{N}^*).$$

From our view-point of the fluctuation-dissipation principle, relation (1.13) should be regarded as a special case of *the fluctuation-fluctuation theorem*, relations among the mutual covariance matrix functions $I(m,n)$ of the forward and backward KM_2O-Langevin forces $\boldsymbol{\nu}_\pm$:

$$(1.14)\qquad I(m,n)\equiv E(\nu_+(m)^t\nu_-(-n))\qquad (m,n\in\mathbf{N}^*).$$

The purpose of this paper is to prove these relations that will be used to build a useful algorithm in applications to data analysis. The precise statement of our results is as follows:

Fluctuation-Fluctuation Theorem.
(i) $I(0,0)=V_+(0)$
(ii) $I(m,0)=I(0,m)=0\qquad (m\in\mathbf{N})$
(iii) $I(m,1)=I(1,m)=-\delta_+(m+1)V_-(m)\quad (m\in\mathbf{N})$
(iv)
$$I(m,n)=I(m+1,n-1)+\left\{\sum_{k=1}^{n-2}I(m+1,k)^t\delta_+(k+1)\right\}^t\delta_-(n)-$$
$$-\delta_+(m+1)\left\{\sum_{k=1}^{m-1}\delta_-(k+1)I(k,n)\right\}\quad (m,n\geq 2).$$

This theorem has already been announced in [5] and [9], and we can easily form an algorithm to compute all values of $I(m,n)$. In a separate paper, we shall give further discussions to assert that *the fluctuation-fluctuation theorem*, together with *the dissipation-dissipation* and *fluctuation-dissipation theorerm*, yields a characterization of the weak stationarity of a stochastic process $\mathbf{X}$ in terms of the KM_2O-Langevin forces $\boldsymbol{\nu}_\pm$.

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§ 2. Proof of Fluctuation-Fluctuation Theorem

For any fixed natural number d, let $\mathbf{X}=(X(n)\,;n\in\mathbf{Z})$ be an $\mathbf{R}^d$-valued weakly stationary process as in § 1. Let us recall the definition of KM_2O-Langevin forces. For any d-dimensional stochastic process $\mathbf{Y}=({}^t(Y_1(n),\cdots,Y_d(n))\,;\,l\leq n\leq r)$ on the basic probability space $(\Omega,\mathscr{B},P)$ $(-\infty\leq l<r\leq\infty)$, we difine, for each n_1, n_2, $l\leq n_1\leq n_2\leq r$, the closed subspace $\mathbf{M}_{n_1}^{n_2}(\mathbf{Y})$ of $L^2(\Omega,\mathscr{B},P)$ by

$$(2.1)\qquad \mathbf{M}_{n_1}^{n_2}(\mathbf{Y})\equiv\text{the closed linear hull of }\{Y_j(n)\,;1\leq j\leq d,\ n_1\leq n\leq n_2\}.$$

Then the forward (resp. backward) KM_2O-Langevin force $\boldsymbol{\nu}_+=(\nu_+(n)\,;n\in\mathbf{N}^*)$ (resp. $\boldsymbol{\nu}_-=(\nu_-(l)\,;l\in-\mathbf{N}^*)$) is an $\mathbf{R}^d$-valued stochastic process given by

$$(2.2)\qquad \begin{cases}\nu_+(n) \equiv X(n)-P_{\mathbf{M}_0^{n-1}(\mathbf{X})}X(n) & (n\in\mathbf{N}^*)\\ \nu_-(-n) \equiv X(-n)-P_{\mathbf{M}_{-n+1}^{0}(\mathbf{X})}X(-n) & (n\in\mathbf{N}^*),\end{cases}$$

where $\mathbf{M}_0^{-1}(\mathbf{X})=\mathbf{M}_1^0(\mathbf{X})=\{0\}$ and $P_{\mathbf{M}_0^{n-1}(\mathbf{X})}$ (resp. $P_{\mathbf{M}_{-n+1}^{0}(\mathbf{X})}$) stands for the orthogonal projection to the space $\mathbf{M}_0^{n-1}(\mathbf{X})$ (resp. $\mathbf{M}_{-n+1}^0(\mathbf{X})$). We have

(2.3) $\nu_+(0)=\nu_-(0)=X(0)$

(2.4) The stochastic processes $\boldsymbol{\nu}_\pm$ are orthogonal with mean vector zero

(2.5) $\mathbf{M}_0^n(\mathbf{X})=\mathbf{M}_0^n(\boldsymbol{\nu}_+)$ and $\mathbf{M}_{-n}^0(\mathbf{X})=\mathbf{M}_{-n}^0(\boldsymbol{\nu}_-)$ $\quad(n\in\mathbf{N}^*)$.

As stated in § 1, the stochastic process $\mathbf{X}$ satisfies the forward (resp. backward) KM_2O-Langevin equation (1.5_+) (resp. (1.5_-)). The dissipation-dissipation theorem $(1.7_\pm)$ and the fluctuation-dissipation theorem $(1.9_\pm)-(1.11)$ are known relations among the KM_2O-Langevin data. On the other hand, the fundamental quantities $\delta_\pm(\cdot)$ can be calculated from the covariance matrix function R by the following algorithm:

Partial Autocorrelation Coefficient ([3], [1], [10], [11], [4]). *For any $n\in\mathbf{N}$,*

$$(2.6_\pm)\qquad \delta_\pm(n)=-\left\{R(\pm n)+\sum_{k=0}^{n-2}\gamma_\pm(n-1,k)R(\pm(k+1))\right\}V_\mp(n-1)^{-1}.$$

Now we are giving to prove Fluctuation-Fluctuation Theorem.

(Step 1) We begin with observing that Burg's relation (1.12) is equivalent to a special case of the fluctuation-fluctuation theorem: for any $m\in\mathbf{N}$,

$$I(m,1)=I(1,m).$$

Multiplying both-hand sides of equation (1.5+) with $n=m$ by ${}^tX(-1)$ from the right and taking an expectation with respect to P, we have

$$(2.7)\qquad R(m+1)=-\sum_{k=0}^{m-1}\gamma_+(m,k)R(k+1)+E(\nu_+(m){}^tX(-1)).$$

Noting that the weak stationarity of $\mathbf{X}$ implies that $R(m+1)=E(X(m){}^tX(-1))=E(X(1){}^tX(-m))$, we then multiply both-hand sides of equation (1.5-) with taking the transpose and putting $n=m$ by $X(1)$ from the left, and similarly obtain

$$(2.8)\qquad R(m+1)=-\sum_{k=0}^{m-1}R(k+1){}^t\gamma_-(m,k)+E(X(1){}^t\nu_-(-m)).$$

Therefore, we apply Burg's relation (1.12) to (2.7) and (2.8), and get

$$(2.9)\qquad E(\nu_+(m){}^tX(-1))=E(X(1){}^t\nu_-(-m)).$$

On the other hand, it follows immediately from ($1.5_\pm$) and (2.3)–(2.5) that

$$(2.10)\qquad E(\nu_+(m){}^tX(-1))=I(m,1)\quad\text{and}\quad E(X(1){}^t\nu_-(-m))=I(1,m).$$

Hence Step 1 follows from (2.9) and (2.10).

(Step 2) We claim that for any $m\in\mathbf{N}$,

$$I(m,1)=I(1,m)=-\delta_+(m+1)V_-(m).$$

Immediately from (2.6_+), we have

$$R(m+1)=-\sum_{k=0}^{m-1}\gamma_+(m,k)R(k+1)-\delta_+(m+1)V_-(m),$$

which, with (2.7) and (2.10), completes the proof of Step 2.

(Step 3) We claim that for any $n\in\mathbf{N}$,

$$\text{(i)}\qquad V_+(n)=R(0)+\sum_{k=0}^{n-1}R(n-k){}^t\gamma_+(n,k)$$

$$\text{(ii)}\qquad V_-(n)=R(0)+\sum_{k=0}^{n-1}\gamma_-(n,k)R(n-k).$$

These are easy versions of (4.5) and (4.6) in the proof of Lemma 4.2 in [4]. Actually we can directly derive them by multipyling both-hand sides of equations ($1.5_\pm$) by ${}^tX(\pm n)$ from the right, taking an expectation with respect to P and using (2.3)–(2.5).

(Step 4) For any $m, n\in\mathbf{N}^*$, put

(2. 11) $$F_n(m) \equiv R(n) + \sum_{k=1}^{m} \gamma_-(m, m-k) R(n+k).$$

Then, rewriting (ii) in Step 3, we have

$$F_0(m) = V_-(m).$$

(Step 5) We are now in a position to prove the following by mathematical induction with respect to n. For any $m, n \in \mathbf{N}$,

$$R(m+n) = -\sum_{k=1}^{n} \delta_+(m+k) F_{n-k}(m+k-1) - \sum_{k=0}^{m-1} \gamma_+(m, k) R(k+n).$$

By (2. 6_+), we have

$$R(m+n) = -\delta_+(m+n)\, V_-(m+n-1) - \sum_{k=0}^{m+n-2} \gamma_+(m+n-1, k) R(k+1),$$

which, using Step 4, implies that Step 5 holds for any $m \in \mathbf{N}$ and $n=1$. Let us assume that Step 5 holds for any $m \in \mathbf{N}$ and $n = n_0 - 1$, $n_0 \geq 2$. It then follows that

(2. 12) $$R(m+n_0) = R((m+1)+(n_0-1)) = \\ = -\sum_{k=1}^{n_0-1} \delta_+(m+1+k) F_{n_0-1-k}(m+k) - \sum_{k=0}^{m} \gamma_+(m+1, k) R(k+n_0-1).$$

By relation (1. 7_+) in the dissipation-dissipation theorem, we have

(2. 13) $$\sum_{k=0}^{m} \gamma_+(m+1, k) R(k+n_0-1) \\ = \delta_+(m+1) F_{n_0-1}(m) + \sum_{k=0}^{m-1} \gamma_+(m, k) R(k+n_0).$$

Therefore, we see from (2. 12) and (2. 13) that Step 5 holds for any $m \in \mathbf{N}$ and $n = n_0$. Hence, we complete the proof of Step 5 by mathematical induction.

(Step 6) We claim that for any $m, n \in \mathbf{N}$,

$$E(\nu_+(m)\, {}^tX(-n)) = -\sum_{k=1}^{n} \delta_+(m+k) F_{n-k}(m+k-1).$$

Taking an analogous manipulation when we got (2. 7) from equation (1. 5_+), we have

$$R(m+n) = -\sum_{k=0}^{m-1} \gamma_+(m, k) R(k+n) + E(\nu_+(m)\, {}^tX(-n)),$$

which, with Step 5, yields Step 6.

(Step 7) We claim that for any $m, n \in \mathbf{N}$, $m \geq 2$,

$$F_{n-1}(m)=F_{n-1}(m-1)+\delta_-(m)E(\nu_+(m-1)^t X(-n)).$$

Applying (1. 7$_-$) to each term $\gamma_-(m, m-k)$ $(1 \leq k \leq m-1)$ in the definition of $F_{n-1}(m)$, and using Step 5, we have

$$\begin{aligned}F_{n-1}(m) &= R(n-1)+\delta_-(m)R(n-1+m)+\\ &\quad +\sum_{k=1}^{m-1}(\gamma_-(m-1, m-1-k)+\delta_-(m)\gamma_+(m-1, k-1))R(n-1+k)\\ &= F_{n-1}(m-1)+\delta_-(m)\left\{R(n-1+m)+\sum_{l=0}^{m-2}\gamma_+(m-1, l)R(n+l)\right\}\\ &= F_{n-1}(m-1)-\delta_-(m)\left\{\sum_{k=1}^{n}\delta_+(m-1+k)F_{n-k}(m+k-2)\right\}.\end{aligned}$$

Hence, Step 7 follows from Step 6.

(Step 8) We claim that for any $m, n \in \mathbf{N}$, $m \geq 2$,

$$\begin{aligned}E(\nu_+(m)^t X(-n)) &= E(\nu_+(m+1)^t X(-n+1))-\\ &\quad -\delta_+(m+1)\left\{\sum_{k=2}^{m}\delta_-(k)E(\nu_+(k-1)^t X(-n))\right\}-\\ &\quad -\delta_+(m+1)F_{n-1}(1).\end{aligned}$$

By Step 6, we have

$$E(\nu_+(m)^t X(-n))=E(\nu_+(m+1)^t X(-n+1))-\delta_+(m+1)F_{n-1}(m).$$

Hence, a repeat substitution of Step 7 into the last term above concludes Step 8.

(Step 9) We claim that for any $m, n \in \mathbf{N}$, $m, n \geq 2$,

$$\begin{aligned}E(\nu_+(m)^t \nu_-(-n)) &= E(\nu_+(m+1)^t X(-n+1))-\\ &\quad -\delta_+(m+1)\left\{\sum_{k=2}^{m}\delta_-(k)E(\nu_+(k-1)^t \nu_-(-n))\right\}-\\ &\quad -\delta_+(m+1)F_{n-1}(1)+\sum_{l=0}^{n-1}\{E(\nu_+(m+1)^t X(-l+1))-\\ &\quad -\delta_+(m+1)F_{l-1}(1)\}^t\gamma_-(n, l).\end{aligned}$$

Substituing the right-hand side of equation (1. 5$_-$) into the terms including $X(-n)$ in Step 8, we have

$$\begin{aligned}E(\nu_+(m)^t \nu_-(-n)) &= E(\nu_+(m+1)^t X(-n+1))-\\ &\quad -\delta_+(m+1)\left\{\sum_{k=2}^{m}\delta_-(k)E(\nu_+(k-1)^t \nu_-(-n))\right\}-\delta_+(m+1)F_{n-1}(1)+\\ &\quad +\sum_{l=1}^{n-1}\left\{E(\nu_+(m)^t X(-l))+\delta_+(m+1)(\sum_{k=2}^{m}\delta_-(k)E(\nu_+(k-1)^t X(-l)))\right\}^t\gamma_-(n, l).\end{aligned}$$

On the other hand, it follows also from Step 8 that

$$\text{the coefficient of } {}^t\gamma_-(n, l) \text{ in the above equality} \\ = E(\nu_+(m+1)^t X(-l+1)) - \delta_+(m+1)F_{l-1}(1).$$

Hence, we get Step 9.

(Step 10) We claim that for any $m, n \in \mathbf{N}$, $n \geq 2$,

$$E(\nu_+(m+1)^t X(-n+1)) + \sum_{l=1}^{n-1} E(\nu_+(m+1)^t X(-l+1))^t\gamma_-(n, l) \\ = E(\nu_+(m+1)^t\nu_-(-n+1)) + \sum_{k=1}^{n-2} E(\nu_+(m+1)^t\nu_-(-k))^t(\delta_-(n)\delta_+(k+1)).$$

It is easy to see from (2. 2)−(2. 5) that Step 10 holds for $n=2$. Let $n \geq 3$. By using equation (1. 5_-) for $X(-n+1)$ and (1. 7_-), we can write

$$\text{the upper-hand side of Step 10} \\ = E(\nu_+(m+1)^t\nu_-(-(n-1))) + \sum_{k=0}^{n-2} E(\nu_+(m+1)^t X(-k))^t(\gamma_-(n, k+1) - \\ - \gamma_-(n-1, k)) \\ = E(\nu_+(m+1)^t\nu_-(-(n-1))) + \\ + E(\nu_+(m+1)^t X(-(n-2)))^t(\delta_-(n)\delta_+(n-1)) + \\ + \sum_{k=0}^{n-3} E(\nu_+(m+1)^t X(-k))^t(\delta_-(n)\gamma_+(n-1, n-k-2)).$$

Using equation (1. 5_-) for $X(-n+2)$ and (1. 7_+) again, we get

$$\text{the upper-hand side of Step 10} \\ = E(\nu_+(m+1)^t\nu_-(-(n-1))) + E(\nu_+(m+1)^t\nu_-(-(n-2)))^t(\delta_-(n)\delta_+(n-1)) + \\ + \sum_{k=0}^{n-3} E(\nu_+(m+1)^t X(-k))^t\{\delta_-(n)(\gamma_+(n-1, n-k-2) - \delta_+(n-1)\gamma_-(n-2, k))\} \\ = E(\nu_+(m+1)^t\nu_-(-(n-1))) + E(\nu_+(m+1)^t\nu_-(-(n-2)))^t(\delta_-(n)\delta_+(n-1)) + \\ + E(\nu_+(m+1)^t X(-(n-3)))^t(\delta_-(n)\delta_+(n-2)) + \\ + \sum_{k=0}^{n-4} E(\nu_+(m+1)^t X(-k))^t(\delta_-(n)\gamma_+(n-2, n-k-3)).$$

Repeating the same prodedure, we arrive at the conclusion of Step 10.

(Step 11) Now, we are going to exhibit the key formula: For any $n \in \mathbf{N}$, $n \geq 2$,

$$F_{n-1}(1) + \sum_{l=1}^{n-1} F_{l-1}(1)^t\gamma_-(n, l) = 0.$$

By the definition of $F_m(n)$ in Step 4, we see that for any $m \in \mathbf{N}$,

$$F_m(1) = R(m) + \delta_-(1)R(m+1),$$

which yields

(2. 14) $$F_{n-1}(1)+\sum_{l=1}^{n-1} F_{l-1}(1)^t\gamma_-(n, l)=I+\delta_-(1)II,$$

where

$$I=R(n-1)+\sum_{l=1}^{n-1} R(l-1)^t\gamma_-(n, l)$$

and

$$II=R(n)+\sum_{l=1}^{n-1} R(l)^t\gamma_-(n, l).$$

We first show

(2. 15) $$II=-R(0)^t\delta_-(n).$$

By (1. 7_-) and (2. 6_-),

$$\begin{aligned} II&=-V_+(n-1)^t\delta_-(n)+\sum_{l=1}^{n-1} R(l)^t(\gamma_-(n, l)-\gamma_-(n-1, l-1)) \\ &=-\left\{V_+(n-1)-\sum_{l=1}^{n-1} R(l)^t\gamma_+(n-1, n-l-1)\right\}^t\delta_-(n). \end{aligned}$$

Hence, (2. 15) follows from (i) in Step 3.
The next task is to show

(2. 16) $$I=R(0)^t\delta_+(1)^t\delta_-(n).$$

When $n=2$, it follows from (1. 7_-) and (2. 6_+) that

$$\begin{aligned} I&=R(1)+R(0)^t\gamma_-(2, 1) \\ &=-\delta_+(1)R(0)+R(0)^t(\delta_-(1)+\delta_-(2)\delta_+(1)). \end{aligned}$$

Hence, by (1. 10), we see that (2. 16) holds for $n=2$.

Let $n\geq 3$. By (1. 7_-) and (2. 6_-),

$$I=-V_+(n-2)^t\delta_-(n-1)-\sum_{k=0}^{n-3} R(k+1)^t\gamma_-(n-2, k)+\sum_{l=1}^{n-1} R(l-1)^t\gamma_-(n, l).$$

Applying (1. 7_-) to the last term in the above equality, we have

$$\begin{aligned} I=&-V_+(n-2)^t\delta_-(n-1)+R(0)^t\delta_-(n-1)+ \\ &+\sum_{k=0}^{n-3} R(k+1)^t(\gamma_-(n-1, k+1)-\gamma_-(n-2, k))+ \\ &+\sum_{l=1}^{n-1} R(l-1)^t\gamma_+(n-1, n-l-1)^t\delta_-(n). \end{aligned}$$

By using (1. 7_-) again, we have

$$(2.17)\quad I=-\left\{V_+(n-2)-R(0)-\sum_{k=0}^{n-3}R(k+1)^t\gamma_+(n-2,\,n-k-3)\right\}{}^t\delta_-(n-1)+ \\ +\sum_{l=1}^{n-1}R(l-1)^t\gamma_+(n-1,\,n-l-1)^t\delta_-(n).$$

It follows from (i) in Step 3 that

(2. 18) the coefficient of ${}^t\delta_-(n-1)$ in (2. 17)$=0$.

So it suffices to show the following:

(2. 19) the coefficient of ${}^t\delta_-(n)$ in (2. 17)$=R(0)^t\delta_+(1)$.

By (1. 7_+),

$$\text{the coefficient of } {}^t\delta_-(n) \text{ in } (2.\,17) \\ =\sum_{k=0}^{n-3}R(k)^t\gamma_+(n-2,\,n-3-k)+\left\{R(n-2)+\sum_{k=0}^{n-3}R(k)^t\gamma_-(n-2,\,k)\right\}{}^t\delta_+(n-1).$$

On the other hand, by $(3.\,5)_0$ in [4], we get

$$R(n-2)=-\sum_{k=0}^{n-3}R(k)^t\gamma_-(n-2,\,k),$$

which is also seen by multiplying both-hand sides of equation (1. 5_-) for $X(-n+2)$ by ${}^tX(0)$ from the right and taking an expectation with respect P. Therefore, we get

$$\text{the coefficient of } {}^t\delta_-(n) \text{ in } (2.\,17)=\sum_{k=0}^{n-3}R(k)^t\gamma_+(n-2,\,n-3-k).$$

By repeating the same procedure and using (1. 7_+), (1. 10) and (2. 6_+), we can write

$$\begin{aligned}\text{the coefficient of } {}^t\delta_-(n) \text{ in } (2.\,17)&=\sum_{k=0}^{1}R(k)^t\gamma_+(2,\,1-k)\\ &=R(0)^t(\delta_+(1)+\delta_+(2)\delta_-(1))-\\ &\qquad-\delta_+(1)R(0)^t\delta_+(2)\\ &=R(0)^t\delta_+(1)\end{aligned}$$

and so (2. 19) holds. This completes the proof of (2. 16).

In conclusion, we see from (2. 14)−(2. 16) and (1.10) that Step 11 holds.

(Step 12) We come to the final position to complete the proof of Fluctuation-Fluctuation Theorem. (i) and (ii) are clear. (iii) has been proved in Step 2. By Step 9 and Step 10, we have

$$
\begin{aligned}
&I(m, n) \\
&=I(m+1, n-1)+\{\sum_{k=1}^{n-2} I(m+1, k)^t\delta_+(k+1)\}^t\delta_-(n)- \\
&\qquad -\delta_+(m+1)\{\sum_{k=2}^{m}\delta_-(k)I(k-1, n)\}- \\
&\qquad -\delta_+(m+1)\{F_{n-1}(1)+\sum_{l=1}^{n-1}F_{l-1}(1)^t\gamma_-(n, l)\}.
\end{aligned}
$$

Hence, by virtue of Step 11, (iv) holds

Thus we have completed the proof of Fluctuation-Fluctuation Theorem. (Q. E. D.)

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