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The extremal case in Toponogov's comparison theorem and gap-theorems

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Dedicated to Professor Toponogov on his 60th birthday

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1. Introduction and results.

The extremal case in Toponogov's comparison theorem asserts that:

PROPOSITION 1. *If for a triangle* $\triangle pqr$ *in a riemannian manifold* M^{n} *with sectional curvature* $K_{\sigma}\geq k$ *we have*:

$$\angle p=\angle p', \angle q=\angle q', \angle r=\angle r',$$

where $\triangle p'q'r'$ *is a triangle with the same sides in a space form* S_{k}^{2} *of constant curvature* k; *then the closed curve, consisting of minimal geodesics* pq, pr *and some geodesic from* q *to* r, *is a boundary of a totally geodesic film in* M^{n} *obtained as an image of isometric embedding of a part of a surface* S_{k}^{2}, *bounded by the triangle* $\triangle p'q'r'$.

If the triangle $\triangle pqr$ is small — say it vertices lie in r_{in}-neighborhood of the vertex p (where r_{in} is the injectivity radius of M^{n}), then the mentioned geodesic from q to r coincides with the minimal geodesic qr, and we will also have the extremal case in the "inverse-Toponogov's" theorem:

PROPOSITION 2. *If for a small triangle* $\triangle pqr$ *in a riemannian manifold* M^{n} *with sectional curvature* $K_{\sigma}\leq k$ *we have*:

$$\angle p=\angle p', \angle q=\angle q', \angle r=\angle r',$$

where $\triangle p'q'r'$ *is a triangle with the same sides in a space form* S_{k}^{2} *of constant curvature* k; *then the closed curve, consisting of minimal geodesics* pq, pr, qr, *is a boundary of a totally geodesic film in* M^{n} *obtained as an image of isometric embedding of a part of a surface* S_{k}^{2}, *bounded by the triangle* $\triangle p'q'r'$.

There is an easy consequence from Proposition 1 for surfaces:

THEOREM 1 [T]. *If in the closed surface* M^{2} *with a curvature* $K\geq k$ *there exists a closed geodesic* S^{1} *with length equal to* $2\pi/\sqrt{k}$, *then* M^{2} *is* *isometric to* S_k^2.

According to [S] this result may be considered as a gap-theorem about manifolds in a neighborhood of the space form. The author of [S] conjectured that the following generalization of Theorem 1 is true:

CONJECTURE 4 [S]. Let M^n be an open hemisphere of the S_1^n. Then its structure cannot be changed in any compact subset with $K_M \geq 1$.

In this paper we show that this conjecture is an easy consequence of Proposition 1 and a method of continuation of isometries to convex sets, given in [SZ] for symmetric spaces of rank≥ 3.

THEOREM 2. *If the riemannian manifold* M^n *with continuous sectional curvature* $K_\sigma \geq k > 0$ *contains some isometrically embedded open neighborhood of* S_k^{n-1} *in* S_k^n, *then* M^n *is isometric to* S_k^n.

Using Proposition 2 instead of Proposition 1 we also obtained:

THEOREM 3. *If the riemannian manifold* M^n, $n \geq 3$ *with continuous sectional curvature* $K_\sigma \leq k$ *contains some isometrically embedded open neighborhood of* S_k^{n-1} *in* S_k^n, *then* M^n *is isometric to* S_k^n.

For $n=2$ this result is obviously false.

For negatively curved manifolds the same construction gives us the proof of the conjecture 3 in [S] in the case $n \geq 3$:

THEOREM 4. *Let* M^n *be a complete riemannian manifold of dimension*≥ 3. *Suppose that* $K_\sigma \geq -1$ *and* M^n *is isometric at infinity to the n-dimensional hyperbolic space* $H^n(-1)$ *of constant curvature* -1 (*that is*: $M^n \setminus W$ *is isometric to* $H^n(-1) \setminus V$ *outside some compact sets* W *and* V). *Then* M^n *is isometric to* $H^n(-1)$.

and also:

THEOREM 5. *Let* M^n *be a complete riemannian manifold of dimension* ≥ 3. *Suppose that* $K_\sigma \leq -1$ *and* M^n *is isometric at infinity to the n-dimensional hyperbolic space* $H^n(-1)$ *of constant curvature* -1 (*that is*: $M^n \setminus W$ *is isometric to* $H^n(-1) \setminus V$ *outside some compact sets* W *and* V). *Then* M^n *is isometric to* $H^n(-1)$.

This was also proved in [S]—see Theorem 1. 2.

2. Proofs of Theorems 2 and 3.

Let M^n be a riemannian manifold with sectional curvature $K_\sigma \geq k > 0$

which contains an embedded $2d$-neighborhood of S_k^{n-1} isometric to the standard one in S_k^n. Let us denote by S the image of S_k^{n-1} under this isometry. Then S is a totally geodesic hypersurface in M, we have two parallel vector fields of normals ν^+ and ν^- to S, and M is a union of equidistants S_t^+ and S_t^-, $0 \leq t \leq diam(M)$:

$$S_t^{\pm} = \{a | \rho(a, S) = t, \ \overline{ba} = \nu^{\pm}(b)\},$$

where b is some point on S which is nearest to a. The whole M is a union of two sets: $M = C^+ \cup C^-$where $C^{\pm} = \bigcup_{o \leq t} S_t^{\pm}$(we don't exclude the case that $int(C^+ \cap C^-) \neq \emptyset$) and if

$$C_t^{\pm} = \bigcup_{t \leq t'} S_{t'}^{\pm}$$

then $S_t^{\pm} = \partial C_t^{\pm}$. We easily see that $\partial C_t = S_t$ for $0 < t < 2d$ are standard spheres, isometric to the corresponding one in S_k^n. And from $K_\sigma \geq k$ for every $t \geq 2d$ all normal geodesic curvatures of ∂C_t at every point in its regular part is not less then $2kd$. This enable us to pierce C_t across every point q which is not far from ∂C_t by two-dimensional balls, which have small size and almost tangent directions. Define this procedure more precisely:

For a point $a \in C_t$, $t > 2d$, find some point b on ∂C_d nearest to a:

$$\rho(a, b) = \rho(a, \partial C_d)$$

and denote by $\overline{ab}$ the unit vector tangent to the minimal geodesic ab which connects points a and b. Denote by N_a^δ all unit vectors which are δ-normal to some vector $\overline{ab}$ (that is $v \in N_a^\delta$ iff $|(v, \overline{ab})| \leq \delta$), where b is some point on S_d which is nearest to a. For $v \in N_a^\delta$ define a geodesic $a_v(s) = \exp(sv)$.

LEMMA 1. *For a distance function $f(s) = \rho(a(s), S_d)$ the following is true:*

$$|f'(0)| \leq \delta, \ f''(s) \leq -kd(1-(f'(s)^2)^{1/2}$$

for every s, when $a(s)$ lies in C_d.

PROOF. It easily follows from the first and second variation formulas — see [CG] or [MT].

In what follows we will assume that $\delta < 1/4$. Let us also denote by $s(2d)$ the first moment when $a(s)$, $s \geq 0$ leaves C_{2d} and by $s(d)$ the first moment when $a(s)$, $s \geq 0$ leaves C_d.

LEMMA 2. *For every v of $N_a^{2\delta}\setminus N_a^{\delta}$*

$$s(2d)\leq 3((t-2d)/\delta+16\delta/kd). \tag{1}$$

PROOF. Let us denote by

$$s^*=(t-2d)/\delta+16\delta/kd,$$

and suppose for the moment that $a(s)$ belongs to C_{2d} (i.e. $f(s)-d\geq 0$) for all $0\leq s\leq 3s^*$.

a). If for every $0\leq s\leq s^*$ the absolute value of $f'(s)$ is less than $1/2$, then according to Lemma 1

$$f''(s)\leq -kd(1-1/4)<-kd/2$$

and

$$f'(s)\leq f'(0)\leq 2\delta.$$

Therefore

$$\begin{aligned}f(s)-d&=f(0)-d+f'(0)s+\int_0^s\int_0^\theta f''(\mu)d\mu d\theta\\&\leq(t-2d)+2\delta s-kds^2/4.\end{aligned}$$

But for $s=s^*$ the right side of the last inequality is negative:

$$\begin{aligned}2\delta s^*&=(kds^*/8)(16\delta/kd)<(kds^*/8)((t-2d)/\delta+16\delta/kd)\\&=(kd(s^*)^2/4)/2\end{aligned}$$

and

$$\begin{aligned}(t-2d)&=((t-2d)/2\delta)2\delta<2\delta((t-2d)/\delta+16\delta/kd)\\&=2\delta s^*<(kd(s^*)^2/4)/2.\end{aligned}$$

b). If for some $0<s_0<s^*$

$$|f'(s_0)|=1/2$$

then (from $\delta<1/4$) it follows that $f'(s_0)=-1/2$, and from the second statement of Lemma 1 we hold for $0\leq s\leq s^*$

$$f'(s)\leq f'(0)\leq 2\delta,$$

and for $s^*\leq s\leq 3s^*$

$$f'(s)\leq f'(s^*)\leq -1/2.$$

Therefore

$$f(3s^*)-d=f(0)-d+\int_0^{s^*}f'(s)ds+\int_{s^*}^{3s^*}f'(s)ds$$
$$\leq(t-2d)+2\delta s^*-2s^*/2<0,$$

because $\delta<1/4$ implies $2\delta s^*<s^*/2$ and

$$t-2d<((t-2d)/\delta)/2<((t-2d)/\delta+16\delta/kd)/2=s^*/2.$$

So in all considered cases $f(3s^*)-d<0$. This contradiction proves our statement $s(2d)<3s^*$. Lemma 2 is proved.

From the definition we see that $s(d)-s(2d)\geq d$. Therefore the following statement follows from the estimate on $s(2d)$ in Lemma 2:

LEMMA 3. *For any given $\bar{K}$, there exist $\bar{\delta}$ and some $\bar{\tau}>0$, such that for all $0<\tau<\bar{\tau}$ and all a of $C_{2d}\backslash C_{2d+\tau}$ and all v of $N_a^{2\bar{\delta}}\backslash N_a^{\bar{\delta}}$*

$$s(d)/s(2d)>\bar{K}.$$

The proof is obvious and easily follows from Lemma 2: take for example

$$\delta=\bar{K}^{-1}kd^2/32 \text{ and } \bar{\tau}=\bar{K}^{-1}\ \delta d/2.$$

Choose at the point a unit vectors u, v, w of $N_a^{2\bar{\delta}}\backslash N_a^{\bar{\delta}}$ so that they lie in some two-dimensional direction σ and:

$$\angle(u, v)=\angle(u, w)=\angle(v, w)$$
$$u+v+w=0. \tag{2}$$

For $s>s(2d)$ denote by $\triangle(s)$ the triangle $\triangle pqr$ with vertices: $p=a_u(s)$, $q=a_v(s)$, $r=a_w(s)$. If $s\leq r_{in}$, then the triangle $\triangle(s)$ is a small one: all vertices and sides pq, pr, qr lie in r_{in}-neighborhood of the vertex p and we may use propositions 1 and 2. Consider this triangle: On the minimal geodesic pq choose an arbitrary point e. From the continuity of the curvature it easily follows that the vector $\overline{ae}$ has continuous dependence of e and almost lies in a plane σ generated by $\overline{ap}$ and $\overline{aq}$:

LEMMA 4. *For some consnant L*

$$\angle(\overline{ae}, \sigma)\leq Ls^2.$$

PROOF. Let $(x^1, \dots x^n)$ be a normal coordinate system with a center at the point a. That is: a point q has coordinates $(x^1, \dots x^n)$ if q is the image under the exponential map $\exp_a$ of a point in T_aM with the same coordinates in some euclidean coordinate system. Without loss of generality we may assume that σ coincides with a plane generated by first coordinate vectors e_1 and e_2 of this system, and points p and q have following coordinates: $p=(s, 0 \ldots 0)$, $q=(-s/2, \sqrt{3}s/2, 0 \ldots 0)$. If $x(\theta)=(x^1(\theta), \ldots x^n(\theta))$, $0 \leq \theta \leq \theta_0$ is a minimal geodesic connecting these points and parameterized by an arc length, then:

$$\ddot{x}^k(\theta)+\Gamma^k_{ij}(x(\theta))\dot{x}^i(\theta)\dot{x}^j(\theta)=0.$$

It is well known that in this setting $\exp_a$ is a quasi isometry, so in some $\bar{s}$-neighborhood of a point a for some constants K, k' depending only on M:

$$|\dot{x}^i(\theta)| \leq K, \quad |\Gamma^k_{ij}(x(\theta))| \leq K\rho(a, x(\theta)),$$
$$\rho(a, x(\theta)) \geq k's. \tag{3}$$

So

$$|\ddot{x}^k(\theta)| \leq Ks \text{ and } 0 \leq \theta_0 \leq Ks. \tag{4}$$

But for every $k>2$ $x^k(0)=x^k(\theta_0)=0$, therefore for some θ_k $\dot{x}^k(\theta_k)=0$ and from (4) it follows:

$$|\dot{x}^k(\theta)| \leq K\theta_0^2 \text{ and } |x^k(\theta)| \leq K\theta_0^3 \tag{5}$$

this obviously leads to the following inequality for the angle between the plane σ and the vector $\overline{ax}(\theta)$:

$$\angle(\sigma, ax(\theta)) \leq K(\Sigma(x^k(\theta))^2)^{1/2}/\rho(a, x(\theta))$$

or

$$\angle(\overline{ae}, \sigma) \leq Ls^2,$$

where the constant L may be chosen the same for all points a of M.

So, if vectors u and v lie in N^δ_a, then for all points e of pq vector $\overline{ae}$ lies in $N^{\delta'}_a$ with $\delta'=\delta+Ls^2$. At last we may define all needed constants: take δ' so small that

$$(192)^2L(k')^{-2}\delta'/(kd)^2<1/8$$

and for $\bar{K}=(k')^{-1}$ find $\bar{\delta} \leq \delta'$ and $\bar{\tau}$ according to Lemma 3. Then for $s_1=3(k')^{-1} s^*$ for $s^*=(\tau/\bar{\delta}+16\bar{\delta}/kd)$, find $\tau_1<\bar{\tau}$ so that for all $\tau<\tau_1$

$$L(s_1)^2<\bar{\delta}/4 \tag{6}$$

(To find τ_1 consider last inequality:

$$L(s_1)^2=L(3(k')^{-2}(\tau/\bar{\delta}+16\bar{\delta}/kd)^2 \leq$$

$$\leq 18L(k')^{-2}\tau^2/\bar{\delta}^2+18L(k')^{-2}(16\bar{\delta}/kd)^2\leq$$
$$\leq 18L(k')^{-2}\tau^2/\bar{\delta}^2+\bar{\delta}/8$$

so for $\tau<\tau_1<(\bar{\delta}^3(k')^2/144L)^{1/2}$ we have (6). Hence for all u, v, w of $N_a^{\delta_1}\backslash N_a^{\delta_2}$, where $\delta_1=3\bar{\delta}/2$ and $\delta_2=\bar{\delta}/2$ we have:

$$\rho(a, \exp_a(s_1(\overline{ae})))\geq s(2d) \text{ and } \rho(\exp_a(s_1(\overline{ae}), S_d))\leq d.$$

So all sides of the triangle $\triangle(s_1)$ lie outside C_{2d} where the sectional curvature of M^n is equal to k. It is easy to see that the angles of $\triangle(s_1)$ are equal to angles of the triangle with the same sides in S_k^n. To see this one may choose the nearest point $\bar{a}$ on S_d to the point p and construct the family of triangles $\triangle_\mu$, $0\leq\mu\leq 1$ with vertices p_μ, q_μ, r_μ, which lie on $\bar{a}p$, $\bar{a}q$, $\bar{a}r$ and divide them in ratio $\mu:(1-\mu)$. Then all $p_\mu q_\mu$, $p_\mu r_\mu$, $q_\mu r_\mu$ lie in $2d$-neighborhood of S_k^{d-1} where M^n is isometric to S_k^n, and therefore all angles of $\triangle(s_1)$ are equal to the corresponding one of the triangle in S_k^n with the same sides. Using Proposition 1 we obtain:

LEMMA 5. *There exist a totally geodesic film π of constant curvature k, which has the following boundary*: $\partial\pi=\triangle(s_1)=pq\cup qr\cup rp$.

LEMMA 6. *The point a belongs to π.*

PROOF. From the $\triangle(s_1)$ construction we see that the point a' — the nearest point on π to the point a lies in the interior of π. $\triangle(s_1)$ is small, so if $a'\neq a$ then we have three small triangles $\triangle aa'p$, $\triangle aa'q$, $\triangle aa'r$ in which all angles a' are equal to $\pi/2$. For small triangles, when aa' doesn't contain focal points to p, q and r this means that all other angles are strictly less then $\pi/2$. So:

$$(\overline{aa'}, \overline{ap})>0,\ (\overline{aa'}, \overline{aq})>0,\ (\overline{aa'}, \overline{ar})>0$$

or

$$(\overline{aa'}, \overline{ap})+(\overline{aa'}, \overline{aq})+(\overline{aa'}, \overline{ar})>0,$$

but this inequality obviously contradicts (2).

So we find some $\tau>0$ such that we can construct a totally geodesic film π across every point a in $C_{2d}\backslash C_{2d+\tau}$ in every direction σ, generated by vectors from $N_a^{\delta_1}\backslash N_a^{\delta_2}$. But this set of directions has non-empty interior, so the sectional curvature of M^n in all points in $C_{2d}\backslash C_{2d+\tau}$ and in every direction is equal to k.

By standard continuation arguments, we can easily prove that M has constant curvature in the complement to some set with empty interior, or

using the continuity of curvature of M — that M is a manifold of constant curvature. But the only manifold of constant curvature which contains some neighborhood of S_k^{n-1} is a standard sphere. This completes the proof of Theorem 2.
To prove Theorem 3 it is sufficient to repeat all arguments above using Proposition 2 instead of Proposition 1.

3. Proofs of Theorems 4 and 5.

To obtain Theorems 4 and 5 we proceed by the same way: in H^n we can find a ball B which contains W and consider $i(B\setminus W)$, where i is supposed an isometry at infinity. So B is a convex set in H^n, then $C=i(B\setminus W)\cup V$ is also a convex set. Then repeating previous consideration, we can extend isometry to $C\setminus C_d$, until C_d is a convex set. But B_d is convex for all d, so is C_d, and we obtain an isometry between M^n and H^n.

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