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The symmetric "doughnut" evolving by its mean curvature

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1. Introduction

Let M_0 be a compact manifold without boundary given by a smooth immersion $\tilde{F}_0: M_0 \to \boldsymbol{R}^{m+1}$ immersing M_0 as a hypersurface in $\boldsymbol{R}^{m+1}$. Then we want to find a family of smooth immersions $\tilde{F}(\tilde{x}, t)$ corresponding to hypersurfaces $M_t = \tilde{F}(\cdot, \mathrm{t})$ such that

$$(1) \qquad \begin{aligned} \frac{\partial}{\partial t}\tilde{F}(\tilde{x}, t) &= -\tilde{H}(\tilde{x}, t) \cdot \tilde{\nu}(\tilde{x}, t) \\ \tilde{F}(\tilde{x}, 0) &= \tilde{F}_0(\tilde{x}), \end{aligned}$$

where $\tilde{\nu}(\tilde{x}, t)$ is the outer unit normal at $\tilde{x} \in M_t$ and $\tilde{H}(x, t)$ is the mean curvature of M_t at $(\tilde{x}, t)$.

In the case of a convex hypersurface M_0 in $\boldsymbol{R}^{m+1}$ with $m \geq 2$, Huisken [Hu1] showed that (1) has a solution on a finite time interval and that the M_t's converge to a single point. In the case of convex plane curves Gage and Hamilton [GH] proved that equation (1) shrinks M_t to a point within finite time.

In this paper we want to discuss the behaviour of M_t when M_0 is the embedding of a m-doughnut $\tilde{F}_0: S^1 \times C_0 \to \boldsymbol{R}^{m+1}$ such that M_0 is invariant under all rotations in the (z_1, z_{m+1})-plane. This means the following:

Let $\boldsymbol{R}^m_{>0} := \{(z_1, \cdots, z_m, 0) \in \boldsymbol{R}^{m+1} | z_1 > 0\}$ and let C_0 be a compact manifold without boundary smoothly immersed as a hypersurface into $\boldsymbol{R}^m_{>0}$. Then we define the "doughnut" M_0 generated by C_0 to be the manifold

$$M_0 := \{(z_1 \cdot \cos(\varphi), z_2, \cdots, z_m, z_1 \cdot \sin(\varphi)) | \varphi \in [0, 2\pi), (z_1, \cdots, z_m, 0) \in C_0\}.$$

The so defined "doughnuts" are not convex, since the eigenvalue in rotational direction assumes positive and negative values. Since equation (1) is isotropic we know that the M_t's must be rotationally symmetric too and we can define the "generating manifolds" C_t as the intersections of M_t with the half-space $\boldsymbol{R}^m_{>0}$.

One can show that M_t stays embedded as long as a smooth solution of (1) exists, if this was true for M_0, so the manifold C_t is an embedding of a hypersurface in $\boldsymbol{R}^m$ and since M_t is generated by C_t the behaviour of M_t is totally determined by the behaviour of C_t. If $\tilde{\nu}(\tilde{x}, t)$ is the outer unit normal on M_t at a point $(\tilde{x}, t)$ of C_t and $\nu(\tilde{x}, t)$ is the outer unit normal on C_t (as a submanifold of the half-space $\boldsymbol{R}^m_{>0}$) at the same point, we have $\tilde{\nu}(\tilde{x}, t)=\nu(\tilde{x}, t)$ since M_t is rotationally symmetric. This is not true for the mean curvature of M_t and C_t respectively. In fact we have $\tilde{H}(\tilde{x}, t)=H(\tilde{x}, t)+\lambda(\tilde{x}, t)$, where $H(\tilde{x}, t)$ is the mean curvature of C_t and $\lambda(\tilde{x}, t)$ is the eigenvalue of the second fundamental form on M_t that belongs to the rotational direction. If we introduce cylindrical coordinates for $\boldsymbol{R}^{m+1}$, i.e.

$$(r, \varphi, z_2, \cdots, z_m) \text{ with } z_1=r\cdot\cos(\varphi),\ z_{m+1}=r\cdot\sin(\varphi),$$
$$r>0,\ \varphi\in[0, 2\pi)$$

and $e_r=\cos(\varphi)\cdot\frac{\partial}{\partial z_1}+\sin(\varphi)\cdot\frac{\partial}{\partial z_{m+1}}$, then it is easy to see that

$$\lambda=\frac{(\nu, e_r)}{r},$$

where $(\cdot, \cdot)$ denotes the inner product of $\boldsymbol{R}^{m+1}$. So we have $\tilde{H}\cdot\tilde{\nu}=(H+\lambda)\cdot\nu$.

This means that the generating manifolds C_t are evolving by a somewhat different evolution equation, i.e. they are solutions of:

$$\frac{\partial}{\partial t}F(x, t)=-(H+\lambda)(x, t)\cdot\nu(x, t) \tag{2}$$
$$F(x, 0)=F_0(x),\ F(x, t)\in\boldsymbol{R}^m_{>0}$$

where $F(\cdot, t)$ is an embedding of C_t. Any solution of (1) gives a solution of (2) and the opposite is true as well. So the maximal time interval $[0, \boldsymbol{T})$ where a smooth solution of (1) exists is the same maximal time interval for which a smooth solution of (2) exists. We remark that for $(z_1, \cdots, z_m, 0)\in\boldsymbol{R}^m_{>0}$ we have $z_1=r$, $e_r=\frac{\partial}{\partial z_1}$ and state our main theorem:

THEOREM 1.1. *Let $n:=m-1$ and $n\geq 2$, assume C_0 is a compact manifold without boundary smoothly embedded into $\boldsymbol{R}^{n+1}_{>0}$ and that all eigenvalues of the second fundamental form λ_i, $i=1, \cdots, n$ satisfy the condition*:

(a) $\quad \lambda_i>\frac{a}{r}$

with a positive constant a satisfying

(b) $\quad na^3-a-\frac{4}{3\sqrt{3}}>0$

then equation (2) *has a smooth solution for a short time and the solutions* C_t *converge to a single point* p *in* $\boldsymbol{R}^{n+1}_{>0}$ *in finite time and in the same time the " doughnut ", generated by* C_0, *converges to a* S^1_r, $r=\|p\|$ *under the mean curvature flow.*

If we examine the normalizations $\bar{C}_t$ in $\boldsymbol{R}^{n+1}$ given by

$$\bar{F}=\Psi(F-p)$$

where $\Psi(t)$ is a factor such that the total area of $\bar{C}_t$ is equal to C_0 for all time, then with the new time variable $\bar{t}:=\int_0^t \Psi^2(\tau)d\tau$ we have

THEOREM 1.2. *The normalized manifolds* $\bar{C}_{\bar{t}}$ *exist for* $\bar{t}\in[0, \infty)$. *They are translated homothetic expansions of the* C_t*'s and converge in the* C^∞*-topology to a sphere of area* $|C_0|$ *as* $\bar{t}\to\infty$.

REMARK. We recently learned that Ahara and Ishimura [AI], [I] have studied the problem of " thin " doughnuts in the case $m=2$, i.e. $n=1$. They used the techniques of Gage and Hamilton [GH] to show that the doughnuts converge to a circle in finite time, if they assume initial conditions very similiar to ours. Both results do not overlap but our result can be seen as an extension to higher dimensions. Our result does not work in case $n=1$ since we make use of the Codazzi-equations which are worthless for $n=1$.

2. Notation and preliminary results

We will follow the notations in [Hu1]. Vectors on C_t will be denoted by $X=\{X^i\}$, convectors by $Y=\{Y_i\}$ and mixed tensors by $T=\{T^{ij}_{kl}\}$. The induced metric will be denoted by $g=\{g_{ij}\}$, the second fundamental form by $A=\{h_{ij}\}$. The summation convention is understood from 1 to n and we will use brackets $\langle\cdot, \cdot\rangle$ for the inner product on C_t, i.e.

$$\langle T^{ij}_k, S^{ij}_k\rangle = g_{is}g_{jt}g^{kl}T^{ij}_k S^{st}_l$$
$$|T|^2 = \langle T^{ij}_k, T^{ij}_k\rangle.$$

The induced connection on C_t is given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{il,j}+g_{jl,i}-g_{ij,l}),$$

where $g_{ij,l} = \frac{\partial}{\partial x_l}g_{ij}$. Then we have for the covariant derivative of a vector $\{X^i\}$

$$\nabla_j X^i = \frac{\partial}{\partial x_j} X^i + \Gamma^i_{jk} X^k.$$

The interchange of two covariant derivatives is given by

$$\nabla_i \nabla_j X^h - \nabla_j \nabla_i X^h = R^h_{ijk} X^k = (h_{lj} h_{ik} - h_{lk} h_{ij}) g^{hl} X^k$$

and

$$\nabla_i \nabla_j Y_k - \nabla_j \nabla_i Y_k = R_{ijkl} g^{lm} Y_m = (h_{ik} h_{jl} - h_{il} h_{jk}) g^{lm} Y_m.$$

The inner product on $\boldsymbol{R}^{n+1}_{>0}$ will be simply denoted by $(\cdot, \cdot)$. Then we have the following relations:

$$H = g^{ij} h_{ij}$$
$$|A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}$$
$$g_{ij} = \left(\frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j}\right)$$
$$h_{ij} = \left(\frac{\partial \nu}{\partial x_i}, \frac{\partial F}{\partial x_j}\right)$$
$$r = (F, e_r)$$
$$e_r = g^{ij}\left(e_r, \frac{\partial F}{\partial x_i}\right)\frac{\partial F}{\partial x_j} + (\nu, e_r)\nu.$$

The Laplacian of a tensor is given by

$$\Delta T^k_{ij} = g^{st} \nabla_s \nabla_t T^k_{ij}.$$

Furtheron we have the Gauss-Weingarten equations

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \Gamma^k_{ij} \frac{\partial F}{\partial x_k} - h_{ij} \nu$$

and

$$\frac{\partial \nu}{\partial x_i} = h_{il} g^{lk} \frac{\partial F}{\partial x_k}.$$

LEMMA 2.1. (a) $\Delta h_{ij} = \nabla_i \nabla_j H + H h_{jm} g^{ml} h_{li} - |A|^2 h_{ij}$

(b) $|\nabla r|^2 = 1 - \lambda^2 r^2$

(c) $\nabla_i \lambda = \frac{1}{r}(g^{kl} h_{li} \nabla_k r - \lambda \nabla_i r)$

(d) $$\nabla_i \nabla_j \lambda = \frac{1}{r}\langle \nabla_k h_{ij}, \nabla_k r\rangle - \lambda h_{in} g^{nm} h_{mj} + \lambda^2 h_{ij} - \frac{1}{r^2} g^{nm}(h_{mj} \nabla_i r + h_{mi} \nabla_j r) \nabla_n r + \frac{2\lambda}{r^2} \nabla_i r \nabla_j r.$$

PROOF. (a) This is 2.1 (i) of [Hul]

(b) We have $\nabla_i r = \frac{\partial r}{\partial x_i} = \left(\frac{\partial F}{\partial x_i}, e_r\right)$ and thus $e_r = \nabla r + (\nu, e_r)\nu$ this yields

$$1=(e_r, e_r)=|\nabla r|^2+(\nu, e_r)^2=|\nabla r|^2+(\lambda r)^2$$

(c) can be proved by a simple calculation whereas (d) is a consequence of the Codazzi-equations, i. e. $\nabla_i h_{jk}=\nabla_j h_{ki}=\nabla_k h_{ij}$.

As in [Hu1] we have

LEMMA 2.2. (a) $|\nabla A|^2 \geq \frac{3}{n+2}|\nabla H|^2$

(b) $|\nabla A|^2-\frac{1}{n}|\nabla H|^2 \geq \frac{2(n-1)}{3n}|\nabla A|^2$.

We define:

$$C := g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}$$
$$z := HC-|A|^4$$
$$Q^2 := |\nabla_i h_{kl}H-\nabla_i H h_{kl}|^2.$$

The following Lemma is Lemma 2.3 in [Hu1]

LEMMA 2.3. *If* $H>0$ *and* $h_{ij}\geq \varepsilon H g_{ij}$ *with some* $\varepsilon>0$, *then we have*:

(a) $Z \geq n\varepsilon^2 H^2(|A|^2-\frac{1}{n}H^2)$

(b) $Q^2 \geq \frac{1}{2}\varepsilon^2 H^2|\nabla H|^2$.

3. The evolution equations

We will denote derivatives with respect to time t with a point, e.g.

$$\dot{g}_{ij} = \frac{\partial}{\partial t}g_{ij}.$$

Then we have

LEMMA 3.1. $\dot{g}_{ij}=-2(H+\lambda)h_{ij}$.

PROOF.

$$\begin{aligned}\dot{g}_{ij}&=\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial x_j}\right)^{\cdot} = \left(\frac{\partial \dot{F}}{\partial x_i}, \frac{\partial F}{\partial x_j}\right)+\left(\frac{\partial F}{\partial x_i}, \frac{\partial \dot{F}}{\partial x_j}\right)\\ &=-\left(\frac{\partial (H+\lambda)\nu}{\partial x_i}, \frac{\partial F}{\partial x_j}\right)-\left(\frac{\partial F}{\partial x_i}, \frac{\partial (H+\lambda)\nu}{\partial x_j}\right)\\ &=-2(H+\lambda)\left(\frac{\partial \nu}{\partial x_i}, \frac{\partial F}{\partial x_j}\right)=-2(H+\lambda)h_{ij}.\end{aligned}$$

LEMMA 3.2. $\dot{\nu}=\nabla(H+\lambda)$.

PROOF.

$$
\begin{aligned}
\dot{\nu} &= g^{ij}\Big(\dot{\nu}, \frac{\partial F}{\partial x_i}\Big)\frac{\partial F}{\partial x_j} = -g^{ij}\Big(\nu, \frac{\partial \dot{F}}{\partial x_i}\Big)\frac{\partial F}{\partial x_j} \\
&= g^{ij}\Big(\nu, \frac{\partial (H+\lambda)\nu}{\partial x_i}\Big)\frac{\partial F}{\partial x_j} \\
&= g^{ij}\frac{\partial (H+\lambda)}{\partial x_i}\frac{\partial F}{\partial x_j} = \nabla(H+\lambda).
\end{aligned}
$$

LEMMA 3.3.

$$
\begin{aligned}
\dot{h}_{ij} = \Delta h_{ij} &+ \frac{1}{r}\langle \nabla_k h_{ij}, \nabla_k r\rangle - 2(H+\lambda)h_{jm}g^{ml}h_{li} + (|A|^2+\lambda^2)h_{ij} \\
&- \frac{1}{r^2}g^{nm}(h_{mi}\nabla_j r + h_{mj}\nabla_i r)\nabla_n r + \frac{2\lambda}{r^2}\nabla_i r\nabla_j r.
\end{aligned}
$$

PROOF.

$$
\begin{aligned}
\dot{h}_{ij} &= -\Big(\frac{\partial^2 F}{\partial x_i \partial x_j}, \nu\Big)^{\cdot} = \Big(\frac{\partial^2 (H+\lambda)\nu}{\partial x_i \partial x_j}, \nu\Big) - \Big(\frac{\partial^2 F}{\partial x_i \partial x_j}, \nabla(H+\lambda)\Big) \\
&= \frac{\partial^2 (H+\lambda)}{\partial x_i \partial x_j} + (H+\lambda)\Big(\frac{\partial^2 \nu}{\partial x_i \partial x_j}, \nu\Big) - \Big(\Gamma_{ij}^k \frac{\partial F}{\partial x_k} - h_{ij}\nu, \nabla(H+\lambda)\Big) \\
&= \frac{\partial^2 (H+\lambda)}{\partial x_i \partial x_j} + (H+\lambda)h_{jm}g^{ml}\Big(\frac{\partial^2 F}{\partial x_i \partial x_l}, \nu\Big) - \Gamma_{ij}^k \nabla_k (H+\lambda) \\
&= \nabla_i \nabla_j (H+\lambda) - (H+\lambda)h_{jm}g^{ml}h_{li}
\end{aligned}
$$

and the Lemma follows from Lemma 2.1 (a) and (d).

LEMMA 3.4. $\dot{H} = \Delta H + \frac{1}{r}\langle \nabla_i H, \nabla_i r\rangle + H(|A|^2+\lambda^2) - \frac{2}{r}\langle \nabla_i \lambda, \nabla_i r\rangle.$

PROOF.

$$
\begin{aligned}
\dot{H} &= -g^{is}g^{jt}\dot{g}_{st}h_{ij} + g^{ij}\dot{h}_{ij} \\
&= 2(H+\lambda)|A|^2 + \Delta H + \frac{1}{r}\langle \nabla_i H, \nabla_i r\rangle - 2(H+\lambda)|A|^2 \\
&\quad + (|A|^2+\lambda^2)H - \frac{2}{r^2}\langle h_{ij}, \nabla_i r \nabla_j r\rangle + \frac{2\lambda}{r^2}|\nabla r|^2 \\
&= \Delta H + \frac{1}{r}\langle \nabla_i H, \nabla_i r\rangle + H(|A|^2+\lambda^2) \\
&\quad - \frac{2}{r^2}(\langle h_{ij}, \nabla_i r \nabla_j r\rangle - \lambda|\nabla r|^2)
\end{aligned}
$$

and we can conclude with $\langle \nabla_i \lambda, \nabla_i r\rangle = \frac{1}{r}(\langle h_{ij}, \nabla_i r \nabla_j r\rangle - \lambda|\nabla r|^2)$.

LEMMA 3.5.

$$
\begin{aligned}
(|A|^2)^{\cdot} = \Delta|A|^2 - 2|\nabla A|^2 &+ \frac{1}{r}\langle \nabla_i |A|^2, \nabla_i r\rangle + 2|A|^2(|A|^2+\lambda^2) \\
&- 4|\nabla\lambda|^2 - \frac{4\lambda}{r}\langle \nabla_i \lambda, \nabla_i r\rangle.
\end{aligned}
$$

PROOF.

$$\begin{aligned}(|A|^2)^{\cdot} &= -2g^{in}g^{jm}\dot{g}_{nm}g^{kl}h_{ik}h_{jl}+2g^{ij}g^{kl}h_{ik}\dot{h}_{jl}\\ &= 4(H+\lambda)g^{in}g^{jm}g^{kl}h_{nm}h_{ik}h_{jl}+2g^{ij}g^{kl}h_{ik}(\Delta h_{jl}+\frac{1}{r}\langle\nabla_s h_{jl}, \nabla_s r\rangle\\ &\quad -2(H+\lambda)h_{jm}g^{mn}h_{nl}+(|A|^2+\lambda^2)h_{lj}\\ &\quad -\frac{1}{r^2}g^{nm}(h_{ml}\nabla_j r+h_{mj}\nabla_l r)\nabla_n r+\frac{2\lambda}{r^2}\nabla_l r\nabla_j r)\\ &= \Delta|A|^2-2|\nabla A|^2+\frac{1}{r}\langle\nabla_i|A|^2, \nabla_i r\rangle+2|A|^2(|A|^2+\lambda^2)\\ &\quad -\frac{4}{r^2}\langle h_{ik}\nabla_j r, h_{jk}\nabla_i r\rangle+\frac{4\lambda}{r^2}\langle h_{ij}, \nabla_i r\nabla_j r\rangle\\ &= \Delta|A|^2-2|\nabla A|^2+\frac{1}{r}\langle\nabla_i|A|^2, \nabla_i r\rangle+2|A|^2(|A|^2+\lambda^2)-4|\nabla\lambda|^2\\ &\quad -\frac{4\lambda}{r}\langle\nabla_i\lambda, \nabla_i r\rangle.\end{aligned}$$

LEMMA 3.6.

$$\begin{aligned}(H^2)^{\cdot} &= \Delta H^2-2|\nabla H|^2+\frac{1}{r}\langle\nabla_i H^2, \nabla_i r\rangle+2H^2(|A|^2+\lambda^2)\\ &\quad -\frac{4H}{r}\langle\nabla_i\lambda, \nabla_i r\rangle.\end{aligned}$$

PROOF. This is an easy consequence of $(H^2)^{\cdot}=2H\dot{H}$ and the fact that $\Delta H^2=2H\Delta H+2|\nabla H|^2$ and Lemma 3.4.

Lemma 3.5 and Lemma 3.6 yield

LEMMA 3.7.

$$\begin{aligned}(|A|^2-\frac{1}{n}H^2)^{\cdot} = \Delta(|A|^2-\frac{1}{n}H^2)&-2(|\nabla A|^2-\frac{1}{n}|\nabla H|^2)\\ &+\frac{1}{r}\langle\nabla_i(|A|^2-\frac{1}{n}H^2), \nabla_i r\rangle\\ &+2(|A|^2+\lambda^2)(|A|^2-\frac{1}{n}H^2)\\ &-4|\nabla\lambda|^2+\frac{4}{nr}(H-n\lambda)\langle\nabla_i\lambda, \nabla_i r\rangle.\end{aligned}$$

LEMMA 3.8. $\dot{r} = -(H+\lambda)\lambda r = \Delta r-\lambda^2 r.$

PROOF. Since $\Delta r=-H\nu_r$, we get

$$\dot{r} = (F, e_r)^{\cdot}=(\dot{F}, e_r)=-(H+\lambda)(\nu, e_r)=-(H+\lambda)\lambda r=\Delta r-\lambda^2 r.$$

LEMMA 3.9. $\dot{\lambda} = \Delta\lambda+\lambda(|A|^2+\lambda^2)+\frac{3}{r}\langle\nabla_i\lambda, \nabla_i r\rangle.$

PROOF.

$$\dot{\lambda} = \left(\frac{(\nu, e_r)}{r}\right)^{\cdot} = \frac{(\dot{\nu}, e_r)}{r} - \frac{(\nu, e_r)}{r^2}\dot{r}$$
$$= \frac{\langle \nabla_i(H+\lambda), \nabla_i r\rangle}{r} + (H+\lambda)\lambda^2.$$

From Lemma 2.1 (d) we get

$$\Delta\lambda = g^{ij}\nabla_i\nabla_j\lambda = \frac{\langle \nabla_i H, \nabla_i r\rangle}{r} - \lambda|A|^2 + \lambda^2 H - \frac{2}{r}\langle \nabla_i\lambda, \nabla_i r\rangle.$$

This gives Lemma 3.9.

LEMMA 3.10. $\dot{\tilde{H}} = \Delta\tilde{H} + \frac{1}{r}\langle \nabla_i\tilde{H}, \nabla_i r\rangle + \tilde{H}(|A|^2+\lambda^2)$.

PROOF. This is a simple consequence of Lemma 3.4 and Lemma 3.9.

REMARK. One can show that for a rotationally symmetric function f on M_t we have the relation $\Delta_M f = \Delta_C f|_C + \frac{1}{r}\langle \nabla_C f|_C, \nabla_C r\rangle$, where Δ_M, Δ_C are the Laplace-Beltrami operators on M_t and C_t respectively. So Lemma 3.10 is an easy consequece of Corollary 3.5 (i) in [Hu1].

We will need the evolution equation for the following function:

$$f_\sigma := \frac{|A|^2 - \frac{1}{n}H^2}{H^{2-\sigma}}, \quad 2 > \sigma > 0.$$

We have

LEMMA 3.11.

$$\dot{f}_\sigma = \Delta f_\sigma + \frac{2(1-\sigma)}{H}\langle \nabla_i f_\sigma, \nabla_i H\rangle + \frac{1}{r}\langle \nabla_i f_\sigma, \nabla_i r\rangle - 2\frac{Q^2}{H^{4-\sigma}}$$
$$-\sigma(1-\sigma)\frac{f_\sigma}{H^2}|\nabla H|^2 + \sigma f_\sigma(|A|^2+\lambda^2) + 2(2-\sigma)\frac{f_\sigma}{H_r}\langle \nabla_i\lambda, \nabla_i r\rangle$$
$$-\frac{4}{H^{2-\sigma}}|\nabla\lambda|^2 + \frac{4}{nrH^{2-\sigma}}(H-n\lambda)\langle \nabla_i\lambda, \nabla_i r\rangle.$$

PROOF.

$$\dot{f}_\sigma = H^{\sigma-2}\{\Delta(|A|^2 - \frac{1}{n}H^2) - 2|\nabla A|^2 + \frac{2}{n}|\nabla H|^2 + \frac{1}{r}\langle \nabla_i(|A|^2 - \frac{1}{n}H^2), \nabla_i r\rangle$$
$$+2(|A|^2+\lambda^2)(|A|^2 - \frac{1}{n}H^2) - 4|\nabla\lambda|^2 + \frac{4}{nr}(H-n\lambda)\langle \nabla_i\lambda, \nabla_i r\rangle\}$$
$$+(\sigma-2)H^{\sigma-3}\{\Delta H + \frac{1}{r}\langle \nabla_i H, \nabla_i r\rangle$$
$$+H(|A|^2+\lambda^2) - \frac{2}{r}\langle \nabla_i\lambda, \nabla_i r\rangle\}(|A|^2 - \frac{1}{n}H^2).$$

Now we have

$$\Delta f_\sigma = H^{\sigma-2}\Delta(|A|^2-\frac{1}{n}H^2)+(|A|^2-\frac{1}{n}H^2)\{(\sigma-2)H^{\sigma-3}\Delta H$$
$$+(\sigma-2)(\sigma-3)H^{\sigma-4}|\nabla H|^2\}+2(\sigma-2)H^{\sigma-3}\langle\nabla_i(|A|^2-\frac{1}{n}H^2), \nabla_i H\rangle.$$

This yields

$$\begin{aligned}\dot{f}_\sigma = \Delta f_\sigma &-(2-\sigma)(3-\sigma)\frac{f_\sigma}{H^2}|\nabla H|^2+2(2-\sigma)H^{\sigma-3}\langle\nabla_i(|A|^2-\frac{1}{n}H^2), \nabla_i H\rangle\\ &-2H^{\sigma-2}|\nabla A|^2+\frac{2}{n}H^{\sigma-2}|\nabla H|^2+\frac{1}{r}H^{\sigma-2}\langle\nabla_i(|A|^2-\frac{1}{n}H^2), \nabla_i r\rangle\\ &+2f_\sigma(|A|^2+\lambda^2)-4H^{\sigma-2}|\nabla\lambda|^2+\frac{4}{nrH^{2-\sigma}}(H-n\lambda)\langle\nabla_i\lambda, \nabla_i r\rangle\\ &-(2-\sigma)\frac{f_\sigma}{rH}\langle\nabla_i H, \nabla_i r\rangle-(2-\sigma)f_\sigma(|A|^2+\lambda^2)+2(2-\sigma)\frac{f_\sigma}{rH}\langle\nabla_i\lambda, \nabla_i r\rangle.\end{aligned}$$

Now taking the relations

$$\begin{aligned}|\nabla A|^2 &= \frac{1}{H^2}(Q^2-|A|^2|\nabla H|^2+H\langle\nabla_i|A|^2, \nabla_i H\rangle)\\ \frac{1}{r}\langle\nabla_i f_\sigma, \nabla_i r\rangle &= \frac{H^{\sigma-2}}{r}\langle\nabla_i(|A|^2-\frac{1}{n}H^2), \nabla_i r\rangle-(2-\sigma)\frac{f_\sigma}{rH}\langle\nabla_i H, \nabla_i r\rangle\\ \frac{1}{H}\langle\nabla_i f_\sigma, \nabla_i H\rangle &= H^{\sigma-3}\langle\nabla_i(|\mathrm{A}|^2-\frac{1}{n}H^2), \nabla_i H\rangle-(2-\sigma)\frac{f_\sigma}{H^2}|\nabla H|^2\end{aligned}$$

into account, we get

$$\begin{aligned}\dot{f}_\sigma = \Delta f_\sigma &-(2-\sigma)(3-\sigma)\frac{f_\sigma}{H^2}|\nabla H|^2-2H^{\sigma-4}Q^2+2\frac{f_\sigma}{H^2}|\nabla H|^2\\ &+2(1-\sigma)H^{\sigma-3}\langle\nabla_i(|A|^2-\frac{1}{n}H^2), \nabla_i H\rangle+\frac{1}{r}\langle\nabla_i f_\sigma, \nabla_i r\rangle+\sigma f_\sigma(|A|^2+\lambda^2)\\ &+2(2-\sigma)\frac{f_\sigma}{rH}\langle\nabla_i\lambda, \nabla_i r\rangle-4H^{\sigma-2}|\nabla\lambda|^2+\frac{4}{nrH^{2-\sigma}}(H-n\lambda)\langle\nabla_i\lambda, \nabla_i r\rangle\\ =\Delta f_\sigma &+\frac{2(1-\sigma)}{H}\langle\nabla_i f_\sigma, \nabla_i H\rangle-[(2-\sigma)(3-\sigma)-2-2(1-\sigma)(2-\sigma)]\frac{f_\sigma}{H^2}|\nabla H|^2\\ &-2H^{\sigma-4}Q^2+\frac{1}{r}\langle\nabla_i f_\sigma, \nabla_i r\rangle+\sigma f_\sigma(|A|^2+\lambda^2)+2(2-\sigma)\frac{f_\sigma}{rH}\langle\nabla_i\lambda, \nabla_i r\rangle\\ &-4H^{\sigma-2}|\nabla\lambda|^2+\frac{4}{nrH^{2-\sigma}}(H-n\lambda)\langle\nabla_i\lambda, \nabla_i r\rangle.\end{aligned}$$

This gives Lemma 3. 11.

LEMMA 3.12.

$$
\begin{aligned}
(|\nabla \widetilde{H}|^2)^{\cdot} = {} & \Delta(|\nabla \widetilde{H}|^2) - 2|\nabla^2 \widetilde{H}|^2 + \frac{1}{r}\langle \nabla_i |\nabla \widetilde{H}|^2, \nabla_i r\rangle \\
& + 2\langle \nabla_i \widetilde{H} h_{jl}, \nabla_j \widetilde{H} h_{il}\rangle - \frac{2}{r^2}\langle \nabla_i \widetilde{H}, \nabla_i r\rangle^2 \\
& + 2(|A|^2 + \lambda^2)|\nabla \widetilde{H}|^2 + 2\widetilde{H}\langle \nabla_i \widetilde{H}, \nabla_i(|A|^2 + \lambda^2)\rangle.
\end{aligned}
$$

PROOF.

$$
\begin{aligned}
& (|\nabla \widetilde{H}|^2)^{\cdot} \\
& = 2\widetilde{H} g^{is} g^{jt} h_{st} \nabla_i \widetilde{H} \nabla_j \widetilde{H} + 2 g^{ij} \nabla_i \widetilde{H} \nabla_j \left(\Delta \widetilde{H} + \frac{1}{r}\langle \nabla_k \widetilde{H}, \nabla_k r\rangle + \widetilde{H}(|A|^2 + \lambda^2)\right).
\end{aligned}
$$

Now we need the following formula. For any smooth function on C_t we have:

$$
\nabla_j \Delta f = \Delta \nabla_j f - g^{kl} \nabla_k f (H h_{jl} - h_{jm} g^{mn} h_{nl}).
$$

With $f = \widetilde{H}$ we get

$$
\begin{aligned}
(|\nabla \widetilde{H}|^2)^{\cdot} = {} & 2\widetilde{H}\langle h_{ij}, \nabla_i \widetilde{H} \nabla_j \widetilde{H}\rangle + \Delta |\nabla \widetilde{H}|^2 - 2|\nabla^2 \widetilde{H}|^2 \\
& - 2 g^{ij} \nabla_i \widetilde{H} g^{kl} \nabla_k \widetilde{H} (H h_{jl} - h_{jm} g^{mn} h_{nl}) \\
& + 2 g^{ij} \nabla_i \widetilde{H} \nabla_j \left(\frac{1}{r}\langle \nabla_k \widetilde{H}, \nabla_k r\rangle\right) + 2\langle \nabla_i \widetilde{H}, \nabla_i(\widetilde{H}(|A|^2 + \lambda^2))\rangle \\
= {} & 2\lambda\langle h_{ij}, \nabla_i \widetilde{H} \nabla_j \widetilde{H}\rangle + \Delta |\nabla \widetilde{H}|^2 - 2|\nabla^2 \widetilde{H}|^2 + 2\langle \nabla_i \widetilde{H} h_{jl}, \nabla_j \widetilde{H} h_{il}\rangle \\
& - \frac{2}{r^2}\langle \nabla_i \widetilde{H}, \nabla_i r\rangle^2 + \frac{2}{r} g^{ij} \nabla_i \widetilde{H} g^{kl} \nabla_j \nabla_k \widetilde{H} \nabla_l r \\
& + \frac{2}{r} g^{ij} \nabla_i \widetilde{H} g^{kl} \nabla_k \widetilde{H} \nabla_j \nabla_l r + 2(|A|^2 + \lambda^2)|\nabla \widetilde{H}|^2 \\
& + 2\widetilde{H}\langle \nabla_i \widetilde{H}, \nabla_i(|A|^2 + \lambda^2)\rangle.
\end{aligned}
$$

The relation

$$
\frac{1}{r}\langle \nabla_i |\nabla \widetilde{H}|^2, \nabla_i r\rangle = \frac{2}{r} g^{ij} \nabla_i \widetilde{H} g^{kl} \nabla_j \nabla_k \widetilde{H} \nabla_l r
$$

yields

$$
\begin{aligned}
(|\nabla \widetilde{H}|^2)^{\cdot} = {} & \Delta |\nabla \widetilde{H}|^2 - 2|\nabla^2 \widetilde{H}|^2 + \frac{1}{r}\langle \nabla_i |\nabla \widetilde{H}|^2, \nabla_i r\rangle \\
& + 2\lambda\langle h_{ij}, \nabla_i \widetilde{H} \nabla_j \widetilde{H}\rangle + 2\langle \nabla_i \widetilde{H} h_{jl}, \nabla_j \widetilde{H} h_{il}\rangle \\
& - \frac{2}{r^2}\langle \nabla_i \widetilde{H}, \nabla_i r\rangle^2 + \frac{2}{r}\langle \nabla_i \nabla_j r, \nabla_i \widetilde{H} \nabla_j \widetilde{H}\rangle \\
& + 2(|A|^2 + \lambda^2)|\nabla \widetilde{H}|^2 + 2\widetilde{H}\langle \nabla_i \widetilde{H}, \nabla_i(|A|^2 + \lambda^2)\rangle
\end{aligned}
$$

and we can continue the proof with the Gauss-Weingarten relation

$$\nabla_i\nabla_j r = -h_{ij}\nu_r = -\lambda r h_{ij}.$$

LEMMA 3.13.

$$\begin{aligned}\Big[\Big(|A|^2-\frac{1}{n}H^2\Big)\tilde{H}\Big]^{\cdot} &= \Delta\Big[\Big(|A|^2-\frac{1}{n}H^2\Big)\tilde{H}\Big]-2\Big\langle\nabla_i\tilde{H},\nabla_i\Big(|A|^2-\frac{1}{n}H^2\Big)\Big\rangle\\ &\quad -4\tilde{H}|\nabla\lambda|^2-2\tilde{H}\Big(|\nabla A|^2-\frac{1}{n}|\nabla H|^2\Big)\\ &\quad +\frac{1}{r}\Big\langle\nabla_i\Big[\Big(|A|^2-\frac{1}{n}H^2\Big)\tilde{H}\Big],\nabla_i r\Big\rangle\\ &\quad +3(|A|^2+\lambda^2)\Big(|A|^2-\frac{1}{n}H^2\Big)\tilde{H}\\ &\quad +\frac{4\tilde{H}}{nr}(H-n\lambda)\langle\nabla_i\lambda,\nabla_i r\rangle.\end{aligned}$$

PROOF. This is an easy consequence of Lemma 3.7 and Lemma 3.10.

Let $M_{ij}:=h_{ij}-\Big(\varepsilon H+\dfrac{a}{r}\Big)g_{ij}$.

LEMMA 3.14.

$$\begin{aligned}\dot{M}_{ij} &= \Delta M_{ij}+\frac{1}{r}\langle\nabla_k M_{ij},\nabla_k r\rangle-2(H+\lambda)h_{jm}g^{ml}h_{li}+(|A|^2+\lambda^2)h_{ij}\\ &\quad -\frac{1}{r^2}g^{nm}(h_{mi}\nabla_j r+h_{mj}\nabla_i r)\nabla_n r+\frac{2\lambda}{r^2}\nabla_i r\nabla_j r\\ &\quad +2(H+\lambda)\Big(\varepsilon H+\frac{a}{r}\Big)h_{ij}-\varepsilon H(|A|^2+\lambda^2)g_{ij}\\ &\quad +\frac{2\varepsilon}{r}\langle\nabla_k\lambda,\nabla_k r\rangle g_{ij}+\Big(\frac{a}{r^3}|\nabla r|^2-\frac{a\lambda^2}{r}\Big)g_{ij}.\end{aligned}$$

PROOF. From Lemma 3.1, 3.3, 3.4 and 3.8 we get

$$\begin{aligned}\dot{M}_{ij} &= \Delta h_{ij}+\frac{1}{r}\langle\nabla_k h_{ij},\nabla_k r\rangle-2(H+\lambda)h_{jm}g^{ml}h_{li}+(|A|^2+\lambda^2)h_{ij}\\ &\quad -\frac{1}{r^2}g^{nm}(h_{mi}\nabla_j r+h_{mj}\nabla_i r)\nabla_n r+\frac{2\lambda}{r^2}\nabla_i r\nabla_j r+2(H+\lambda)\Big(\varepsilon H+\frac{a}{r}\Big)h_{ij}\\ &\quad -\Big(\varepsilon\Delta H+\frac{\varepsilon}{r}\langle\nabla_k H,\nabla_k r\rangle+\varepsilon H(|A|^2+\lambda^2)-\frac{2\varepsilon}{r}\langle\nabla_k\lambda,\nabla_k r\rangle\\ &\qquad\qquad +\Delta\frac{a}{r}-\frac{2a}{r^3}|\nabla r|^2+\frac{a\lambda^2}{r}\Big)g_{ij}.\end{aligned}$$

The lemma then follows from

$$\frac{1}{r}\langle\nabla_k M_{ij},\nabla_k r\rangle = \frac{1}{r}\langle\nabla_k h_{ij},\nabla_k r\rangle-\frac{\varepsilon}{r}\langle\nabla_k H,\nabla_k r\rangle g_{ij}+\frac{a}{r^3}|\nabla r|^2 g_{ij}$$

$$\Delta M_{ij} = \Delta h_{ij} - g_{ij}\Delta\left(\varepsilon H + \frac{a}{r}\right)$$

and reorganizing terms.

4. First results

Condition (a) in Theorem 1.1 means that $\tilde{H} = H + \lambda \geq n\frac{a}{r} - |\lambda| \geq \frac{na-1}{r} > 0$, since condition (b) implies that $a > \frac{1}{n}$.

LEMMA 4.1. *If $\tilde{H} \geq \tilde{H}_0 > 0$ on C_0 then $\tilde{H} \geq \tilde{H}_0$ on C_t for all $t \in [0, T)$, where $[0, T)$ is the maximal time interval on which a smooth solution of (2) exists.*

PROOF. This follows from the parabolic maximum principle and Lemma 3.10.

LEMMA 4.2. *If $\tilde{H} > 0$ and $r > r_0 > 0$ on C_0, then this is true on C_t, $t \in [0, T)$.*

PROOF. Otherwise there would be a first time t_0 and a point $p \in C_{t_0}$ such that $r(p, t_0) = r_0$ and $\dot{r}(p, t_0) \leq 0$, $\nabla r(p, t_0) = 0$. This yields, since ν is the outer unit normal, that $\nu = -e_r$ and from Lemma 3.8 and 4.1 we would get that $\dot{r}(p, t_0) = -\tilde{H}(\nu, e_r) = \tilde{H} > 0$ which is a contradiction.

Lemma 4.2 means in particular that

$$r(p, t) \geq r_0 := \min_{p \in C_0} r(p, 0).$$

REMARK. If $M_{ij}dx^i \otimes dx^j$ is a tensor then we will write $M_{ij} \geq 0$, if $M_{ij}v^iv^j \geq 0$ for all $v \in TC$.

From our main condition (a) of Theorem 1.1 we obtain that

$$h_{ij} - \frac{a}{r}g_{ij} > 0$$

on C_0 and so we can find an $\varepsilon > 0$, with $\varepsilon < \min\left(\frac{1}{n}, \frac{1}{2}\left(na^3 - a - \frac{4}{3\sqrt{3}}\right)\right)$ such that

$$\text{(c)} \qquad h_{ij} - \left(\varepsilon H + \frac{a}{r}\right)g_{ij} \geq 0$$

on C_0. We want to show that (c) remains true as long as a smooth solution of (2) exists. For that purpose we need the following maximum principle, which was proved in $[H]$:

Let u_k be a convector field and let g_{ij}, M_{ij} and N_{ij} be symmetric tensors on C which may all depend smoothly on time t. Assume that N_{ij} satisfies a null-eigenvector condition, i.e. for any null-eigenvector $\{v^i\}$ of M_{ij} we have $N_{ij}v^iv^j\geq 0$. Then we have

THEOREM 4.1. (HAMILTON): *Suppose that on* $[0, T)$ *holds*

$$\dot{M}_{ij}=\Delta M_{ij}+\langle u_k, \nabla_k M_{ij}\rangle+N_{ij}$$

where N_{ij} *satisfies the null-eigenvector condition above. If* $M_{ij}\geq 0$ *at* $t=0$ *then this is true for all* $t\in[0, T)$.

From this we get

LEMMA 4.3. *If condition* (c) *is true on* C_0 *then it remains true as long as a smooth solution of* (2) *exists.*

PROOF. This is a consequence of Theorem 4.1 with $u_k=\frac{1}{r}\nabla_k r$ and

$$\begin{aligned}N_{ij}=&-2(H+\lambda)h_{jm}g^{ml}h_{li}+(|A|^2+\lambda^2)h_{ij}\\&-\frac{1}{r^2}g^{nm}(h_{mi}\nabla_j r+h_{mj}\nabla_i r)\nabla_n r+\frac{2\lambda}{r^2}\nabla_i r\nabla_j r+2(H+\lambda)\Big(\varepsilon H+\frac{a}{r}\Big)h_{ij}\\&-\varepsilon H(|A|^2+\lambda^2)g_{ij}+\frac{2\varepsilon}{r}\langle\nabla_k\lambda, \nabla_k r\rangle g_{ij}+\Big(\frac{a}{r^3}|\nabla r|^2-\frac{a\lambda^2}{r}\Big)g_{ij}.\end{aligned}$$

We must only proof that N_{ij} satisfies the null-eigenvector condition. Let t_0 be the first time where at some point $p\in C_{t_0}$ a zero eigenvector $\{v^i\}$ of M_{ij} occurs. Choose an orthonormal basis $(e_1, \cdots, e_n)$ for $T_pC_{t_0}$ such that h_{ij} and thus M_{ij} becomes diagonal. We assume that $v=e_1$ and that $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of h_{ij} at p. Then it follows that at p

$$\lambda_1=\varepsilon H+\frac{a}{r}$$

and we obtain

$$\begin{aligned}N_{ij}v^iv^j&=N_{11}\\&=-2(H+\lambda)\lambda_1^2+(|A|^2+\lambda^2)\lambda_1-\frac{2}{r^2}\lambda_1(\nabla_1 r)^2\\&\quad+\frac{2\lambda}{r^2}(\nabla_1 r)^2+2(H+\lambda)\lambda_1^2-\varepsilon H(|A|^2+\lambda^2)\\&\quad+\frac{2\varepsilon}{r}\langle\nabla_k\lambda, \nabla_k r\rangle+\frac{a}{r^3}|\nabla r|^2-\frac{a\lambda^2}{r}\\&=\frac{a}{r}|A|^2+\frac{a}{r^3}|\nabla r|^2+\frac{2(\lambda-\lambda_1)}{r^2}(\nabla_1 r)^2+\frac{2\varepsilon}{r}\langle\nabla_k\lambda, \nabla_k r\rangle.\end{aligned}$$

From Lemma 2.1 (c) we get

$$\frac{2\varepsilon}{r}\langle\nabla_k\lambda, \nabla_k r\rangle = \frac{2\varepsilon}{r^2}(\lambda_i-\lambda 1_i)(\nabla_i r)^2$$

where $1_i:=1$, $i=1,\cdots,n$. Since $H=\lambda_i 1_i\geq n\left(\varepsilon H+\frac{a}{r}\right)>n\varepsilon H$ and $n\varepsilon<1$ we have $H>0$ and thus $\lambda_i>0$, $i=1,\cdots,n$. Furtheron we have $|\lambda|=\left|\frac{(\nu, e_r)}{r}\right|\leq\frac{1}{r}$. Then we obtain together with Lemma 2.1 (b)

$$\frac{2\varepsilon}{r}\langle\nabla_k\lambda, \nabla_k r\rangle\geq-\frac{2\varepsilon}{r^3}.$$

Since $H>0$ we get $|A|^2=\lambda_i^2 1_i>n\frac{a^2}{r^2}+2n\varepsilon H\frac{a}{r}$ and thus

$$N_{ij}v^iv^j \geq n\frac{a^3}{r^3}+2n\varepsilon H\frac{a^2}{r^2}+\frac{a}{r^3}|\nabla r|^2+2\left(\lambda-\varepsilon H-\frac{a}{r}\right)\frac{(\nabla_1 r)^2}{r^2}-\frac{2\varepsilon}{r^3}.$$

Since $na^3-a-\frac{4}{3\sqrt{3}}>0$, $a>0$ implies that $na^2-1>0$, we obtain

$$N_{ij}v^iv^j \geq n\frac{a^3}{r^3}-\frac{a}{r^3}|\nabla r|^2-\frac{2|\lambda r|}{r^3}|\nabla r|^2-\frac{2\varepsilon}{r^3}$$

and with Lemma 2.1 (b) finally

$$N_{ij}v^iv^j \geq \frac{1}{r^3}\left(na^3-a-\frac{4}{3\sqrt{3}}-2\varepsilon\right)$$

and we can continue the proof since $2\varepsilon<na^3-a-\frac{4}{3\sqrt{3}}$ by assumption.

COROLLARY 4.1. *There are positive constants d_1, d_2, d_3 independent of time such that*

$$d_1<H<d_2\tilde{H}<d_3H.$$

PROOF. This is immediate from Lemma 4.1, 4.2 and Lemma 4.3, since $\tilde{H}=H+\lambda$ and $|\lambda|\leq\frac{1}{r}$.

As in [Hu1] we have

COROLLARY 4.2. $T<\infty$.

5. A bound for f_σ

As in [Hu1] we want to show that the eigenvalues of the second fundamental form asymptotically approach each other at those points, where the mean curvature tends to infinity. The idea is to bound the function $f_\sigma = \frac{|A|^2 - \frac{1}{n}H^2}{H^{2-\sigma}}$ for a positive constant σ. We do this since $|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}^{n}(\lambda_i - \lambda_j)^2$ measures how far the eigenvalues of $\{h_{ij}\}$ diverge from each other.

THEOREM 5.1. *There are constants d_0, $\sigma > 0$ depending only on the initial surface C_0 such that*

$$f_\sigma \leq d_0$$

for all $t \in [0, T)$.

To prove Theorem 5.1 we want to show that high L^p-norms of f_σ are bounded and therefore we will need the following Lemma which is Lemma 5.3 in [Hu1].

LEMMA 5.1. *Let $p \geq 2$. Then for any $\eta > 0$ and any $0 \leq \sigma \leq \frac{1}{2}$ we have the estimate*

$$n\varepsilon^2 \int f_\sigma^p H^2 d\mu \leq (2\eta p + 5) \int f_\sigma^p H^{\sigma-2} |\nabla H|^2 d\mu + \eta^{-1}(p-1) \int f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu.$$

PROOF of 5.1. From Lemma 4.3 we get that $h_{ij} - \varepsilon H g_{ij} \geq 0$ for all $t \in [0, T)$, with $\varepsilon < \frac{1}{n}$. Then we can use Lemma 2.3 to prove Lemma 5.1 exactly as in [Hu1].

LEMMA 5.2. *Let $0 \leq \sigma \leq 1$. Then we have*

$$\begin{aligned} \dot{f}_\sigma \leq & \Delta f_\sigma + \frac{2(1-\sigma)}{H} \langle \nabla_i f_\sigma, \nabla_i H \rangle + \frac{1}{r} \langle \nabla_i f_\sigma, \nabla_i r \rangle \\ & - \varepsilon^2 |\nabla H|^2 H^{\sigma-2} + 2\sigma f_\sigma H^2 + \frac{16n}{r_0^2} H^\sigma. \end{aligned}$$

PROOF. Since $\sigma \leq 1$ we get from Lemma 3.11 and Lemma 2.3 (b)

$$\begin{aligned} \dot{f}_\sigma \leq & \Delta f_\sigma + \frac{2(1-\sigma)}{H} \langle \nabla_i f_\sigma, \nabla_i H \rangle + \frac{1}{r} \langle \nabla_i f_\sigma, \nabla_i r \rangle - \varepsilon^2 |\nabla H|^2 H^{\sigma-2} \\ & + \sigma f_\sigma (|A|^2 + \lambda^2) + \frac{4}{nr} H^{\sigma-2} (H - n\lambda) \langle \nabla_i \lambda, \nabla_i r \rangle \\ & + 2(2-\sigma) \frac{f_\sigma}{Hr} \langle \nabla_i \lambda, \nabla_i r \rangle. \end{aligned}$$

We have $\lambda_i \geq \frac{a}{r} \geq a|\lambda|$ and thus $|\lambda| \leq \frac{H}{na}$. This gives

$$|\langle\nabla_i\lambda, \nabla_i r\rangle| = \frac{1}{r}|\lambda_i - \lambda 1_i|(\nabla_i r)^2 \leq \frac{1}{r}|\lambda_i - \lambda 1_i| 1_i \leq \frac{H}{r}\left(1+\frac{1}{a}\right).$$

Therefore and since $f_\sigma \leq H^\sigma$, $\sigma < 2$ we get

$$\begin{aligned}
&\frac{4}{nr}H^{\sigma-2}(H-n\lambda)\langle\nabla_i\lambda, \nabla_i r\rangle + 2(2-\sigma)\frac{f_\sigma}{Hr}\langle\nabla_i\lambda, \nabla_i r\rangle \\
&\quad\leq 4\frac{H^{\sigma-1}}{r}|\langle\nabla_i\lambda, \nabla_i r\rangle|\left(\frac{1}{n}+\frac{1}{na}+1\right) \\
&\quad\leq \frac{4H^\sigma}{r^2}\left(1+\frac{1}{a}\right)\left(\frac{1}{n}+\frac{1}{na}+1\right) < 16n\frac{H^\sigma}{r_0^2}
\end{aligned}$$

because $r \geq r_0$, $a > \frac{1}{n}$, $n > 2$.

Now taking into account that $\lambda^2 \leq \frac{H^2}{n^2a^2}$, $|A|^2 \leq H^2$, $1+\frac{1}{n^2a^2} \leq 2$, we get Lemma 5.2. If we multiply this inequality with pf_σ^{p-1} and integrate, we get:

$$\begin{aligned}
\text{(i)}\qquad \left(\int f_\sigma^p d\mu\right)^\cdot = \int((f_\sigma^p)^\cdot - H\tilde{H}f_\sigma^p)d\mu \leq &-p(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu \\
&+2(1-\sigma)p\int\frac{f_\sigma^{p-1}}{H}\langle\nabla_i f_\sigma, \nabla_i H\rangle d\mu + p\int\frac{f_\sigma^{p-1}}{r}\langle\nabla_i f_\sigma, \nabla_i r\rangle d\mu \\
&-\varepsilon^2 p\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu + 2\sigma p\int f_\sigma^p H^2 d\mu + \frac{16np}{r_0^2}\int f_\sigma^{p-1}H^\sigma d\mu.
\end{aligned}$$

REMARK. Let us denote by d any positive constant that only depends on C_0.

LEMMA 5.3. *There is a positive constant $d = d(H_0, r_0, n, \varepsilon)$ such that for all $0 \leq \sigma \leq 1$*

$$\frac{16np}{r_0^2}\int f_\sigma^{p-1}H^\sigma d\mu \leq \frac{n\sqrt{p}\,\varepsilon^3}{16}\int f_\sigma^p H^2 d\mu + dp^2\int f_\sigma^p d\mu + dp^2.$$

PROOF. We have $f_\sigma^{p-1}H^\sigma \leq \frac{1}{2\eta}f_\sigma^p H^2 + \frac{\eta}{2}f_\sigma^{p-2}H^{2\sigma-2}$ and since $x^{p-2} \leq x^p + 1$ for all $x \geq 0$ and $H \geq H_0 > 0$, $\sigma \leq 1$ we get with $\eta := \frac{128\sqrt{p}}{r_0^2\varepsilon^3}$

$$\begin{aligned}
f_\sigma^{p-1}H^\sigma &\leq \frac{r_0^2\varepsilon^3}{256\sqrt{p}}f_\sigma^p H^2 + \frac{64\sqrt{p}}{r_0^2\varepsilon^3}H_0^{2\sigma-2}(f_\sigma^p+1) \\
&\leq \frac{r_0^2\varepsilon^3}{256\sqrt{p}}f_\sigma^p H^2 + dp(f_\sigma^p+1)
\end{aligned}$$

with a positive constant d depending only on n, ε, r_0, H_0 but not on p, t or σ.

LEMMA 5.4. *Let $p>2$. Then we have*

$$p\int\frac{f_\sigma^{p-1}}{r}\langle\nabla_i f_\sigma, \nabla_i r\rangle d\mu \leq \frac{p(p-1)}{4}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu+\frac{p^2}{r_0^2}\int f_\sigma^p d\mu.$$

PROOF. First we take into account that for all $\eta>0$

$$\frac{f_\sigma^{p-1}}{r}\langle\nabla_i f_\sigma, \nabla_i r\rangle \leq \frac{f_\sigma^{p-1}}{r}|\nabla f_\sigma||\nabla r| \leq \frac{f_\sigma^{p-2}|\nabla f_\sigma|^2}{2\eta}+\frac{\eta}{2}f_\sigma^p\frac{|\nabla r|^2}{r^2}$$

and since $r\geq r_0$, $\dfrac{1}{p-1}<p$, $|\nabla r|^2\leq 1$ we can continue the poof with $\eta:=\dfrac{2}{p-1}$.

LEMMA 5.5. *If $\sigma<2$ then*

$$2(1-\sigma)p\int\frac{f_\sigma^{p-1}}{H}\langle\nabla_i f_\sigma, \nabla_i H\rangle d\mu$$
$$\leq\frac{p(p-1)}{2}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu+2\frac{p}{p-1}\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu.$$

PROOF. This is immediate from $f_\sigma\leq H^\sigma$ and

$$\frac{f_\sigma^{p-1}}{H}|\nabla H||\nabla f_\sigma| \leq \frac{1}{2\eta}f_\sigma^{p-2}|\nabla H|^2+\frac{\eta}{2}\frac{f_\sigma^p}{H^2}|\nabla H|^2,$$

if we take $\eta:=\dfrac{2}{p-1}$.

We are now going to prove the following theorem:

THEOREM 5.2. *There is a constant $d=d(C_0, n, \varepsilon)$ such that for all $p\geq\max\left(\dfrac{25}{\varepsilon^2}, 2\right)$, $\sigma\leq\min\left(1, \dfrac{n\varepsilon^3}{32\sqrt{p}}\right)$*

$$\left(\int f_\sigma^p d\mu\right)^{\frac{1}{p}} \leq d.$$

PROOF. In a first step we obtain from (i), Lemma 5.3, 5.4 and 5.5 that

$$\left(\int f_\sigma^p d\mu\right)^{\cdot} \leq -p(p-1)\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu+\frac{p(p-1)}{2}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu$$
$$+\frac{2p}{p-1}\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu+\frac{p(p-1)}{4}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu+\frac{p^2}{r_0^2}\int f_\sigma^p d\mu$$

$$-\varepsilon^2 p\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu+2\sigma p\int f_\sigma^p H^2 d\mu$$
$$+\frac{n\sqrt{p}\,\varepsilon^3}{16}\int f_\sigma^p H^2 d\mu+dp^2\int f_\sigma^p d\mu+dp^2$$

and since $\frac{2}{p-1}\leq\frac{\varepsilon^2}{2}$ we obtain with $2\sigma p\leq\frac{n\sqrt{p}\,\varepsilon^3}{16}$

$$\Big(\int f_\sigma^p d\mu\Big)^{\cdot}\leq-\frac{p(p-1)}{4}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu-\frac{\varepsilon^2 p}{2}\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$+\frac{n\sqrt{p}\,\varepsilon^3}{8}\int f_\sigma^p H^2 d\mu+dp^2\int f_\sigma^p d\mu+dp^2.$$

Then we get with Lemma 5.1

$$\Big(\int f_\sigma^p d\mu\Big)^{\cdot}\leq-\frac{p(p-1)}{4}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu-\frac{\varepsilon^2 p}{2}\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$+\frac{\varepsilon\sqrt{p}}{8}(2\eta p+5)\int\frac{f_\sigma^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$+\frac{\varepsilon\sqrt{p}(p-1)}{8\eta}\int f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu+dp^2\int f_\sigma^p d\mu+dp^2$$

for all $\eta>0$. We choose $\eta:=\frac{\varepsilon}{2\sqrt{p}}$ and get, since $5\leq\varepsilon\sqrt{p}$

$$\Big(\int f_\sigma^p d\mu\Big)^{\cdot}\leq dp^2\int f_\sigma^p d\mu+dp^2$$

and therefore

$$\int f_\sigma^p d\mu\leq\Big(\int_{C_0} f_\sigma^p d\mu+1\Big)e^{dp^2 t}-1$$

and the theorem follows from Corollary 4.2.

COROLLARY 5.1. *For all* $p\geq\max\left(\left(\frac{64m}{\varepsilon^3 n}\right)^2,2\right)$, $\sigma\leq\min\left(1,\frac{\varepsilon^3 n}{64\sqrt{p}}\right)$ *we have*

$$\Big(\int H^m f_\sigma^p d\mu\Big)^{\frac{1}{p}}\leq d.$$

PROOF. This is immediate from Theorem 5.2 since

$$\int H^m f_{\sigma'}^p d\mu=\int f_\sigma^p d\mu$$

with $\sigma'=\sigma+\frac{m}{p}\leq\frac{n\varepsilon^3}{32\sqrt{p}}$.

Now we can proceed exactly as in [Hu1] section 5 to show that f_σ is bounded for suffciently small σ. This ends the proof of Theorem 5. 1.

6. The bound for $|\nabla H|^2$

As in [Hu1] we want to show:

THEOREM 6. 1. *For any $\eta>0$ there is a constant $d(\eta, C_0, n)$ such that*

$$|\nabla H|^2 \leq \eta H^4 + d(\eta, C_0, n).$$

PROOF. First we get from Lemma 3. 12 and Schwartz' inequality that

$$(1)\qquad (|\nabla \tilde{H}|^2)^{\cdot} \leq \Delta|\nabla\tilde{H}|^2 - 2|\nabla^2\tilde{H}|^2 + \frac{1}{r}\langle\nabla_i|\nabla\tilde{H}|^2, \nabla_i r\rangle + 4(|A|^2+\lambda^2)|\nabla\tilde{H}|^2 + 2\tilde{H}\langle\nabla_i\tilde{H}, \nabla_i(|A|^2+\lambda^2)\rangle.$$

If we set $h_{ij}^0 := h_{ij} - \frac{1}{n}Hg_{ij}$, we get

$$\begin{aligned}
2|\langle\nabla_i\tilde{H}, \nabla_i(|A|^2-\tfrac{1}{n}H^2)\rangle| &\leq 2|\langle\nabla_i H, \nabla_i(|A|^2-\tfrac{1}{n}H^2)\rangle| \\
&\quad + 2|\langle\nabla_i\lambda, \nabla_i(|A|^2-\tfrac{1}{n}H^2)\rangle| \\
&= 4|\langle\nabla_i Hh_{kl}^0, \nabla_i h_{kl}^0\rangle| + 4|\langle\nabla_i\lambda h_{kl}^0, \nabla_i h_{kl}^0\rangle| \\
&\leq 4|\nabla H||h_{kl}^0||\nabla A| + 4|\nabla\lambda||h_{kl}^0||\nabla A| \\
&\leq 4|\nabla H||h_{kl}^0||\nabla A| + 4\tilde{H}|\nabla\lambda|^2 + 4\frac{1}{4\tilde{H}}|h_{kl}^0|^2|\nabla A|^2
\end{aligned}$$

and since $|h_{kl}^0|^2 = |A|^2 - \frac{1}{n}H^2 \leq dH^{2-\sigma}$, $|\nabla H| \leq n|\nabla A|$, we obtain

$$2|\langle\nabla_i\tilde{H}, \nabla_i(|A|^2-\tfrac{1}{n}H^2)\rangle| \leq 4nd^{\frac{1}{2}}H^{1-\frac{\sigma}{2}}|\nabla A|^2 + 4\tilde{H}|\nabla\lambda|^2 + d\frac{H^{2-\sigma}}{\tilde{H}}|\nabla A|^2.$$

Now we use Corollary 4. 1 to derive

$$\begin{aligned}
2|\langle\nabla_i\tilde{H}, \nabla_i(|A|^2-\tfrac{1}{n}H^2)\rangle| &\leq d\tilde{H}^{1-\frac{\sigma}{2}}|\nabla \mathrm{A}|^2 + 4\tilde{H}|\nabla\lambda|^2 \\
&\leq \frac{2(n-1)}{3n}\tilde{H}|\nabla A|^2 + 4\tilde{H}|\nabla\lambda|^2 + d|\nabla A|^2.
\end{aligned}$$

Finally taking into account that

$$\frac{4}{nr}(H-n\lambda)\langle\nabla_i\lambda, \nabla_i r\rangle \leq dH^2$$

and

$$|\nabla A|^2 - \frac{1}{n}|\nabla H|^2 \geq \frac{2(n-1)}{3n}|\nabla A|^2$$

we get from Lemma 3.13 that

$$\begin{aligned}(2) \qquad [(|A|^2 - \tfrac{1}{n}H^2)\tilde{H}]^{\cdot} \leq & \Delta((|A|^2 - \tfrac{1}{n}H^2)\tilde{H}) + \frac{1}{r}\langle \nabla_i((|A|^2 - \tfrac{1}{n}H^2)\tilde{H}), \nabla_i r\rangle \\ & - \frac{2(n-1)}{3n}\tilde{H}|\nabla A|^2 + 3(|A|^2+\lambda^2)(|A|^2 - \tfrac{1}{n}H^2)\tilde{H} + d\tilde{H}H^2 + d|\nabla A|^2.\end{aligned}$$

Now we want to find a constant $N>0$ such that for $\eta>0$ the following function is bounded for all $t\in[0, T)$

$$f := \frac{|\nabla \tilde{H}|^2}{\tilde{H}} + N\left(|A|^2 - \frac{1}{n}H^2\right)\tilde{H} + Nd|A|^2 - \eta\tilde{H}^3$$

with the same d as in (2). As in [Hu1] we get the inequality

$$\begin{aligned}(3) \qquad \left(\frac{|\nabla \tilde{H}|^2}{\tilde{H}}\right)^{\cdot} \leq & \Delta\frac{|\nabla \tilde{H}|^2}{\tilde{H}} + \frac{1}{r}\langle \nabla_i \frac{|\nabla \tilde{H}|^2}{\tilde{H}}, \nabla_i r\rangle \\ & + 3(|A|^2+\lambda^2)\frac{|\nabla \tilde{H}|^2}{\tilde{H}} + 2\langle \nabla_i \tilde{H}, \nabla_i(|A|^2+\lambda^2)\rangle.\end{aligned}$$

Then (1), (2) and (3) yield

$$\begin{aligned}(4) \quad \dot{f} \leq & \nabla f + \frac{1}{r}\langle \nabla_i f, \nabla_i r\rangle + 3(|A|^2+\lambda^2)\frac{|\nabla \tilde{H}|^2}{\tilde{H}} \\ & + 2\langle \nabla_i \tilde{H}, \nabla_i(|A|^2+\lambda^2)\rangle \\ & - 2N\frac{n-1}{3n}\tilde{H}|\nabla A|^2 + 3N(|A|^2+\lambda^2)(|A|^2 - \frac{1}{n}H^2)\tilde{H} \\ & + Nd\tilde{H}H^2 + Nd|\nabla A|^2 - 2Nd|\nabla A|^2 - 4Nd|\nabla\lambda|^2 \\ & + 2Nd|A|^2(|A|^2+\lambda^2) - 4Nd\frac{\lambda}{r}\langle \nabla_i\lambda, \nabla_i r\rangle + 6\eta\tilde{H}|\nabla\tilde{H}|^2 - 3\eta\tilde{H}^3(|A|^2+\lambda^2).\end{aligned}$$

Now we have

$$0 \leq |\nabla_i \tilde{H} h_{kl} - \nabla_i h_{kl}\tilde{H}|^2 = |\nabla \tilde{H}|^2|A|^2 + |\nabla A|^2\tilde{H}^2 - \tilde{H}\langle \nabla_i|A|^2, \nabla_i\tilde{H}\rangle$$

and using Corollary 4.1, Lemma 2.1 (c), Lemma 2.2 (a) and the fact that

$$|\nabla \tilde{H}|^2 \leq 2|\nabla H|^2 + 2|\nabla\lambda|^2, \ |A|^2 \leq H^2,$$

we get

$$\begin{aligned}2\langle \nabla_i \tilde{H}, \nabla_i(|A|^2+\lambda^2)\rangle & \leq 2\frac{|A|^2}{\tilde{H}}|\nabla \tilde{H}|^2 + 2\tilde{H}|\nabla A|^2 + 4|\lambda|\left(\frac{|\nabla \tilde{H}|^2}{2} + \frac{|\nabla\lambda|^2}{2}\right) \\ & \leq dH^3 + dH|\nabla A|^2.\end{aligned}$$

Using this we can proceed exactly as in [Hu1] to find a constant $N(\eta, C_0, n)$ such that

$$\dot{f} \leq \Delta f + \frac{1}{r}\langle \nabla_i f, \nabla_i r \rangle + d$$

which implies that $f \leq d$. From this we obtain together with Theorem 5. 1 that

$$|\nabla \tilde{H}|^2 \leq \eta \tilde{H}^4 + d\tilde{H} \leq 2\eta \tilde{H}^4 + d$$

and since η is arbitrary and $|\nabla H|^2 = |\nabla \tilde{H} - \nabla \lambda|^2 \leq 2|\nabla \tilde{H}|^2 + 2|\nabla \lambda|^2$ we get with the help of Lemma 2. 1 (b) and (c) and Corollary 4. 1 that

$$|\nabla H|^2 \leq \eta H^4 + d$$

which proves Theorem 6. 1.

7. Proof of Theorem 1. 1

THEOREM 7. 1.

$$\lim_{t \to T} \max_{C_t} |A|^2 = \infty.$$

PROOF. Since $\lambda^2 \leq \frac{1}{r^2}$ is bounded by Lemma 4. 2, this can be easily derived from Theorem 8. 1 in [Hu1].

If we use Corollary 4. 1 we can calculate exactly as in [Hu1] to show:

THEOREM 7. 2.

$$\lim_{t \to T} \frac{H_{max}}{H_{min}} = 1.$$

THEOREM 7. 3.

$$\int_0^T H_{max}^2(\tau) d\tau = \infty.$$

THEOREM 7. 4. *If* $h := \frac{\int_{C_t} H\tilde{H} d\mu}{\int_{C_t} d\mu}$, *then*

$$\int_0^T h(\tau) d\tau = \infty.$$

COROLLARY 7. 1.

$$\lim_{t \to T} \frac{|A|^2}{H^2} = \frac{1}{n}.$$

These theorems together with Theorem 6.1 then prove Theorem 1.1 in the same way as in [Hu1].

8. The normalized equation

We have just seen that the solution of the unnormalized equation

$$\dot{F} = -(H+\lambda)\cdot\nu$$

shrinks down to a single point $p\in \boldsymbol{R}^{n+1}_{>0}$ after finite time. As in [Hu1] we want to rescale the solutions by a constant factor Ψ depending on t such that the total area $|\widehat{C}_t|$ of the rescaled surfaces are equal to $|C_0|$ for all $t\in[0, T)$. So if we set $\widehat{F}:=\Psi F$ and introduce the new time variable $\hat{t}(t):=\int_0^t \Psi^2(\tau)d\tau$ we get proceeding as in [Hu1]:

$$\frac{\partial \widehat{F}}{\partial \hat{t}} = -(\widehat{H}+\hat{\lambda})\cdot\hat{\nu}+\frac{1}{n}\hat{h}\cdot\widehat{F}$$

where here $h(t):=\dfrac{\int_{C_t} H(H+\lambda)d\mu}{\int_{C_t} d\mu}$. Then we can prove the next Lemma in the same way as this was done for Lemma 9.2 in [Hu1]

LEMMA 8.1. (a) $\hat{h}_{ij}\geq \varepsilon\widehat{H}\hat{g}_{ij}$

(b) $\dfrac{\widehat{H}_{max}}{\widehat{H}_{min}}\rightarrow 1$ *as* $\hat{t}\rightarrow\widehat{T}$

(c) $\dfrac{|\widehat{A}|^2}{\widehat{H}^2}\rightarrow\dfrac{1}{n}$ *as* $\hat{t}\rightarrow\widehat{T}$.

LEMMA 8.2. *There are constants c_1 and c_2 such that for $0\leq\hat{t}<\widehat{T}$.*

$$0<c_1\leq\widehat{H}_{min}\leq\widehat{H}_{max}\leq c_2<\infty.$$

PROOF. Suppose that the unnormalized equation shrinks down to the point $p\in\mathbf{R}^{n+1}_{>0}$. Then we define:

$$\overline{F}:=\Psi(F-p)=\widehat{F}-\Psi p.$$

Then we get from Lemma 8.1 (a) that $(\overline{F},\overline{F})\leq c$ with a constant c independent of time and we have $\bar{h}_{ij}=\hat{h}_{ij}$, $\bar{\nu}=\hat{\nu}$ and so on. If V denotes the enclosed volume then the same calculations as in [HU1] show that

$$\widehat{V}_{\hat{t}}=\overline{V}_{\hat{t}}\geq\frac{1}{n+1}\overline{H}^{-1}_{max}\int_{\overline{C}_{\hat{t}}}(\overline{F},\bar{\nu})\overline{H}d\bar{\mu}=\frac{n}{n+1}\overline{H}^{-1}_{max}|C_0|=\frac{n}{n+1}\widehat{H}^{-1}_{max}|C_0|$$

and

$$\widehat{V}_{\hat{t}}\leq c_n(\varepsilon\widehat{H}_{min})^{-(n+1)}$$

which gives the upper bound in view of Lemm 8.1 (b). The upper bound together with Lemma 8.1 (b) implies that

$$\lim_{\hat{t}\to\hat{T}}(\hat{H}_{max}-\hat{H}_{min}) = 0.$$

From the isoperimetric inequality we get

$$\hat{V}_{\hat{t}} \leq c_n|C_0|^{\frac{n}{n-1}}$$

and the first variation formula yields

$$\begin{aligned}|\bar{C}_{\hat{t}}| &= \frac{1}{n}\int_{\bar{C}\hat{t}}\bar{H}(\bar{F},\, \bar{\nu})d\bar{\mu}\\ &\leq \frac{1}{n}\left(\bar{H}_{max}\int_{\bar{C}\hat{t}}(\bar{F},\, \bar{\nu})d\bar{\mu}+(\bar{H}_{min}-\bar{H}_{max})\int_{\bar{C}\hat{t}\cap\{(\bar{F},\, \bar{\nu})\leq 0\}}(\bar{F},\, \bar{\nu})d\bar{\mu}\right).\end{aligned}$$

This gives the lower bound since by the divergence theorem

$$\hat{V}_{\hat{t}} = \int_{\hat{C}\hat{t}}(\hat{F},\, \hat{\nu})d\hat{\mu}$$

and $\hat{V}=\bar{V}$, $\hat{H}=\bar{H}$, $|\bar{F}|\leq c$, $\lim_{\hat{t}\to\hat{T}}(\hat{H}_{max}-\hat{H}_{min})=0$ and $|\hat{C}_{\hat{t}}|=|\bar{C}_{\hat{t}}|=|C_0|$.
Since $\hat{h}\leq const\,\hat{H}^2_{max}\leq const$ we get again as in [Hu1]

COROLLARY 8.1. $\hat{T}=\infty$.

9. Proof of Theorem 1.2

THEOREM 9.1. *There is a constant $c>0$ such that for all $\hat{t}\in[0,\infty)$*

$$\Psi(\hat{t})\geq e^{c\hat{t}}.$$

PROOF. We have

$$\frac{\frac{\partial\Psi}{\partial t}}{\Psi} = \frac{1}{n}h$$

therefore we obtain

$$\frac{\frac{\partial\Psi}{\partial\hat{t}}}{\Psi} = \Psi^{-2}\frac{1}{n}h = \frac{1}{n}\hat{h}$$

and since Corollary 4.1 takes over unchanged to the normalized case we get from Lemma 8.2 that

$$\frac{\frac{\partial \Psi}{\partial \hat{t}}}{\Psi} \geq c > 0$$

for all $\hat{t} \in [0, \infty)$ and the theorem follows since $\Psi(0)=1$.

THEOREM 9.2. *There are constasts c, $\delta>0$ such that*

$$|\hat{A}|^2 - \frac{1}{n}\hat{H}^2 \leq ce^{-\delta \hat{t}}.$$

PROOF. By Theorem 5.1 we have

$$|A|^2 - \frac{1}{n}H^2 \leq d_0 H^{2-\sigma}$$

with positive constants d_0 and σ. Now we multiply this by Ψ^{-2} and get

$$|\hat{A}|^2 - \frac{1}{n}\hat{H}^2 \leq d_0 \hat{H}^{2-\sigma}\Psi^{-\sigma}$$

and the theorem follows from Lemma 8.2 and Theorem 9.1.

Then we can proceed exactly as in [H] section 17 to conclude

THEOREM 9.3. *There are constants $\delta>0$ and $c<\infty$ such that*

(i) $\hat{H}_{max} - \hat{H}_{min} \leq ce^{-\delta \hat{t}}$

(ii) $|\hat{h}_{ij}\hat{H} - \frac{1}{n}\hat{h}\hat{g}_{ij}| \leq ce^{-\delta \hat{t}}$

(iii) $\max_{\hat{C}} |\nabla^m \hat{A}| \leq c_m e^{-\delta \hat{t}}$,

and since $\bar{C}$ stays in a bounded region and $\bar{h}_{ij} = \hat{h}_{ij}$, $\bar{g}_{ij} = \hat{g}_{ij}$ the same arguments as in [Hu1] section 10 show that $\bar{C}_{\hat{t}}$ converges to a sphere in the C^∞-topology as $\hat{t} \to \infty$. This completes the proof of Theorem 1.2.

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