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Selfsimilar solutions of the porous medium equation without sign restriction

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Abstract

We consider radially symmetric selfsimilar solutions $u(x, t)= t^{-\alpha}U(|x|t^{-\beta})$ of the porous medium equation $u_t-\Delta(|u|^{m-1}u)=0$. If $m\in \left(\frac{(N-2)_+}{N}, 1\right)$, we show that the resulting ODE allows global solutions with rapid decay for a sequence of parameters $k=\alpha/\beta$, denoted by $\{k_i^g(m, N)\}_{i\in N}\subset \boldsymbol{R}^+$. The corresponding solution U_i has exactly $(i-1)$ simple zeroes in $\boldsymbol{R}^+$. This case was left open by previous papers, where the result for the degenerate case was given.
Besides the existence result in the singular case $m<1$ for arbitrary space dimension N we prove continuity of the $k_i^g(., N)$ at functions of m in $\left(\frac{(N-2)_+}{N}, 1\right)$.
In one space dimension there also exist antisymmetric solutions with rapid decay for certain values $\{k_i^u(m)\}_{i\in N}$. We show that these values as well as the $k_i^g(., 1)$ are continuous functions of m in $\boldsymbol{R}^+$ and identify their limits $m\rightarrow\infty$ with compactly supported solutions of a limit problem.

Chapter 1. Introduction

In this paper we consider a special class of solutions of the Porous Medium Equation

$$(PME) \qquad u_t-\Delta(|u|^{m-1}u)=0 \qquad \text{in } \boldsymbol{R}^N\times(0, T)$$

with $m>0$ and $N\geq 1$. These radially symmetric, so called selfsimilar solutions

$$u(x, t)=t^{-\alpha}U(|x|t^{-\beta}),\ \alpha, \beta>0,$$

play an important role in describing the large time behaviour of initial value problems related to (PME). We stress the fact that we impose no sign restriction on the solution.
The above ansatz leads to an initial value problem for $U=U(\eta)$:

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$$(P_g)\qquad \begin{aligned} &V''+\frac{N-1}{\eta}V'+\beta\eta U'+\alpha U=0 \qquad\qquad \text{in } \boldsymbol{R}^+,\\ &V=|U|^{m-1}U, \qquad \alpha(m-1)+2\beta=1,\\ &V'(0)=0, \qquad V(0)=1. \end{aligned}$$

It is sufficient to examine the case $V(0)=1$ due to the scaling invariance of (P_g) – if U is a solution, so is

$$\hat{U}(\eta) := \lambda^{-2}U(\lambda^{m-1}\eta).$$

The question of existence and uniqueness of solutions of (P_g) was solved by Hulshof in [H]. He showed that for every $k>0$ there is a unique solution (U, V) with

$$U\in C^0(\boldsymbol{R}^+), \quad V\in C^1(\boldsymbol{R}^+), \quad U, V' \text{ absolutely continuous.}$$

The sign change condition is given by $V=0$ and $V'\neq 0$: Given a zero η_0 of V, the solution has to be continued by the zero function if and only if $V'(\eta_0)=0$.

In what follows the most important parameter will be

$$k := \frac{\alpha}{\beta}.$$

Noting the above relation between α and β in (P_g) and choosing suitable parameter ranges there are bijections to α and β, given by

$$\alpha=\frac{k}{2+k(m-1)}, \qquad \beta=\frac{1}{2+k(m-1)}.$$

Let us mention that (P_g) can be transformed into the equivalent equation

$$V''+\frac{N-1}{\hat{\eta}}V'+\hat{\eta}U'+kU=0 \qquad \text{in } \boldsymbol{R}^+$$

by the scaling $U=U(\hat{\eta})$, $\hat{\eta} := \beta^{1/2}\eta$. This equation only contains k as a parameter. Both versions will be used.

Concerning the structure of solutions of (P_g) we have the following

THEOREM: ([H] and [BHV]) *Let $N=1$ and $m>0$. Then there exists a strictly increasing sequence $\{k_i^g(m, N)\}_{i\in N}\subset\boldsymbol{R}^+$, such that the corresponding solutions $V(\eta)$ of (P_g) have the following properties:*

(i) *If $k\in(k_{i-1}^g, k_i^g]$, then V has exactly $(i-1)$ zeroes in $\boldsymbol{R}^+\cap \operatorname{supp} V$.*

(ii) *If k does not belong to the sequence and is less than its supremum, then*

$$\eta^{km}V(\eta)\to C\neq 0 \qquad \text{if } \eta\to\infty.$$

(iii) *If* $k=k_i^g(m, N)$, *then*

$$\eta^{km}V(\eta)\to 0 \qquad \text{if } \eta\to\infty.$$

REMARK: First, in the degenerate case $m>1$ the sequence k_i^g converges to infinity (see [H]); in the case of singular diffusion $m<1$ we have an upper bound $k_i^g<\frac{2}{1-m}$. It is not known whether this value is also the limit of the sequence.
Secondly, the solution V in (iii) is called i-th eigenfunction and $k_i^g(m, N)$ the i-th eigenvalue. The large time behaviour of the eigenfunction is known to be as follows:

(i) If $m>1$, then the eigenfunctions have compact support.
(ii) If $m=1$, then $\eta^{-2(i-1)}e^{\eta^2/2}V(\eta)\to C\neq 0$ for $\eta\to\infty$.
(iii) If $m<1$, then $\eta^{\frac{2m}{1-m}}V(\eta)\to C\neq 0$ for $\eta\to\infty$.

In arbitrary space dimension only the degenerate case $m>1$ was solved. Hulshof showed that the theorem is also valid in this case.
In this paper we will fill the gap by proving that the above theorem holds for $N>1$ and $m\in\left(\frac{(N-2)_+}{N}, 1\right)$.

This parameter range is the interesting one, as in the case $m<\frac{(N-2)_+}{N}$ extinction in finite time for solutions of (PME) occurs (see [BC]).
The proof of this result is given in chapter 2.

The first eigenfunction is known explicitely; it is the famous Barenblatt-solution (see [B])

$$U(\eta)=\left(1-\frac{m-1}{2m(m+1)}|\eta|^2\right)_+^{\frac{1}{m-1}},$$

and the corresponding eigenvalue is given by $k_1^g(m, N)=N$.
Kamin and Vasquez in [KV] proved that the solution of the initial value problem related to (PME) with

$$u(.,0)=u_0 \qquad \text{in } \boldsymbol{R}^N,$$
$$u_0\in L^1(\boldsymbol{R}^N), \qquad M:=\int_{\boldsymbol{R}^N}u_0(x)dx\neq 0$$

converges to the Barenblatt-solution with mass M for large times. Using strictly one dimensional techniques, Bernis, Hulshof and Vasquez in [BHV] proved that the large time behaviour in case of

$$\int_R u_0(x)dx=0, \qquad \int_R\int_{-\infty}^{x} u_0(r)dr\,dx=0,$$
$$\int_{-\infty}^{s}\int_{-\infty}^{x} u_0(r)drdx\geq 0 \qquad \text{for all } s\in \boldsymbol{R}$$

can be described by the second eigenfunction.

As shown in [H] and [BHV], in one space dimension there are also not radially symmetric eigenfunctions, that is solutions

$$u(x, t)=t^{-\alpha}U(xt^{-\beta})$$

of (PME) with rapid decay for $\eta\to\infty$ as described above. Such eigenfunctions occur in case of antisymmetric initial data replacing the symmetric ones in (P_g). Again normalized due to the scaling invariance of the equation the problem now reads

$$(P_u) \qquad \begin{gathered} V''+\beta\eta U'+\alpha U=0 \quad \text{in } \boldsymbol{R}^+, \\ V=|U|^{m-1}U, \quad \alpha(m-1)+2\beta=1, \\ V'(0)=1, \quad V(0)=0. \end{gathered}$$

Again the theorem on the qualitative structure of solutions is valid; the antisymmetric eigenvalues are called $k_i^u(m)$. They alternate with the $k_i^g(m, 1)$

$$k_{i-1}^g(m, 1)<k_{i-1}^u(m)<k_i^g(m, 1)<k_i^u(m).$$

The first antisymmetric eigenfunction is the dipole solution (see [BZ])

$$U(\eta)=\eta^{\frac{1}{m}}\left(1-\frac{m-1}{2m(m+1)}|\eta|^{1+\frac{1}{m}}\right)_+^{\frac{1}{m-1}}.$$

It is positive in $\boldsymbol{R}^+$ and the eigenvalue is $k_1^u(m)=2$.
As shown in [KV] the dipole solution describes the large time behaviour of solutions of the initial value problem related to (PME) with

$$\int_R u_0(x)dx=0 \qquad \text{and} \qquad \int_R xu_0(x)dx\neq 0.$$

This result fills the gap arising in the results concerning large time behaviour with symmetric eigenfunctions in one space dimension. The existence of solutions of dipole-type in arbitary space dimension in the degenerate case was proved by Hulshof and Vasquez in [HV]. Such solutions are antisymmetric in one space direction, are positive in the halfspace given by this direction and have compact support.
In this paper, however, we want to concentrate on the radially symmetric

case and the case of one space dimension.

Results concerning the continuity and the limit behaviour $m \to \infty$ of all these eigenvalues are subject of the third and fourth chapters of this paper.
Besides the eigenvalues $k_1^g(m, N)=N$ and $k_1^u(m)=2$ the eigenvalues in the linear case are explicitely known to be

$$k_i^g(1, N)=N+2(i-1),$$
$$k_i^u(1)=2i.$$

Moreover in [HV] the estimate

$$k_2^g(m, N)>N+2 \quad \text{if } m>1$$

was proved and in [BHV] the continuity of $k_2^g(., 1)$ as a function of m in $\boldsymbol{R}^+$ as well as the limit

$$\lim_{m\to\infty} k_2^g(m, 1)=4$$

could be stated.

By generalization of the techniques in [BHV] we are able to prove the continuity of all eigenvalues as functions of m in $\boldsymbol{R}^+$ in one space dimension. In arbitrary space dimension we have to restrict the result to the singular case $m<1$. Only the case $k_2^g(., N)$ can be handled in the degenerate case, due to the above lower bound.

In order to examine the limit $m \to \infty$ of the eigenvalues, we first observe that all eigenvalues remain bounded as functions of m.
Following [BHV], who showed the above result for $k_2^g(., 1)$, we define

$$Z(\eta):=-\frac{1}{k-2}+\int_0^\eta x^{1-N}\int_0^x t^{N-1}U(t)\,dtdx.$$

This function satisfies

$$Z''+\frac{N-1}{\eta}Z'=U=|V|^{\frac{1}{m}-1}V \qquad \text{in } \boldsymbol{R}^+$$

and, combined with the ODE from (P_g) in a twice integrated version we are led to a system with unknown functions V and Z. Passing to the limit in this problem we arrive at

$$V+\eta Z'+(k-2)Z=0 \qquad \text{in } \boldsymbol{R}^+,$$

$$
(P_g^\infty)\quad
\begin{aligned}
&Z''+\frac{N-1}{\eta}Z'\in sgn\, V \qquad \text{a.e. in } \boldsymbol{R}^+,\\
&Z(0)=-\frac{1}{k-2}, \qquad Z'(0)=0,\\
&V(0)=1, \quad V'(0)=0,\\
&\frac{1}{2}|V'(\eta)|^2+k|V(\eta)| \text{ is nonincreasing in } \boldsymbol{R}^+.
\end{aligned}
$$

This will be shown in detail in chapter 4.
Concerning the structure of solutions the limit problem has the same properties as (P_g). In particular it allows solutions with compact support for certain discrete values of k, which will be called eigenvalues of (P_g^∞). In fact we will prove that there is a strictly increasing sequence $\{k_i^g(\infty, N)\}_{i\in \boldsymbol{N}}$, such that V, Z have compact support and V has precisely $(i-1)$ zeroes, if and only if $k=k_i^g(\infty, N)$. The same result holds for the limit problem derived from (P_u).
As in chapter 3, in one space dimension we are able to identify these eigenvalues with the limits $m\to\infty$ of the eigenvalues of (P_g) and (P_u), whereas in arbitrary space dimension only the case k_2^g can be treated.
More as a curiosity we present the value of this limit:

$$
\lim_{m\to\infty} k_2^g(m, N)=\begin{cases} 2+\dfrac{2(N-2)}{N+2-N\sqrt[N]{4}}, & \text{if } N\neq 2,\\[2mm] 2+\dfrac{2}{2\ln 2-1}, & \text{if } N=2.\end{cases}
$$

Chapter 2. Existence and uniqueness of the eigenvalues for the case $m<1$

We start giving the precise result and recall, that k is an eigenvalue, if and only if the corresponding solution V of (P_g) tends faster to zero than η^{-km}.

THEOREM 2.1: *Let $N\in\boldsymbol{N}$ and $m\in\left(\frac{(N-2)_+}{N}, 1\right)$. Then there is a strictly increasing sequence $\{k_i^g(m, N)\}_{i\in\boldsymbol{N}}$, bounded by $\frac{2}{1-m}$, such that a solution V of (P_g) is i-th eigenfunction, iff $k=k_i^g(m, N)$.*
If $k<\sup_{i\in N} k_i^g(m, N)$ and k is not contained in the sequence, V behaves like η^{-km} for $\eta\to\infty$ and if $k\in(k_{i-1}^g, k_i^g)$, V has exactly $(i-1)$ zeroes in $\boldsymbol{R}^+$.

PROOF: As mentioned in chapter 1, (P_g) possesses a unique global solution for all $k>0$. To go on we change variables

$$t:=\log\eta,\quad f(t):=e^{\frac{2m}{1-m}t}V(e^t)$$

and arrive at an autonomous first order system:

$$g'=c_1g-\frac{\beta}{m}|f|^{\frac{1}{m}-1}g-\alpha|f|^{\frac{1}{m}-1}f,$$
$$f'=g+\frac{2m}{1-m}f.$$

The constant c_1 is given by $c_1:=\frac{2}{1-m}-N$ and is positive due to the assumption on m.
The phase plane contains three critical points: the origin - a repeller with eigenvalues $\frac{2m}{1-m}$ and c_1 and eigenvectors $v_1:=(0,1)$ and $v_2:=(2-N,1)$, respectively - and two saddle points $\pm P$, given by

$$|f|^{\frac{1}{m-1}}=2mc_1,\qquad g=-\frac{2m}{1-m}f.$$

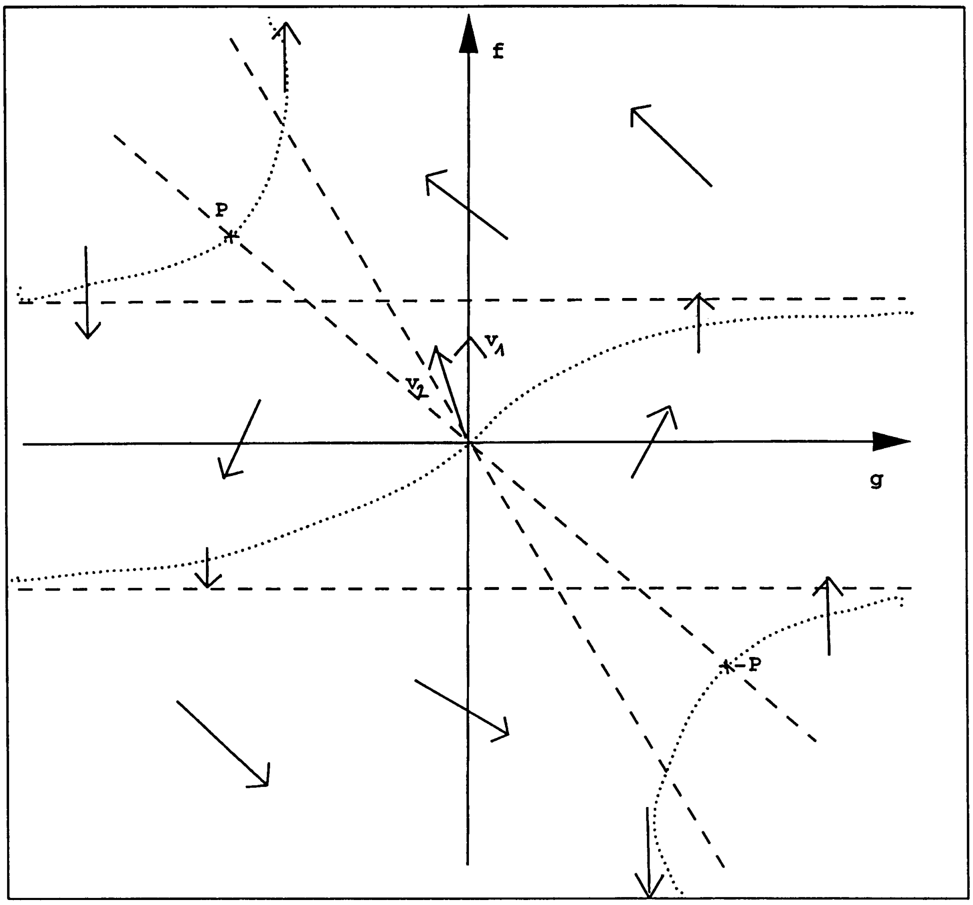
To identify orbits in the phase plane that originate from solutions of (P_g), let us mention that $\lim\limits_{t\to-\infty}f,g=0$ holds whenever $V(0)$, $V'(0)$ are finite and that the scaling invariance of (P_g) corresponds to the translational invariance of the system. Moreover

$$\frac{g(t)}{f(t)}=\frac{\eta V'(\eta)}{V(\eta)}\to 0\qquad\text{for } \eta\to 0,$$

if $V(0)\neq 0$. Hence the (P_g)-orbits leave the origin along the eigenvector v_1. If $N>2$, then $\frac{2m}{1-m}$ is the largest eigenvalue in $(0,0)$ and the orbit leaving there along v_1 is unique. If $N\leq 2$ we have a parametrization of these orbits via

$$\lim_{t\to-\infty}\frac{g(t)}{|f(t)|^{\frac{1+m}{2m}}sgn\,f(t)}=\lim_{\eta\to 0}\frac{V'(\eta)}{|V(\eta)|^{\frac{1+m}{2m}}sgn\,V(\eta)}=:p.$$

In particular the (P_g)-orbit corresponds to $p=0$.



In order to describe the global behaviour of the (P_g)-orbits, we have to distinguish three possible cases (see sketch of the phase plane): First, an orbit may escape to infinity after a finite number of rotations; this implies

$$\frac{g(t)}{f(t)} \to -km \qquad \text{if } t \to \infty$$

and thereby slow decay η^{-km} of V as $\eta \to \infty$. Secondly, it may turn around the origin forever and therefore have maxima of order $\eta^{-\frac{2m}{1-m}}$. Such solutions, however, do not satisfy the eigenfunction property. The existence of such oscillatory solutions is possible and strongly related to the question whether $\frac{2}{1-m}$ equals the limit $i \to \infty$ of the eigenvalues or not. We must leave this an open problem.

Finally an orbit may converge to the saddle points $+P$ or $-P$ after a finite number of rotations. These orbits correspond to eigenfunctions V of (P_g). As an example we have the Barenblatt solution, emerging from the origin along v_1 and going to $+P$ without leaving the second quadrant. To prove existence of a second eigenfunction, we have to look for an orbit again leaving the origin along v_1, intersecting the negative g-axis

and then going into $-P$.
In order to do so, we first establish a monotonicity property of the system (see [H]):
Expressing the system for (f, g) in polar coordinates (r, ϕ),

$$f=:\left(\frac{r}{1-r}\right)^{\frac{m}{1-m}}\sin\phi,\quad g=:\left(\frac{r}{1-r}\right)^{\frac{m}{1-m}}\cos\phi,$$

we get

$$\begin{aligned} r'&=\left(\frac{1}{m}-1\right)r(1-r)F(r,\phi,m,\alpha,N)\\ \phi'&=G(r,\phi,m,\alpha,N), \end{aligned}$$

F and G given by

$$\begin{aligned} F(r,\phi,m,\alpha,N):&=\sin\phi\cos\phi+\frac{2m}{1-m}+(2-N)\cos^2\phi\\ &\quad-\frac{r}{1-r}|\sin\phi|^{\frac{1}{m}-1}\cos\phi\left(\alpha\sin\phi+\frac{\beta}{m}\cos\phi\right),\\ G(r,\phi,m,\alpha,N):&=\cos^2\phi+(N-2)\sin\phi\cos\phi\\ &\quad+\frac{r}{1-r}|\sin\phi|^{\frac{1}{m}-1}\sin\phi(\alpha\sin\phi+\frac{\beta}{m}\cos\phi). \end{aligned}$$

Then

$$FG_\alpha-F_\alpha G=\frac{2m}{1-m}\left(\sin\phi+\frac{1-m}{2m}\cos\phi\right)^2\geq 0,$$

which means that the flow turns monotonically in one direction as α increases. Since k depends monotonically on α the same is true as k increases.
Bearing this in mind, we define two disjoint sets of values k, such that the corresponding (P_g)-orbits do not have the behaviour of a second eigenfunction:

$A=\{N<k<\frac{2}{1-m}|$ The (P_g)-orbit intersects the negative g-axis exactly once and escapes to infinity through the fourth quadrant.$\}$,

$B=\{N<k<\frac{2}{1-m}|$ The (P_g)-orbit intersects the positive g-axis.$\}$.

Both sets are open. They are also not empty, as can be seen as follows: If $k=N\neq 2$, the orbits leaving the origin along v_2 have to escape to infinity

through the fourth quadrant, and as this is an open property, it holds also for k slightly larger than N. Due to the monotonicity property this in turn implies that the same must hold for the (P_g)-orbits in this parameter range. If $k=N=2$ one only has to replace the v_2-orbit by the orbit leaving along v_1 with $p=-1$ (see page 481). Thus A is not the empty set.
To show that B is not empty, we consider the problem related to the upper bound $k=\frac{2}{1-m}$. Then $f=f(t)$ rescaled by $\hat{\eta}:=\beta^{\frac{1}{2}}\eta$, $t=\log\hat{\eta}$, satisfies the second order ODE

$$f''+\left\{\frac{1}{m}|f|^{\frac{1}{m}-1}+\left(N-\frac{2}{1-m}(1+m)\right)\right\}f'+\frac{2m}{1-m}c_1 f=0.$$

As proved for instance is [SC], p.330, this equation has a periodic solution. Therefore the solutions for k slighty less than $\frac{2}{1-m}$ must have arbitrarily many zeroes.
Consequently there is a $k>N$, which is neither contained in A nor in B. Due to the monotonicity property it is unique; hence it is the second eigenvalue $k_2^g(m, N)$.
Iterating this process with the obvious modifications concerning A and B, the theorem is proved.

q.e.d

REMARK 2.2: One easily derives the limit behaviour $m\to\frac{(N-2)_+}{N}$ of the eigenvalues -

$$\lim_{m\to\frac{(N-2)_+}{N}} k_i^g(m, N)=\begin{cases}2 & \text{for } N=1 \text{ and } i\geq 2,\\ N & \text{otherwise.}\end{cases}$$

But, as already mentioned in the proof, it is not known whether the limit $i\to\infty$, keeping m and N fixed, equals the upper bound $\frac{2}{1-m}$.

Chapter 3. Continuity

As already mentioned in the introduction, $k_1^g(m, N)=N$ and $k_1^u(m)=2$ are explicitely known and obviously continous functions of m. In what follows we can therefore concentrate on the parameter range

$$k>\max\{2, N\}, \qquad i\geq 2,$$

even if its not explicitely stated.

To proceed further we define

$$W(\eta):=\eta^{1-N}\int_0^\eta x^{n-1}U(x)dx,$$
$$Z(\eta):=-\frac{1}{k-2}+\int_0^\eta W(x)dx.$$

Integrating twice the ODE from (P_g), these definitions lead to

$$V'+\eta U+(k-N)W=0, \qquad (*)$$
$$V+\eta W+(k-2)Z=0. \qquad (**)$$

LEMMA 3.1: *(a-priori-estimates)*
Let V be a solution of (P_g). Then the following functions are nonincreasing in η:

(i) $E_1(\eta):=\frac{1}{2}V'(\eta)^2+\frac{km}{m+1}|V(\eta)|^{1+\frac{1}{m}}$,

(ii) $E_2(\eta):=\frac{1}{2}W(\eta)^2+(k-2)^{\frac{1}{m}}\frac{m}{m+1}|Z(\eta)|^{1+\frac{1}{m}}$ if $k>2$,

(iii) $E_3(\eta):=\frac{k-1}{2}W(\eta)^2+\frac{m}{m+1}|V(\eta)|^{1+\frac{1}{m}}$ if $k\geq 1+\frac{N-1}{4}$.

REMARK: Due to the relation

$$1+\frac{N-1}{4}=N-\frac{3}{4}(N-1)\leq N$$

estimate (iii) is valid in particular for all k under consideration.

PROOF: (in case that $N=1$ see [BHV])
(i): This is the standard energy estimate for (P_g).
(ii): Differentiating E_2 and using $Z''=U+\frac{1-N}{\eta}Z'$ we deduce

$$E_2'(\eta)=\frac{1-N}{\eta}|Z'|^2+(g(V)+g((k-2)Z))Z',$$

where $g(s):=|s|^{\frac{1}{m}}\,sgn\ s$ being a strictly increasing, odd function. Due to

$$V=-(\eta Z'+(k-2)Z)$$

the second term is nonpositive and the assertion follows.
(iii): Differentiating E_3 and using $(*)$ yields

$$E_3'(\eta)=UV'+(k-1)W\left(U-\frac{N-1}{\eta}W\right)$$

$$= U(\underbrace{V'+(k-N)W}_{=-\eta U})+(N-1)UW-(k-1)\frac{N-1}{\eta}W^2.$$

If $N=1$, then the assertion is proved. So assume $N>1$. But in this case Young's inequality gives

$$E_3'(\eta)\leq -\left(1-\frac{N-1}{2}\varepsilon\right)\eta U^2-(N-1)\left(k-1-\frac{1}{2\varepsilon}\right)\frac{W^2}{\eta}$$
$$=-(N-1)\left(k-\left(1+\frac{N-1}{4}\right)\right)\frac{W^2}{\eta}.$$

This completes the proof of the lemma.

q.e.d.

As an immediate consequence we get

(i) $|V|\leq 1$, (ii) $|V'|^2\leq\frac{2km}{m+1}$, (iii) $|Z|\leq\frac{1}{k-2}$,

(iv) $|W|^2\leq\frac{2m}{(k-1)(m+1)}$, (v) $|W'|=|Z''|\leq 2$.

The last inequality follows by (i) and the identity $W=Z'$:

$$|Z''|\leq|U|+(N-1)\frac{Z'}{\eta}\leq 1+(N-1)\eta^{-N}\int_0^\eta t^{N-1}dt$$
$$=1+\frac{N-1}{N}\leq 2.$$

Now suppose there is a sequence $\{m_j\}_{j\in \mathbf{N}}$ converging to $\overline{m}\in\left(\frac{(N-2)_+}{N},\infty\right)$. We first deal with the continuity of the eigenvalues in the singular case $\overline{m}<1$.

THEOREM 3.2: *For all $i, N\in \mathbf{N}$ the eigenvalues $k_i^g(.,N)$ are continuous functions of m in* $\left(\frac{(N-2)_+}{N},1\right)$.

PROOF: Without loss of generality we can assume that the m_j are bounded away from 1 and we have

$$\limsup_{j\to\infty} k_i^g(m_j,N)<\infty.$$

Thus we can choose a convergent subsequence, again denoted by $\{m_j\}_{j\in \mathbf{N}}$,

$$k_i^g(m_j,N)\to\overline{k}\geq 2.$$

Now the a-priori bounds from lemma 3.1 give uniform bounds, such that -

again for a subsequence -

$$V_j \to \bar{V} \text{ in } C^0_{loc}(\boldsymbol{R}^+) \quad \text{and} \quad W_j \to \bar{W} \text{ in } C^0_{loc}(\boldsymbol{R}^+)$$

holds. Furthermore $(*)$ yields

$$\begin{aligned} -V_j' &= \eta U_j + (k_j - N) W_j \\ &\to \eta \bar{U} + (\bar{k} - N) \bar{W} \qquad \text{in } C^0_{loc}(\boldsymbol{R}^+), \end{aligned}$$

hence $V_j \to \bar{V}$ in $C^1_{loc}(\boldsymbol{R}^+)$.
As we deal here with the singular case, we observe $U_j \to \bar{U}$ in $C^1_{loc}(\boldsymbol{R}^+)$. Combining this and the fact that $\left|\frac{1}{\eta} V_j(\eta)\right|$ is uniformly bounded in j and η, we obtain

$$\begin{aligned} V_j'' &= -\frac{N-1}{\eta} V_j' - \eta U_j' - k_j U_j \\ &\to -\frac{N-1}{\eta} \bar{V}' - \eta \bar{U}' - \bar{k} \bar{U} \qquad \text{in } C^0_{loc}(\boldsymbol{R}^+), \end{aligned}$$

hence $\bar{V}$ solves (P_g) with the parameter values $\bar{m}$ and $\bar{k}$.
It remains to prove that $\bar{V}$ has rapid decay and precisely $(i-1)$ zeroes in its support.
We recall the asymptotic behaviour of eigenfunctions derived in the proof of theorem 2.1:

$$\eta^{\frac{2m}{1-m}} V(\eta) \to \left[2m\left(\frac{2}{1-m} - N\right)\right]^{\frac{m}{1-m}} (-1)^{i+1} \quad \text{if } \eta \to \infty.$$

Consequently

$$\eta^{\frac{2\bar{m}}{1-\bar{m}}} |\bar{V}(\eta)| \leq C;$$

thus $\bar{V}$ is an eigenfunction.
The number of zeroes of $\bar{V}$ cannot exceed $(i-1)$, which is the number of zeroes of all the V_j. To show that it equals $(i-1)$, we proceed by induction:

Let $i=2$. Due to the result from 2.1 mentioned above, $\lim_{\eta\to\infty} \eta^{\frac{2m_j}{1-m_j}} V_j(\eta)$ is negative. As this must also hold in the limit, we conclude $\bar{k} = k_2^g(\bar{m}, N)$. As the arguments do not depend on the subsquence, $k_2^g(., N)$ is continuous. If $i>2$, the same argument gives $\bar{k} = k_{i-2}^g(\bar{m}, N)$ or $\bar{k} = k_i^g(\bar{m}, N)$. On the other hand eigenvalues for fixed m and N strictly increase, so by the hypothesis of this induction

$$k_{i-2}^g(\bar{m}, N) < k_{i-1}^g(\bar{m}, N) \leq \bar{k}.$$

Again the arguments are independent of the subsequence; hence the theorem is proved.

q.e.d.

To prove continuity results in the degenerate case, we first have to show that the sequence $\{k_i^g(m_j, N)\}_{j\in N}$ is bounded and that an L^∞-estimate holds, which allows a suitable decay estimate of the limit function $\bar{V}$.

LEMMA 3.3: *If* $\bar{m}\geq 1$, *then*

$$\limsup_{j\to\infty} k_i^g(m_j, N) < \infty.$$

PROOF: Assume there is an unbounded subsequence of the $k_j := k_i^g(m_j, N)$ with corresponding solutions (V_j, Z_j).
Introducing the scaling

$$Z_j(\eta) := \frac{1}{k_j} \bar{Z}_j(\sqrt{k_j}\,\eta), \quad V_j(\eta) := \bar{V}_j(\sqrt{k_j}\,\eta), \quad t := \sqrt{k_j}\,\eta$$

we obtain from $(**)$

$$-\bar{V}_j(t) = \frac{t}{k_j} \bar{Z}_j'(t) + \left(1 - \frac{2}{k_j}\right) \bar{Z}_j(t).$$

Due to the estimate $k_j \geq N+2 \geq 3$ (see [HV]), this implies

$$\bar{Z}_j \to \bar{Z} \quad \text{in} \quad C^1_{loc}(\boldsymbol{R}^+), \quad \bar{Z}_j'' \overset{*}{\rightharpoonup} \bar{Z}'' \quad \text{in } L^\infty_{loc}(\boldsymbol{R}^+)$$

by using the a-priori-estimates of lemma 3.1. Thus the number of zeroes cannot increase and thereby not exceed $(i-1)$.
On the other hand we can pass to the limit in $\bar{Z}_j'' + \frac{N-1}{t} \bar{Z}_j' = \bar{U}_j$. This implies that $\bar{Z}$ solves the initial value problem

$$\bar{Z}'' + \frac{N-1}{t} \bar{Z}' \in -|\bar{Z}|^{\frac{1}{\bar{m}}} sgn\, \bar{Z} \qquad \text{a.e. in } \boldsymbol{R}^+,$$
$$\bar{Z}(0) = 1, \quad \bar{Z}'(0) = 0,$$

whose solution has infinitely many zeroes, as can be seen as follows:

Step 1: For any $a \in \boldsymbol{R}^+$ with $\bar{Z}' = 0$ there is an $\bar{a} \in (a, \infty)$, such that $\bar{Z}(\bar{a}) > 0$.

PROOF: Without loss of generality assume $\bar{Z}(a) > 0$. Due to $\bar{Z}'' = -|\bar{Z}|^{\frac{1}{\bar{m}}} sgn\ \bar{Z} < 0$ in any positive extremum, $\bar{Z}$ cannot attain a positive minimum. Moreover it cannot equal a positive constant within an open interval; thus the assumption $\bar{Z}>0$ in (a,∞) implies that $\bar{Z}$ strictly decreases. Consequently

$$\begin{aligned}\bar{Z}'(\eta)&=-\frac{1}{\eta^{N-1}}\int_a^\eta x^{N-1}\underbrace{\bar{Z}(x)^{\frac{1}{\bar{m}}}}_{\geq \bar{Z}(\eta)}dx\\&\leq-\frac{1}{\eta^{N-1}}\bar{Z}(\eta)^{\frac{1}{\bar{m}}}\frac{1}{N}(\eta^N-a^N).\end{aligned}$$

Provided $\bar{m}>1$ and $N\neq 2$ this yields

$$\begin{aligned}\left(1-\frac{1}{\bar{m}}\right)^{-1}\bar{Z}(\eta)^{1-\frac{1}{\bar{m}}}&\leq C-\frac{1}{2N}\eta^2+\frac{a^N}{N(2-N)}\eta^{2-N}\\&<0,\end{aligned}$$

if $\eta>a$ is chosen large enough; contradition. If $N=2$ the argument still works; one only has to replace $\frac{1}{2-N}\eta^{2-N}$ by $ln\,\eta$.

If the linear case is considered, the above argument only yields exponential decay of $\bar{Z}$, but this in turn implies

$$\eta^{N-1}\bar{Z}'(\eta)\to 0 \qquad \text{if } \eta\to\infty.$$

Integrating the differential equation for $\bar{Z}$ over (a,η) we derive

$$\int_a^\eta x^{N-1}\bar{Z}(x)dx=-\eta^{N-1}\bar{Z}'(\eta)\to 0 \qquad \text{if } \eta\to\infty,$$

which contradicts the fact that due to the assumpton $\bar{Z}>0$ this integral has to be strictly positive.

Step 2: For any $a\in\boldsymbol{R}^+$ with $\bar{Z}(a)=0$ there is an $\bar{a}\in(a,\infty)$, such that $\bar{Z}'(\bar{a})=0$.

PROOF: Without loss of generality assume $\bar{Z}'(a)>0$. Integrating again the ODE for $\bar{Z}$, we obtain

$$\eta^{N-1}\bar{Z}'(\eta)=a^{N-1}\bar{Z}'(a)-\int_a^\eta x^{N-1}\bar{Z}(x)^{\frac{1}{\bar{m}}}dx.$$

Now it is apparent that the last term goes to infinity if we assume $\bar{Z}'$ to be positive for all $\eta>a$. This in turn implies $\bar{Z}'(\eta)<0$, if $\eta>a$ is chosen large enough; contradiction.

Thus the lemma is proved.

q.e.d.

LEMMA 3.4: (L^{∞}—estimate) *Let*

$$m, k>0, \quad 2+(m-1)k>0$$

and $0\leq\bar{a}<a<\infty$. *Mereover let* U *be a solution of* (P_g) *not identically zero in* $(\bar{a}, a)$ *and* $U(a)=\bar{a}U(\bar{a})=0$. *Then*

$$M:=\sup_{x\in(\bar{a},a)} x^{\lambda}|U(x)|\leq C(m, \lambda, N, k)\|U\|_{L^{\infty}((\bar{a},a))}^{\frac{2+\lambda(m-1)}{2}}$$

for all $\lambda\geq k$ *satisfying* $2+(m-1)\lambda\geq 0$.

REMARK: (i) Combining 3.4 and the a-priori-estmates 3.1, we have in particular

$$\sup_{x\in(\bar{a},a)} x^{\lambda}|U(x)|\leq C(m, \lambda, N, k).$$

(ii) This lemma also holds if $a=\infty$, provided $x^{\lambda}U(x)\to 0$ for $x\to\infty$.

PROOF: (in case that $N=1$ see also [BHV]) Without loss of generality let U be positive in $(\bar{a}, a)$. As U is continuous, M is attained for some $\rho\in(\bar{a}, a)$. Hence

$$\begin{aligned}\rho^{k}U(\rho)&=\int_{\rho}^{a}(-r^{k}U(r))'dr\\ &=\int_{\rho}^{a}r^{k-N}(r^{N-1}V'(r))'dr\end{aligned}$$

by using (P_g). Integrating twice we obtain

$$\rho^{k}U(\rho)\leq-\rho^{k-1}V'(\rho)+(k-N)\rho^{k-2}V(\rho)+(k-N)(k-2)\int_{\rho}^{a}r^{k-3}V(r)dr.$$

Due to the choice of ρ

$$0=(x^{\lambda}U(x))'_{|x=\rho}=\rho^{\lambda-1}(\lambda U(\rho)+\rho U'(\rho))$$

holds, which implies

$$\rho V'(\rho)=\rho m|U(\rho)|^{m-1}U'(\rho)=-\lambda m|U(\rho)|^{m-1}U(\rho)=-\lambda mV(\rho).$$

Combining this and $V(x)\leq\frac{M^{m}}{x^{\lambda m}}$, we arrive at

$$\rho^{k}U(\rho)\leq\rho^{k-2-\lambda m}M^{m}\left\{\lambda m+k-N+2\left|\frac{(k-N)(k-2)}{2+\lambda m-k}\right|\right\}.$$

Consequently

$$M = \rho^{\lambda-k}\rho^{k}U(\rho)$$
$$\leq \rho^{\lambda-2-\lambda m}M^{m}\{\dots\},$$

the exponent of ρ being negative due to the assumption on λ. Hence we eliminate ρ using $\rho^{-1} \leq \left(\frac{\|U\|_{\infty}}{M}\right)^{\frac{1}{\lambda}}$ and finally obtain

$$M \leq \left\{\lambda m + k - N + 2\left|\frac{(k-N)(k-2)}{2+\lambda m - k}\right|\right\}^{\frac{\lambda}{2}} \|U\|_{L^{\infty}((\bar{a},a))}^{\frac{2+\lambda(m-1)}{2}}.$$

q.e.d.

PROPOSITION 3.5: *For all $N \in \boldsymbol{N}$ the second eigenvalue $k_2^g(.,N)$ is a continuous function of m in $\left(\frac{(N-2)_+}{N}, \infty\right)$.*

PROOF: As the case $\bar{m}<1$ was already treated in 3.2, we now assume $\bar{m}>1$. By lemma 3.3 we can choose a subsequence

$$k_2^g(m_j, N) \to \bar{k} < \infty \text{ for } j \to \infty.$$

Proceeding as in the singular case yields

$$U_j \to \bar{U} \quad \text{in } C^0_{loc}(\boldsymbol{R}^+), \quad V_j \to \bar{V} \quad \text{in } C^1_{loc}(\boldsymbol{R}^+).$$

Now we have to distinguish two cases: First $m_j > 1$ and secondly $m_j < 1$ with $\bar{m} = 1$.
As the first case is concerned, the solutions (U_j, V_j) are non-classical solutions; U_j and V_j' are only absolutely continuous. But in this sense $(\bar{U}, \bar{V})$ solves (P_g) with parameter values $\bar{m}$ and $\bar{k}$:
Defining $g(s) := |s|^{\frac{1}{m}} sgn\, s$, we have - due to the absolute continuity of g -

$$U_j' = g'(V_j)V_j' \to g'(\bar{V})\bar{V}' = g(\bar{V})' \quad \text{in } L^1_{loc}(\boldsymbol{R}^+)$$

and therefore

$$V_j'' \to -\frac{N-1}{\eta}\bar{V}' - \eta\bar{U}' - \bar{k}\bar{U} \qquad \text{in } L^1_{loc}(\boldsymbol{R}^+).$$

The decay of $\bar{V}$ is faster than $\eta^{-\bar{k}\bar{m}}$, as can be seen using the L^{∞}-estimate 3.4 with an arbitrarily large $\lambda > \bar{k}$ and passing to the limit. This and the existence theorem in [H] this imply that $\bar{V}$ has compact support.
The number of zeroes cannot be larger than one, since $V_j \to \bar{V}$ in C^1; on the other hand we have the estimate (see [HV])

$$k_2^g(m, N) \geq N+2 \quad \text{if } m \geq 1,$$

from which we derive $\bar{k} \neq k_1^g(\bar{m}, N) = N$. If $\bar{m} = 1$ and $m_j < 1$, we conclude that $\bar{V}$ solves (P_g) along the same arguments as in 3. 2. To prove the eigenvalue property, we use 3. 4 again: Without loss of generality assume

$$\frac{2}{1-m_j} > \bar{k} \quad \text{for all } j \geq j_0.$$

Now choose $\lambda \in \left(\bar{k}, \dfrac{2}{1-m_{j_0}} \right)$; due to

$$\eta^{\lambda} |U_j(\eta)| = \eta^{\lambda - \frac{2}{1-m_j}} \underbrace{\eta^{\frac{2}{1-m_j}} |U_j(\eta)|}_{\leq C} \rightarrow 0 \quad \text{if } \eta \rightarrow \infty$$

and the remark in 3. 4 the L^{∞}-estimate is applicable. As above we conclude

$$\eta^{\bar{k}} \bar{U}(\eta) \rightarrow 0 \quad \text{if } \eta \rightarrow \infty.$$

In this case we do not have an estimate from below as above; thus we have to prove the existence of a zero of $\bar{U}$ in a different way:
Integrating (*) and observing $\eta^N |U_j(\eta)|$, $\eta^{N-1} |V_j'(\eta)| \rightarrow 0$ if $\eta \rightarrow \infty$ gives

$$\int_{R^+} x^{N-1} U_j(x) dx = 0$$

for all $j \geq j_0$. Using the Lebesgue theorem this continues to hold in the limit, which guarantees the existence of a zero of $\bar{U}$; therefore $\bar{k} = k_2^g(\bar{m}, N)$.
Again the arguments do not depend on the subsequence; hence the theorem is proved.

q.e.d.

THEOREM 3. 6: *In one space dimension all eigenvalues $k_i^g(.,1)$ and k_i^u are continuous functions of m in $\boldsymbol{R}^+$.*

PROOF: In order to prove this theorem, we can argue along the lines used in 3. 2 and 3. 5 to derive that the limit function $\bar{V}$ solves (P_g) - or (P_u), of course - with the parameter values $\bar{m}$ and $\bar{k}$ and that it has fast decay.
To show that $\bar{V}$ is indeed the i-th eigenfunction, we use induction and the fact that the k_i^g and the k_i^u alternate and cannot be equal (see [H]).
Assuming continuity of $k_l^g(.,1)$ and k_l^u for all $l \leq i-1$ and passing to the limit gives

$$k_{i-1}^g(\bar{m}, 1) < k_{i-1}^u(\bar{m}) \leq \bar{k}.$$

As the number of zeroes of the V_j cannot increase by passing to the limit, $\bar{k}$ equals $k_i^g(\bar{m}, 1)$. Similar considerations can be made for k_i^u; thus the theorem is proved.

q.e.d.

REMARK 3.7: If $N>1$ and $m \geq 1$, this technique - to be precise: lemma 3.3 and 3.4 - only yields that any converging subsequence has a limit which belongs to the set $\{k_l^g(\bar{m}, N)\}_{l \leq i}$. The unsolved problem is to determine the number of zeroes of $\bar{V}$.

Chapter 4. The Limit $m \to \infty$ of the eigenvalues

In this chapter we will still use the notation of chapter 3 and also assume $k > \max(2, N)$. The fundamental lemma of this chapter is the following:

LEMMA 4.1: *If $i, N \in \boldsymbol{N}$ are fixed, then*

$$\limsup_{m \to \infty} k_i^g(m, N) < \infty.$$

PROOF: Proceeding exactly as in lemma 3.3, we arrive at the initial value problem

$$\bar{Z}'' + \frac{N-1}{\eta} \bar{Z}' \in -sgn\, \bar{Z} \qquad \text{a.e. in } \boldsymbol{R}^+,$$
$$\bar{Z}(0) = 1, \quad \bar{Z}'(0) = 0.$$

In order to get a contradiction, we have to show that the solution $\bar{Z}$ has infinitely many zeroes. This, however, is clear as the problem can be solved explicitely (see also 4.3) and is strictly concave in $\{\bar{Z} > 0\}$ and strictly convex in $\{\bar{Z} < 0\}$.

q.e.d.

Thus we will consider sequences $m_j \to \infty$ and $k_j \to k < \infty$ (we stress that the k_j do not have to be eigenvalues) and pass to the limit $j \to \infty$ in (P_g). Due to $U \to sgn\, V$ formally, we first have to express the problem in more convenient terms. In order to do so, we replace U by Z, defined at the beginning of chapter 3, which solves the equation

$$Z'' + \frac{N-1}{\eta} Z' = U = |V|^{\frac{1}{m}-1} V.$$

In view of (**) we now obtain an initial value problem with unknown functions V and Z. It turns out that the limit problem also allows solutions with two essentially different behaviours at infinity, one of these possibilities being a solution with compact support.

PROPOSITION 4.2: *Consider two sequences* $m_j \to \infty$, $k_j \to k$ *and the corresponding solutions* V_j, Z_j *of* (P_g). *Moreover suppose that the number of zeroes of the* V_j *is uniformly bounded.*
Then

$$V_j \to V \text{ in } C^0_{loc}(\boldsymbol{R}^+), \quad Z_j \to Z \text{ in } C^1_{loc}(\boldsymbol{R}^+), \quad Z_j'' \overset{*}{\rightharpoonup} Z'' \text{ in } L^\infty_{loc}(\boldsymbol{R}^+)$$

and (V, Z) *solves the initial value problem*

$$(P_g^\infty) \qquad \begin{aligned} &V+\eta Z'+(k-2)Z=0 \qquad \text{in } \boldsymbol{R}^+, \\ &Z''+\frac{N-1}{\eta}Z' \in sgn\, V \qquad \text{a.e. in } \boldsymbol{R}^+, \\ &Z(0)=-\frac{1}{k-2}, \quad Z'(0)=0, \\ &V(0)=1, \quad V'(0)=0, \\ &\frac{1}{2}|V'(\eta)|^2+k|V(\eta)| \text{ is nonincreasing in } \boldsymbol{R}^+. \end{aligned}$$

PROOF: (in case that $N=1$ also see [BHV]) The convergence results above are an immediate consequence of the a-priori estimates 3.1; they thereby imply

$$V+\eta Z'+(k-2)Z=0 \qquad \text{in } \boldsymbol{R}^+.$$

As the U_j are concerned, the estimate $|U_j| \leq 1$ yield $U_j \overset{*}{\rightharpoonup} U$ in $L^\infty_{loc}(\boldsymbol{R}^+)$. The U_j are continuous and cannot possess positive minima or negative maxima; hence the number of extrema is bounded uniformly in j and so $\{U_j\}$ is a bounded sequence in $BV(\boldsymbol{R}^+)$, consequently compact in $L^p_{loc}(\boldsymbol{R}^+)$ for every $p<\infty$ and we have

$$U_j \to U \qquad \text{a.e. in } \boldsymbol{R}^+.$$

On the other hand $U_j \to sgn\, V$ a.e. in $\boldsymbol{R}^+$ is valid within $\{V \neq 0\}$, from which we conclude $\lim_{j\to\infty} U_j \in sgn\, V$.
Obviously the initial values and the energy estimate are satisfied; therefore the proposition is proved.

q.e.d.

The following proposition is concerned with existence and uniqueness of solutions of (P_g^∞). It turns out that the energy estimate is crucial for uniqueness.

PROPOSITION 4.3: *For every $k>2$ there is a unique solution $V\in C^0(\boldsymbol{R}^+)$, $Z\in C^1(\boldsymbol{R}^+)$ of (P_g^∞).*

PROOF: (in case that $N=1$ see [BHV]) We will construct the solution. As long as V is not equal zero, V and $U=sgn\,V$ are smooth and differentiating twice the first equation in (P_g^∞) we derive

$$V''=\frac{N-1}{\eta}V'=-k\ sgn\,V$$

on intervals where V does not change sign. This equation, taken as initial value problem in $c\in\boldsymbol{R}^+$, can be solved explicitely:

$$V(\eta)=\begin{cases} V(c)+\frac{1}{2-N}\left(V'(c+)+\frac{k}{N}c\right)c\left(\left(\frac{c}{\eta}\right)^{N-2}-1\right)+\frac{k}{2N}(c^2-\eta^2) & \text{if } N\neq 2,\\ V(c)+\left(V'(c+)+\frac{k}{N}c\right)c\log\frac{\eta}{c}+\frac{k}{4}(c^2-\eta^2) & \text{if } N=2.\end{cases}$$

The first part of the solution is a parabola

$$V(\eta)=1-\frac{k}{2N}\eta^2.$$

The question now arising is how the solution is to be continued beyond a zero, or, equivalently, how to determine the right handed derivative of V in a zero.
Thus, let $c\in\boldsymbol{R}^+$ be a zero of V and without loss of generality assume $V'(c-)\leq 0$. First observe that due to the strict convexity of V left of c, $V'(c-)$ cannot equal zero.
Moreover, if we assume $V'(c+)>0$, there is no sign change in a neighbourhood of c, hence $U=sgn\,V=1$ there. Then we derive from $(*)$

$$-V'(c\pm)=cU(c\pm)+(k-N)Z'(c\pm)$$

and therefore

$$V'(c+)=V'(c-)<0,$$

due to the C^1-property of Z. Thus we get a contradiction and $V'(c+)\leq 0$. If $V'(c+)<0$, we have $U=sgn\,V=-1$ right of c and the same calculation yields

$$V'(c+)=V'(c-)+2c.$$

This equation is valid as long as $V'(c-)<-2c$. Otherwise $V'(c+)=0$

and the solution is identically zero in $\{\eta > c\}$ due to the energy estimate.
On the other hand $V'(c+)=0$ implies

$$U(c) = -\frac{1}{c}(k-N)Z'(c) = \frac{1}{c}(V'(c)+c) > -1,$$

if $V'(c-) > -2c$. In this case $U(c) \in sgn\, V$.
Combining the results we have

$$|V'(c+)| = (|V'(c-)| - 2c)_+,$$

the sign of $V'(c+)$ - unless it is zero - being the same as the sign of $V'(c-)$.
Hence V is uniquely determined and has only finitely many zeroes due to the energy estimate and the fact that this energy is reduced by $2c$ in every zero c of V. This also shows that V has compact support regardless of k. As Z is continuously differentiable, it is uniquely determined in $supp\, V$ although the ODE is only valid almost everywhere. In $\boldsymbol{R}^+ \backslash supp\, V$ it is given by

$$\eta Z' + (k-2)Z = 0,$$

whose solution is identically zero if $Z(c)=0$ and equals

$$Z(\eta) = Z(c)\left(\frac{\eta}{c}\right)^{2-k} \neq 0$$

- in this case $supp\, Z = \boldsymbol{R}^+$ - otherwise.
This concludes the proof of the proposition.

q.e.d.

Thus we can define the concept of eigenfunctions of (P_g^∞).

DEFINITION: The pair of functions V, Z is called i-th eigenfunction of (P_g^∞) corresponding to the eigenvalue $k \in \boldsymbol{R}^+$, iff V and Z have compact support and V possesses exactly $(i-1)$ zeroes within its support.

Recall that we only consider $i \geq 2$. The above definition does not cover the case k_1^g, as (P_g^∞) with $k=N$ has a solution Z, which does not have compact support.
In what follows we want to prove existence and uniqueness of i-th eigenvalues of (P_g^∞). In order to do so, we need another characterization:

LEMMA 4.4: *A value $k \in \boldsymbol{R}^+$ is an eigenvalue of (P_g^∞), iff the corresponding solution V has the property*

$$\int_{supp\, V} t^{N-1} sgn\, V(t)\, dt = 0.$$

PROOF: As $V \neq 0$ almost everywhere within its support, we may integrate the equation

$$(t^{N-1} Z'(t))' = sgn\, V(t) t^{N-1} \qquad \text{a.e.} \quad \text{in } \boldsymbol{R}^+.$$

Using the fact that Z and Z' only vanish simultaneously at the boundary of *supp* V and $Z=0$ at this point characterizes the eigenfunctions, the lemma is proved.

q.e.d.

THEOREM 4.5 *For every $N \in \boldsymbol{N}$ there is a strictly increasing sequence $\{k_i^g(\infty, N)\}_{i \in N}$, such that the solutions (V, Z) of (P_g^∞) is i-th eigenfunction, iff $k = k_i^g(\infty, N)$.*

PROOF: It is more convenient to look at the problem in a scaled version. Let $x = \sqrt{k}\,\eta$; then $V = V(x)$ satisfies (see 4.3)

$$\begin{aligned} &V'' + \frac{N-1}{x} V' = -sgn\, V \quad \text{in } \{V \neq 0\}, \\ &V(0) = 1, \qquad V'(0) = 0, \\ &V(c) = 0 \Rightarrow |V'(c+)| = \left(|V'(c-)| - \frac{2}{k} c\right)_+. \end{aligned}$$

The eigenvalue criterion 4.4 remains unchanged. As the problem is symmetric as far as sign changes are concerned, we will only consider the positive arcs of the solutions.

The first part of V is given by

$$V(x) = 1 - \frac{1}{2N} x^2 \qquad \text{in } (0, \sqrt{2N})$$

and the right derivative in $a_1 := \sqrt{2N}$ equals

$$\frac{|V'(a_1+)|}{a_1} = \left(\frac{1}{N} - \frac{2}{k}\right)_+,$$

which is positive provided $k > 2N$.

Further zeroes can only be given implicitely. If $V(c) = 0$ and $\frac{|V'(c+)|}{c} > 0$, the next zero of V is located at $x = sc$, $s = s\left(\frac{|V'(c+)|}{c}\right)$ being the unique solution of

$$\frac{s^2-1}{1-s^{2-N}}=\frac{2}{N-2}\left(N\frac{|V'(c+)|}{c}+1\right)$$

in $(1,\infty)$ - provided $N\neq 2$. If $N=2$, the equation reads

$$\frac{s^2-1}{ln\, s}=2\left(2\frac{|V'(c+)|}{c}+1\right).$$

Using the explicit solution we calculate

$$\frac{|V'(sc-)|}{sc}=-\frac{V'(sc-)}{sc}=-\frac{|V'(c+)|}{c}s^{-N}-\frac{1}{N}s^{-N}+\frac{1}{N},$$

which can be expressed as a function of s only by

$$\frac{|V'(sc-)|}{sc}=-\frac{N-2}{2N}\frac{s^2-1}{1-s^{2-N}}s^{-N}+\frac{1}{\mathrm{N}}.$$

Elementary calculations show that $\frac{|V'(sc-)|}{sc}$ is strictly increasing as a function of s. This fact will be used below.
The above equations led to a recursive calculation of the values that characterize V - the zeroes a_i and the right hand derivatives at these points:

$$a_1:=\sqrt{2N},\quad p_1:=0,\quad \frac{|V'(a_1+)|}{a_1}=\left(\frac{1}{N}-\frac{2}{k}\right)_+.$$

If $\frac{|V'(a_{i-1}+)|}{a_{i-1}}>0$, define $s_{i-1}=s\left(\frac{|V'(a_{i-1}+)|}{a_{i-1}}\right)$ to be the solution of the equation characterising s, inserting $c=a_{i-1}$.

$$a_i:=s_{i-1}a_{i-1},\quad p_i:=(1-p_{i-1})s_{i-1}^{-N},\quad \frac{|V'(a_i+)|}{a_i}=\left(\frac{1}{N}(1-2p_i)-\frac{2}{k}(1-p_i)\right)_+.$$

We want to show now that the number of zeroes of V is strictly increasing in k and that within the range of values k, for which V has a constant number of zeroes, there exists exactly one eigenvalue.
A simple zero, however, occurs, if and only if the right handed derivative is positive there. This - as mentioned above - is valied in a_1 for all $k>k^{(1)}:=2N$. Moreover $\frac{d}{dk}\left(\frac{|V'(a_1+)|}{a_1}\right)=\frac{2}{k^2}>0$ holds.
We will now generalize these observations to arbitrary zeroes of V: Assume $\frac{|V'(a_{i-1}+)|}{a_{i-1}}>0$ and $\frac{d}{dk}\left(\frac{V'(a_{i-1}+)|}{a_{i-1}}\right)>0$ for all $k>k^{(i-1)}$ (which is true in case $i=2$ as pointed out before). From the energy estimate we obtain that at all preceeding zeroes of V the derivatives are positive; thus

V has at least $(i-1)$ simple zeroes for all $k>k^{(i-1)}$.
By elementary calculations we know that s_{i-1} is strictly increasing as a function of $\dfrac{|V'(a_{i-1}+)|}{a_{i-1}}$. Therefore

$$\frac{d}{dk}s_{i-1}=\frac{d}{d\left(\dfrac{|V'(a_{i-1}+)|}{a_{i-1}}\right)}s_{i-1}\frac{d}{dk}\left(\frac{|V'(a_{i-1}+)|}{a_{i-1}}\right)>0.$$

Moreover $\dfrac{|V'(sc-)|}{sc}$ can be expressed as a function of s only, and this function is strictly increasing in s. Setting $s=s_{i-1}$ and $c=a_{i-1}$ we finally arrive at

$$\frac{d}{dk}\left(\frac{|V'(a_i-)|}{a_i}\right)>0.$$

As $\dfrac{2}{k}$ goes to zero if k increases, there exists a $k^{(i)}<\infty$, such that $\dfrac{|V'(a_i+)|}{a_i}=0$, if $k=k^{(i)}$.
Differentiating with respect to k we get

$$\frac{d}{dk}\left(\frac{|V'(a_i+)|}{a_i}\right)=\frac{d}{dk}\left(\frac{|V'(a_i-)|}{a_i}\right)+\frac{2}{k^2}>0,$$

if $k>k^{(i)}$, consequently $\dfrac{|V'(a_i+)|}{a_i}>0$ for all $k>k^{(i)}$; hence the induction works.
Summarising the results up to now, there is a sequence $\{k^{(j)}\}_{j\in N}$, such that V has exactly $(i-1)$ simple zeroes, if and only if $k\in(k^{(i-1)}, k^{(i)})$.
As the eigenvalues are concerned, we can rewrite the criterion of lemma 4.4 as

$$(-1)^{i+1}\prod_{l=1}^{i-1}s_l^N+2\sum_{j=1}^{i-1}(-1)^{j+1}\prod_{l=1}^{j-1}s_l^N=0.$$

This gives

$$s_{i-1}^N=2\sum_{j=1}^{i-1}(-1)^{i+j+1}\prod_{l=j}^{i-2}s_l^{-N}=2(1-p_{i-1}),$$

saying that an i-th eigenvalue corresponds to $p_i=\dfrac{1}{2}$ (see definition of p_i). But in this case

$$\frac{|V'(a_i+)|}{a_i}=\left(-\frac{2}{k}(1-p_i)\right)_+=0,$$

hence the existence of an i-th eigenvalue is proved. Uniqueness follows from monotonicity of p_i as a function of k:
If $k>k^{(i-1)}$,

$$\frac{|V'(a_i-)|}{a_i}=\frac{1}{N}(1-2p_i)+\frac{2}{k}p_i$$

is valid. This implies

$$\frac{d}{dk}p_i=\frac{N^2}{(k-N)^2}\underbrace{\left(\frac{|V'(a_i-)|}{a_i}-\frac{1}{\mathrm{N}}\right)}_{<0}-\frac{1}{2}\frac{kN}{k-N}\frac{d}{dk}\left(\frac{|V'(a_i-)|}{a_i}\right)<0,$$

the negativity of the first term due to the energy estimate.
Thus the theorem is proved.

q.e.d.

Once this theorem, which parallels the result on (P_g), is established, we try to show a connection between eigenfunctions of (P_g) and (P_g^∞). Therefore we consider a sequence $\{m_j\}_{j\in N}$, $m_j\to\infty$ and the corresponding eigenvalues. The next proposition shows that the limit of eigenfunctions of (P_g) is indeed an eigenfunction of (P_g^∞):

PROPOSITION 4.6: *Let (V_j, Z_j) be i-th eigenfunctions corresponding to m_j, and suppose $m_j\to\infty$. Then there is a subsequence $k_i^g(m_j, N)\to\bar{k}$ and*

$$V_j\to\bar{V}\quad in\ C^0_{loc}(\boldsymbol{R}^+),\ Z_j\to\bar{Z}\quad in\ C^1_{loc}(\boldsymbol{R}^+),\quad Z_j''\overset{*}{\rightharpoonup}Z''\ in\ L^\infty_{loc}(\boldsymbol{R}^+),$$

$(\bar{V}, \bar{Z})$ is a solution of (P_g^∞) and both functions have compact support.

PROOF: (in case that $N=1$ see also [BHV]) In view of proposition 4.2 it only remains to prove the property of compact support for $\bar{V}$ and $\bar{Z}$. To simplify the notation, we set $k_j:=k_i^g(m_j, N)$. From (*) (see the beginning of chapter 3) we know that $\eta U_j(\eta)$ is bounded uniformly in η and j. Consequently

$$V_j(\eta)\leq\left(\frac{C}{\eta}\right)^{m_j}\to 0\qquad\text{if } \eta>C \text{ and } j\to\infty,$$

using the definition of V. This implies $\bar{V}(\eta)=0$ for all $\eta>C$.
As to $\bar{Z}$, we have

$$
\begin{aligned}
-V_j &= \eta Z_j' + (k_j-2)Z_j \\
&= \eta^{3-k_j}(\eta^{k_j-2}Z_j)'.
\end{aligned}
$$

All the Z_j have compact support; hence we may integrate this equation and arrive at

$$\eta^{k_j-2}Z_j(\eta)=\int_\eta^\infty t^{k_j-3}V_j(t)dt.$$

Due to $k_j>2$ the integral on the right hand side is finite and we can pass to the limit. We obtain

$$\eta^{\bar{k}-2}\bar{Z}(\eta)=\int_\eta^\infty t^{\bar{k}-3}\bar{V}(t)dt,$$

thereby $\bar{Z}=0$ for all η not contained in *supp* V.

q.e.d.

In order to show that the whole sequence converges - and that the limit is the i-th eigenvalue of (P_g^∞) - we have to face the same problem as in chapter 3: Does $\bar{V}$ possess $(i-1)$ zeroes within its support? As before in arbitrary space dimension only $i=2$ can be settled:

PROPOSITION 4.7:

$$\lim_{m\to\infty} k_2^g(m_j, N)=k_2^g(\infty, N)=\begin{cases} 2+\dfrac{2(N-2)}{N+2-N\sqrt[N]{4}}, & \text{for } N\neq 2 \\ 2+\dfrac{2}{2\,ln2-1}, & \text{for } N=2. \end{cases}$$

PROOF: From proposition 4.6 we know that at least a subsequence k_j, V_j converges to an eigenvalue $\bar{k}$ and an eigenfunction $\bar{V}$ of (P_g^∞). The number of zeroes cannot exceed one, as the V_j only have one zero. Due to the estimate $k_2^g(m_j, N)>N+2>k_1^g(m, N)=N$ (see [HV]), $\bar{k}$ must coincide with $k_2^g(\infty, N)$. The usual arguments carry over this result to the whole sequence.

In order to calculate the value of $k_2^g(\infty, N)$, we have two informations: Using the notation of theorem 4.5, it must satisfy

$$-s_1^N+2=0.$$

Moreover the derivative $|V'(a_1+)|$ is known explicitely, though

$$\frac{s_1^2-1}{1-s_1^{2-N}}=\frac{2}{N-2}\left(N\left(\frac{1}{N}+\frac{2}{k}\right)+1\right) \qquad \text{if } N\neq 2.$$

Eliminating s_1 yields the result. The case $N=2$ can be treated the same way.

q.e.d.

In case that $N=1$ it is possible to identify the limit of all eigenvalues. As in chapter 3 this relies on the existence of antisymmetric eigenfunctions, whose eigenvalues alternate with the $k_i^g(., 1)$. Proceeding as before we arrive at the following antisymmetric limit problem:

$$
(P_u^\infty) \qquad
\begin{aligned}
&V+\eta Z'+(k-2)Z=0 \quad \text{in } \boldsymbol{R}^+,\\
&Z''\in sgn\, V \quad \text{a.e. in } \boldsymbol{R}^+,\\
&Z(0)=0, \quad Z'(0)=-\frac{1}{k-1},\\
&V(0)=0, \quad V'(0)=1,\\
&\frac{1}{2}|V'(\eta)|^2+k|V(\eta)| \text{ ist nonincreasing in } \boldsymbol{R}^+.
\end{aligned}
$$

Due to the initial values in this case we have the eigenvalue criterion

LEMMA 4.8: *A value $k\in\boldsymbol{R}^+$ is an eigenvalue of (P_u^∞), iff the corresponding solution V has the property*

$$\int_{supp\, V} t^{N-1} sgn\, V(t)\, dt=\frac{1}{k-1}.$$

The calculations are exactly those made in 4.4 and thus omitted.

In order to relate the eigenvalues of (P_g^∞) to those of (P_u^∞), we first observe that all calculations made before simplify significantly; in particular all zeroes a_i and derivatives at those points can be stated explicitely. Thus define

$$
\begin{aligned}
&a_1^g=\sqrt{\frac{2}{k}}, && v_1^g=(k-2)\sqrt{\frac{2}{k}},\\
&a_{i+1}^g=a_i^g+\frac{2}{k}v_i^g, && v_{i+1}^g=v_i^g-2a_{i+1}^g,\\
&a_1^u=\frac{2}{k}, && v_1^u=\left(1-\frac{4}{k}\right),\\
&a_{i+1}^u=a_i^u+\frac{2}{k}v_i^u, && v_{i+1}^u=v_i^u-2a_{i+1}^u.
\end{aligned}
$$

The v_i - as long as they are positive - are equal to $|V'(a_i+)|$. But let us first consider them without this restriction as sequences in $\boldsymbol{R}$.

LEMMA 4.9: *The sequences $\{a_i^g\}$, $\{a_i^u\}$, $\{v_i^g\}$ and $\{v_i^u\}$ defined above satisfy*

(i) $v_i^g=-v_{i-1}^g+2\sqrt{\frac{2}{k}}(k-1)v_{i-1}^u,$

(ii) $a_i^u=2\sum_{j=1}^{i-1}a_j^u(-1)^{i-1-j}+(-1)^{i-1}\frac{1}{k-1}+\frac{1}{k(k-1)}-\sqrt{\frac{k}{2}}v_i^g,$

(iii) $v_i^u=-v_{i-1}^u+\sqrt{\frac{2}{k}}v_i^g,$

(iv) $a_{i+1}^g=2\sum_{j=1}^{i}a_j^g(-1)^{i-j}+\sqrt{\frac{2}{k}}v_i^u.$

PROOF: We proceed by induction, simultaneously over all four identities. The case $i=2$ only requires explicit calculations.

(**i**): By definition of v_i^g and using the assumption on v_{i-1}^g we have

$$\begin{aligned}v_i^g+v_{i-1}^g&=v_{i-1}^g-2a_i^g+v_{i-2}^g-2a_{i-1}^g\\&=2\sqrt{\frac{2}{k}}(k-1)v_{i-2}^u-2(a_i^g+a_{i-1}^g).\end{aligned}$$

In order to eliminate a_i^g, we use the assumption on v_{i-1}^u:

$$\begin{aligned}a_i^g-a_1^g&=\frac{2}{k}\sum_{j=1}^{i-1}v_j^g=\sqrt{\frac{2}{k}}\sum_{j=1}^{i-1}(v_j^u+v_{j-1}^u)\\&=-\sqrt{\frac{2}{k}}v_{i-1}^u+\sqrt{\frac{2}{k}}+2\sqrt{\frac{2}{k}}\sum_{j=1}^{i-1}v_j^u\\&=-\sqrt{\frac{2}{k}}v_{i-1}^u+\sqrt{\frac{2}{k}}+2\sqrt{\frac{2}{k}}\left(\frac{k}{2}a_i^u-1\right)\\\Longleftrightarrow\qquad a_i^g&=-\sqrt{\frac{2}{k}}v_{i-1}^u+k\sqrt{\frac{2}{k}}a_i^u.\end{aligned}$$

Combining the two equations we arrive at

$$\begin{aligned}v_1^g+v_{i-1}^g&=2\sqrt{\frac{2}{k}}(kv_{i-2}^u-ka_i^u+v_{i-1}^u-ka_{i-1}^u)\\&=2\sqrt{\frac{2}{k}}((k+1)v_{i-1}^u-k(a_i^u-a_{i-1}^u))\\&=2\sqrt{\frac{2}{k}}(k-1)v_{i-1}^u.\end{aligned}$$

(**ii**): Using (i) and the assumption on a_{i-1}^u we derive

$$\begin{aligned}a_i^u&=a_{i-1}^u+\frac{2}{k}v_{i-1}^u\\&=2a_{i-1}^u-2\sum_{j=1}^{i-2}a_j^u(-1)^{i-2-j}-(-1)^{i-2}\frac{1}{k-1}-\frac{1}{k(k-1)}\sqrt{\frac{k}{2}}v_{i-1}^g+\frac{2}{k}v_{i-1}^u\end{aligned}$$

$$=2\sum_{j=1}^{i-1} a_j^u(-1)^{i-1-j}+(-1)^{i-1}\frac{1}{k-1}+\frac{1}{k(k-1)}\sqrt{\frac{k}{2}}\,v_i^g.$$

(iii) and (iv): Roughly speaking these identities can be derived by interchanging g- and u-indexes in the above calculations.

q.e.d.

THEOREM 4.10: *In one space dimension there exists a strictly increasing sequence $\{k_i^u(\infty)\}_{i\in N}$, such that the corresponding solution (V, Z) is i-th eigenfunction, iff $k=k_i^u(\infty)$. Moreover*

$$k_{i-1}^g(\infty, 1)<k_{i-1}^u(\infty)<k_i^g(\infty, 1)<k_i^u(\infty).$$

PROOF: Using the same technique as in theorem 4.5, we look at the problem in the scaled version and deduce

$$a_1^u=\frac{2}{\sqrt{k}},\quad \frac{|V'(a_1^u+)|}{a_1^u}=\left(\frac{1}{2}-\frac{2}{k}\right)_+.$$

All further considerations can be carried over word by word from 4.5. Due to the calculations made in 4.9 we can say more: (iv) implies that $v_i^u=0$, if and only if k is i-th eigenvalue of (P_g^∞), thus the solution of (P_u^∞) has exactly $(i-1)$ simple zeroes, if and only if $k\in(k_i^g(\infty, 1), k_{i+1}^g(\infty, 1)]$.
On the other hand (ii) implies that the i-th eigenvalue of (P_u^∞) corresponds to $v_i^g=0$, which holds for the $k^{(i)}$, defined in 4.5, which denote the parameter, where the number of simple zeroes of V increases by one. In other words,

$$k_i^u(\infty)=k^{(i)}.$$

q.e.d.

Now we are prepared to state the main result of this chapter.

THEOREM 4.11: *In one space dimension all eigenvalues $k_i^g(m, 1)$ and $k_i^u(m)$, $i\geq 2$ satisfy*

$$\lim_{m\to\infty} k_i^g(m, 1)=k_i^g(\infty, 1),$$
$$\lim_{m\to\infty} k_i^u(m)=k_i^u(\infty).$$

PROOF: Consider a sequence $\{m_j\}_{j\in N}$, $m_j\to\infty$. Following lemma 4.1 and theorem 4.10 we know that the corresponding eigenvalues - keeping i fixed - remain bounded as $j\to\infty$. Thus there exists a subsequence $k_i^g(m_j, 1)\to \bar{k}$ the corresponding solution $\bar{V}$ being an eigenfunction of (P_g^∞) due to lemma 4.6. In order to prove that $\bar{k}$ is indeed the i-th eigenvalue of (P_g^∞) we use again that the $k_i^g(., 1)$ and the k_i^u alternate: From 4.10 we conclude by induction

$$\bar{k} \geq k_{i-1}^{u}(\infty) > k_{i-1}^{g}(\infty, 1)$$

and thereby continuity of the whole sequence.
In the same way we can prove the result for k_i^u

q.e.d.

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