Hokkaido Mathematical Journal Vol. 23 (1994) p. 465-474

Almost periodic solutions of functional differential equations with infinite delays in a Banach space

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(Received December 28, 1993)

§ 1. Introduction and preliminaries

Let E be a Banach space with norm $\|\cdot\|$ and let $J=\boldsymbol{R}=(-\infty, \infty)$ or $\boldsymbol{R}_-=(-\infty, 0]$. We shall mean by $C(J;E)$ the set of E-valued continuous functions defined on J. By $C_B(J;E)$ we denote the set of E-valued functions continuous and bounded on J with the sup-norm $\|\cdot\|_\infty$. For each $t\in\boldsymbol{R}$ and $u\in C_B(\boldsymbol{R};E)$, the symbol u_t is defined by $u_t(s)=u(t+s)$ for $s\in\boldsymbol{R}_-$. Clearly $u_t\in C_B(R_-;E)$.

With these notations, we consider in this paper the following delay-differential equation

(D. D. E) $\quad x'=F(t, x, x_t), \ t\in\boldsymbol{R}.$

Here $F(t, x, \phi)$ is an E-valued function defined on $\boldsymbol{R}\times E\times C_B(\boldsymbol{R}_-;E)$ which satisfies some conditions mentioned precisely later. By a solution of (D. D. E), we mean a continuously differentiable function u defined on $\boldsymbol{R}$ such that $u'(t)=F(t, u(t), u_t)$ for all $t\in\boldsymbol{R}$. In this paper the term "continuous" means "strongly continuous".

Recently, we proved the existence and uniqueness of a solution of (D. D. E) in the case of $E=\boldsymbol{R}^n$, the n-dimensional Euclidean space. Moreover, we showed that if $F(t, x, \phi)$ is almost periodic (a. p. for short) with respect to t uniformly for (x, ϕ) in closed bounded subsets of $\boldsymbol{R}^n\times C_B(\boldsymbol{R}_-;\boldsymbol{R}^n)$, then (D. D. E) has a unique a. p. solution ([4]). These results give an affirmative answer to the open question proposed by G. Seifert [10]. The results of [4] and [10] are essentially based on a result of Medvedev [8] which guarantees the existence of a bounded solution on $\boldsymbol{R}$ of a certain class of differential equation. The result of [8], however, can be treated in the framework of our previous papers [2, 3]. The purpose of this paper is to extend these results to the case of a functional differential equation with infinite delay in a general Banach space.

We define the functional $[\ ,\ ]: E\times E\to\boldsymbol{R}$ by

$$[x, y]=\lim_{h\to+0}(\|x+hy\|-\|x\|)/h.$$

The following two lemmas will be needed later. For the proofs of these lemmas see [2, 3, 7].

LEMMA 1. *Let x, y and z be in E. Then the functional* $[\ ,\ ]$ *has the following properties:*

(1) $[x, y]=\inf\{(\|x+hy\|-\|x\|)/h ; h>0\}$,

(2) $|[x, y]|\leqq\|y\|$, $[0, y]=\|y\|$,

(3) $[x, y+z]\leqq[x, y]+[x, z]$,

(4) *let u be a function from a real interval I into E such that the strong derivative $u'(t_0)$ exists for an interior point t_0 of I, then $D_+\|u(t_0)\|$ exists and*

$$D_+\|u(t_0)\|=[u(t_0), u'(t_0)],$$

where $D_+\|u(t_0)\|$ denotes the right derivative of $\|u(t)\|$ at t_0.

LEMMA 2. *Let Q be a closed subset of E and let $a\in R$. Suppose that f is a continuous function from $[a, \infty)\times Q$ into E satisfying the following conditions:*

(1) $[x-y, f(t, x)-f(t, y)]\leqq\omega(t)\|x-y\|$

for all (t, x), $(t, y)\in[a, \infty)\times Q$, where ω is a real-valued continuous function defined on $[a, \infty)$;

(2) $\liminf_{h\to+0} d(x+hf(t, x), Q)/h=0$

for all $(t, x)\in[a, \infty)\times Q$, where $d(z, Q)$ denotes the distance from $z\in E$ to Q. Then for each $(\tau, z)\in[a, \infty)\times Q$, the Cauchy problem

(1.1) $x'=f(t, x)$, $x(\tau)=z$

has a unique global solution u on $[\tau, \infty)$ such that $u(t)\in Q$ for all $t\in[\tau, \infty)$.

Throughout this paper we assume that the E-valued function F is defined on $\boldsymbol{R}\times E\times C_B(\boldsymbol{R}_- ; E)$ and satisfies the following conditions:

(K_1) for each $r>0$ there exist $M(r)>0$ and $N(r)>0$ such that

$$\|F(t, 0, \phi)\|\leqq M(r) \text{ and } \|F(t, x, 0)\|\leqq N(r)$$

for all $t\in\boldsymbol{R}$, $\|x\|\leqq r$ and $\|\phi\|_\infty\leqq r$, $(\phi\in C_B(\boldsymbol{R}_- ; E))$;

(K_2) if for $x(t)$ uniformly continuous and bounded on $\boldsymbol{R}$, $F(t, x(t), x_t)$ is continuous on $\boldsymbol{R}$ and $F(t, y, x_t)$ is continuous in (t, y) on $\boldsymbol{R}\times B_r(0)$, where $B_r(0)=\{x\in E ; \|x\|\leqq r\}$;

(K_3) if there exist positive numbers p, r, L such that $p>max\{M(r)/r, L\}$, where $M(r)$ is as in (K_1), such that

$$(1.2)\qquad [x-y, F(t, x, \phi)-F(t, y, \phi')]\leqq -p\|x-y\|+L\|\phi-\phi'\|_\infty$$

for all $t\in \boldsymbol{R}$, $\|x\|\leqq r$, $\|y\|\leqq r$, $\|\phi\|_\infty\leqq r$, $\|\phi'\|_\infty\leqq r$ $(\phi, \phi'\in C_B(\boldsymbol{R}_-; E))$.

REMARK 1. It can be easily seen that if F is continuous on $\boldsymbol{R}\times E\times C_B(\boldsymbol{R}_-; E)$, it satisfies (K_2). From (2) in Lemma 1, (K_1) and (1.2) in (K_3) it follows that

$$(1.3)\qquad \|F(t, x, \phi)\|\leqq Lr+N(r)$$

for all $(t, x)\in \boldsymbol{R}\times B_r(0)$, $\|\phi\|_\infty\leqq r$ $(\phi\in C_B(\boldsymbol{R}_-; E))$.

§ 2. Existence of a bounded solution on $\boldsymbol{R}$

The following theorem is both an improvement and a generalization of a result of Medvedev [8] into a general Banach space.

THEOREM 1. *Let A be an E-valued function defined on $\boldsymbol{R}\times E$ and let $g\in C(\boldsymbol{R}; E)$. Suppose that there exist positive numbers p, r, M such that A is continuous on $\boldsymbol{R}\times B_r(0)$, $\|A(t, 0)+g(t)\|\leqq M$ $(t\in \boldsymbol{R})$, $M/p<r$ and*

$$(2.1)\qquad [x-y, A(t, x)-A(t, y)]\leqq -p\|x-y\|$$

for all (t, x), $(t, y)\in \boldsymbol{R}\times B_r(0)$. Then the equation

$$(2.2)\qquad x'=A(t, x)+g(t)$$

has a solution u on $\boldsymbol{R}$ such that $\|u(t)\|\leqq M/p$ for all $t\in \boldsymbol{R}$, and this solution is unique in G_r, where $G_r=\{\varphi\in C_B(\boldsymbol{R}; E); \|\varphi\|_\infty\leqq r\}$. Moreover, if v is any solution of (2.2) such that $\|v(t_0)\|\leqq M/p$ for some $t_0\in \boldsymbol{R}$, then $\|v(t)\|\leqq M/p$ and

$$(2.3)\qquad \|v(t)-u(t)\| \leqq \|v(t_0)-u(t_0)\| \exp(-p(t-t_0))$$

for all $t\in[t_0, \infty)$.

PROOF. If $A(t, 0)\not\equiv 0$ for $t\in \boldsymbol{R}$, we replace $A(t, x)$ and $g(t)$ by $A(t, x)-A(t, 0)$ and $g(t)+A(t, 0)$, respectively. We assume henceforth that $A(t, 0)\equiv 0$ and $\|g(t)\|\leqq M$ for all $t\in \boldsymbol{R}$. Fix a $u_0\in E$ with $\|u_0\|=M/p$ and consider the following Cauchy problem for each positive integer n.

$$(2.4)\qquad x'=A(t, x)+g(t),\ x(-n)=u_0.$$

For each $x\in E$ with $\|x\|=r$, (2.1) and (3) in Lemma 1 imply

$$[x, A(t, x)+g(t)] \leqq [x, A(t, x)]+\|g(t)\| \leqq -p\|x\|+M$$
$$=-pr+M<0.$$

It follows from the definition of the functional [,] that

$$\|x+h(A(t, x)+g(t))\|<\|x\|=r$$

for sufficiently small $h>0$. It therefore follows that, for each $(t, x)\in[-n, \infty)\times B_r(0)$ there exists an $h_0=h_0(t, x)>0$ such that

$$x+h(A(t, x)+g(t))\in B_r(0) \quad \text{for all } 0<h\leqq h_0.$$

Lemma 2 can now be applied to guarantee the existence of a unique global solution u_n of (2.4) on $[-n, \infty)$ such that $u_n(t)\in B_r(0)$ for all $t\in[-n, \infty)$. Moreover, we can show that $\|u_n(t)\|\leqq M/p$ for all $t\in[-n, \infty)$. In fact, (2.1) and (4) in Lemma 1 imply

$$D_+\|u_n(t)\|=[u_n(t), A(t, u_n(t))+g(t)]\leqq -p\|u_n(t)\|+\|g(t)\|$$
$$\leqq -p\|u_n(t)\|+M$$

for all $t\in[-n, \infty)$. Solving this differential inequality we obtain

$$\|u_n(t)\|\leqq\|u_n(-n)\|\exp(-p(t+n))+M\int_{-n}^{t}\exp(-p(t-s))ds$$
$$\leqq\frac{M}{p}\exp(-p(t+n))+\frac{M}{p}(1-\exp(-p(t+n)))=\frac{M}{p}$$

for all $t\in[-n, \infty)$ (see Proposition 1.3 in [7]). We next show that, for an arbitrary fixed number $a<0$, $\{u_n\}$ is a uniformly Cauchy sequence on $[a, \infty)$. Let m, n be positive integers such that $n\geqq m>-a$, then

$$D_+\|u_n(t)-u_m(t)\|=[u_n(t)-u_m(t), A(t, u_n(t))-A(t, u_m(t))]$$
$$\leqq -p\|u_n(t)-u_m(t)\| \quad \text{for all } t\in[-m, \infty).$$

It follows as above that

$$\|u_n(t)-u_m(t)\|\leqq\exp(-p(t+m))\|u_n(-m)-u_m(-m)\|$$
$$\leqq\exp(-p(a+m))\|u_n(-m)-u_0\|$$
$$\leqq\frac{2M}{p}\exp(-p(a+m))$$

for all $t\in[a, \infty)$. Thus, $\{u_n\}$ is a uniformly Cauchy sequence on $[a, \infty)$. Define $u(t)=\lim_{n\to\infty}u_n(t)$ for $t\in[a, \infty)$. Then $\|u(t)\|\leqq M/p$ for all $t\in[a, \infty)$. Since

$$u_n(t)=u_n(a)+\int_a^t(A(s, u_n(s))+g(s))ds$$

for each $t \in [a, \infty)$, letting $n \to \infty$ we have

$$u(t) = u(a) + \int_a^t (A(s, u(s)) + g(s))ds \qquad (t \in [a, \infty)).$$

This shows that u is a solution of (2. 2) on $[a, \infty)$. Since $a < 0$ is arbitrary, we conclude that there is a solution u of (2. 2) on $\boldsymbol{R}$ such that $\|u(t)\| \leqq M/p$ for all $t \in \boldsymbol{R}$.

To prove the uniqueness of a solution of (2. 2) in G_r, let w be another solution of (2. 2) in G_r. Then for each $t \in \boldsymbol{R}$

$$\begin{aligned} D_+\|u(t) - w(t)\| &= [u(t) - w(t), A(t, u(t)) - A(t, w(t))] \\ &\leqq -p\|u(t) - w(t)\|. \end{aligned}$$

This implies

$$\|u(t) - w(t)\| \leqq \exp(-p(t-s))\|u(s) - w(s)\| \leqq 2r \exp(-p(t-s))$$

for all $s \leqq t$. Letting $s \to -\infty$, we get $u(t) \equiv w(t)$, $t \in \boldsymbol{R}$. To show the last assertion of the theorem, let v be any solution of (2. 2) such that $\|v(t_0)\| \leqq M/p$ for some $t_0 \in \boldsymbol{R}$. Then we have

$$\begin{aligned} D_+\|v(t)\| &= [v(t), A(t, v(t)) + g(t)] \leqq -p\|v(t)\| + \|g(t)\| \\ &\leqq -p\|v(t)\| + M \end{aligned}$$

provided $\|v(t)\| \leqq r$ for $t \geqq t_0$, and hence

$$\begin{aligned} \|v(t)\| &\leqq \exp(-p(t-t_0))\|v(t_0)\| + M\int_{t_0}^t \exp(-p(t-s))ds \\ &\leqq \frac{M}{p}\exp(-p(t-t_0)) + \frac{M}{p}(1 - \exp(-p(t-t_0))) = \frac{M}{p}. \end{aligned}$$

Consequently, $\|v(t)\| \leqq M/p$ for all $t \in [t_0, \infty)$. By the same argument as before we have also

$$\|v(t) - u(t)\| \leqq \exp(-p(t-t_0))\|v(t_0) - u(t_0)\|$$

for all $t \in [t_0, \infty)$. This completes the proof.

§ 3. Existence of an almost periodic solution

In this section we consider the delay-differential equation $(D. D. E)$. Let r be as in (K_3) and let $C_r^* = \{\varphi \in C_B(\boldsymbol{R}\,; E)\,; \|\varphi\|_\infty \leqq r$ and φ is uniformly continuous on $\boldsymbol{R}\}$. Then we have the following.

THEOREM 2. *Suppose that* $(K_1)-(K_3)$ *are satisfied. Then there exists a unique solution* u *of* $(D. D. E)$ *in* C_r^*.

PROOF. Let r, L be as in (K_3) and $N(r)$ be as in (K_1). Define $S_r=\{f\in C_B(\boldsymbol{R}\,;E)\,;\|f\|_\infty\leqq r,\ \|f(t)-f(t')\|\leqq(Lr+N(r))|t-t'|$ for all $t, t'\in\boldsymbol{R}\}$, then S_r is a closed bounded subset of the Banach space $C_B(\boldsymbol{R}\,;E)$ with the sup-norm $\|\cdot\|_\infty$. Define

$$A(t,x,\phi)=F(t,x,\phi)-F(t,0,\phi) \text{ and } B(t,\phi)=F(t,0,\phi)$$

for (t,x), $(t,y)\in\boldsymbol{R}\times B_r(0)$ and $\|\phi\|_\infty\leqq r$ ($\phi\in C_B(\boldsymbol{R}_-\,;E)$). We now define a mapping $T: S_r\to S_r$ as follows:
$x(t)=Tf(t)$ is the unique solution in G_r of

$$(2.5)\qquad x'=A(t,x,f_t)+B(t,f_t),$$

where G_r is as in Theorem 1 and $f\in S_r$. Such a solution x exists by Theorem 1 and satisfies $\|x\|_\infty\leqq M/p<r$. In fact, (K_1) and (K_2) imply that $A(t,x,f_t)$ is continuous in (t,x) on $\boldsymbol{R}\times B_r(0)$ and $B(t,f_t)$ is continuous on $\boldsymbol{R}$ such that $\|B(t,f_t)\|\leqq M(r)$ for all $t\in\boldsymbol{R}$. Moreover, (1.2) in (K_3) implies

$$\begin{aligned}&[x-y,\ A(t,x,f_t)-A(t,y,f_t)]\\&\quad=[x-y,F(t,x,f_t)-F(t,y,f_t)]\\&\quad\leqq-p\|x-y\|\qquad\text{for all }(t,x),\ (t,y)\in\boldsymbol{R}\times B_r(0).\end{aligned}$$

Since

$$\|x(t)-x(t')\|=\left\|\int_{t'}^{t}F(s,x(s),f_s)ds\right\|\leqq(Lr+N(r))|t-t'|$$

for all $t, t'\in\boldsymbol{R}$, we see that $x\in S_r$. We next show that T is a strict contraction on S_r. For each f, $g\in S_r$, putting $x=Tf$ and $y=Tg$, we see that

$$\begin{aligned}D_+\|x(t)-y(t)\|&=[x(t)-y(t),\ F(t,x(t),f_t)-F(t,y(t),g_t)]\\&\leqq-p\|x(t)-y(t)\|+L\|f_t-g_t\|_\infty\qquad\text{for all }t\in\boldsymbol{R}.\end{aligned}$$

Solving this differential inequality we have

$$\begin{aligned}\|x(t)-y(t)\|&\leqq\exp(-p(t-s))\|x(s)-y(s)\|\\&\qquad+L\int_s^t\exp(-p(t-\sigma))\|f_\sigma-g_\sigma\|_\infty d\sigma\\&\leqq\frac{2M(r)}{p}\exp(-p(t-s))\\&\qquad+\frac{L}{p}(1-\exp(-p(t-s)))\|f-g\|_\infty\\&=\frac{1}{p}(2M(r)-L\|f-g\|_\infty)\exp(-p(t-s))+\frac{L}{p}\|f-g\|_\infty\end{aligned}$$

for all $s \leqq t$. Letting $s \to -\infty$, we obtain

$$\|x(t)-y(t)\| \leqq \frac{L}{p}\|f-g\|_\infty \quad \text{for all } t \in \boldsymbol{R},$$

and this shows that $\|Tf-Tg\|_\infty \leqq \frac{L}{p}\|f-g\|_\infty$. Since $0<L/p<1$ by (K_3), T has a unique fixed point u in S_r by the contraction principle. Clearly, u is a solution of $(D.D.E)$.

To show that the above solution u is unique in C_r^* (note that $S_r \subset C_r^*$), let $v \in C_r^*$ be another solution of $(D.D.E)$. Then we have for each $t_0 \in R$

$$\begin{aligned} D_+\|u(t)-v(t)\| &= [u(t)-v(t), F(t, u(t), u_t)-F(t, v(t), v_t)] \\ &\leqq -p\|u(t)-v(t)\|+L\|u_t-v_t\|_\infty \quad \text{for all } t \geqq t_0. \end{aligned}$$

Solving this differential inequality we obtain

$$\begin{aligned} \|u(t+s)-v(t+s)\| &\leqq \exp(-p(t-t_0))\|u(t_0+s)-v(t_0+s)\| \\ &\quad + L\int_{t_0+s}^{t+s} \exp(-p(t+s-\sigma))\|u_\sigma - v_\sigma\|_\infty d\sigma \\ &\leqq \exp(-p(t-t_0))\|u(t_0+s)-v(t_0+s)\| \\ &\quad + \frac{L}{p}(1-\exp(-p(t-t_0)))\|u_t-v_t\|_\infty \end{aligned}$$

for all $t \in [t_0, \infty)$ and $s \in \boldsymbol{R}_-$. Here we have used the fact that $\|u_\sigma - v_\sigma\|_\infty$ is nondecreasing in σ. It therefore follows that

$$\|u_t-v_t\|_\infty \leqq \exp(-p(t-t_0))\|u_{t_0}-v_{t_0}\|_\infty + \frac{L}{p}\|u_t-v_t\|_\infty$$

for all $t \in [t_0, \infty)$, and this implies

$$\|u_t-v_t\|_\infty \leqq \frac{p}{p-L}\exp(-p(t-t_0))\|u_{t_0}-v_{t_0}\|_\infty \qquad (t \in [t_0, \infty)) .$$

Consequently, we obtain

$$\|u_{t_0}-v_{t_0}\|_\infty \leqq \|u_t-v_t\|_\infty \leqq \frac{p}{p-L}\exp(-p(t-t_0))\|u_{t_0}-v_{t_0}\|_\infty$$

for all $t \in [t_0, \infty)$. Letting $t \to \infty$, we get $\|u_{t_0}-v_{t_0}\|_\infty = 0$ and this implies, in particular, $u(t_0)=v(t_0)$.

THEOREM 3. *Suppose that $(K_1)-(K_3)$ are satisfied. Suppose further that $F(t, x, \phi)$ is a.p. in t uniformly for (x, ϕ) in closed bounded subsets of $E \times C_B(\boldsymbol{R}_-; E)$. Then $(D.D.E)$ has a unique a.p. solution in C_r^*, where C_r^* is as in Theorem 2. Moreover, if v is any solution of*

(D. D. E) such that $\|v_{t_0}\|_\infty \leqq M(r)/p$ *for some* $t_0 \in \boldsymbol{R}$*, then* $v=u$.

PROOF. Let u be the unique solution of $(D. D. E)$ in C_r^* obtained in Theorem 2. Since $Q=B_r(0)\times\{\phi\in C_B(\boldsymbol{R}_-;\ E);\|\phi\|_\infty\leqq r\}$ is a closed bounded subset of $E\times C_B(\boldsymbol{R}_-;E)$, for each $\varepsilon>0$ there exists a positive number $l=l(\varepsilon, Q)$ such that any interval of length l contains a $\tau=\tau(\varepsilon)$ for which

$$\|F(t+\tau, u(t+\tau),\ u_{t+\tau})-F(t, u(t+\tau), u_{t+\tau})\|<\varepsilon$$

for all $t\in\boldsymbol{R}$. By virtue of (2)−(4) in Lemma 1 and (1.2) in (K_3) we have

$$\begin{aligned} D_+\|u(t+\tau)-u(t)\| &= [u(t+\tau)-u(t), F(t+\tau, u(t+\tau), u_{t+\tau}) \\ &\quad -F(t, u(t), u_t)] \\ &\leqq -p\|u(t+\tau)-u(t)\|+L\|u_{t+\tau}-u_t\|_\infty \\ &\quad +\|F(t+\tau, u(t+\tau), u_{t+\tau})-F(t, u(t+\tau), u_{t+\tau})\| \\ &\leqq -p\|u(t+\tau)-u(t)\|+L\|u_{t+\tau}-u_t\|_\infty+\varepsilon \\ &\quad \text{for all}\quad t\in\boldsymbol{R}. \end{aligned}$$

Solving this differential inequality we obtain

$$\begin{aligned} \|u(t+\tau+s)-u(t+s)\| &\leqq \exp(-p\beta\|u(t+\tau-\beta+s)-u(t-\beta+s)\| \\ &\quad +L\int_{t+s-\beta}^{t+s}\exp(-p(t+s-\sigma))\|u_{\sigma+\tau}-u_\sigma\|_\infty d\sigma \\ &\quad +\frac{\varepsilon}{p}(1-\exp(-p\beta)) \\ &\leqq 2r\exp(-p\beta)+\frac{L}{p}(1-\exp(-p\beta))\|u_{t+\tau}-u_t\|_\infty \\ &\quad +\frac{\varepsilon}{p} \\ &\leqq 2r\exp(-p\beta)+\frac{L}{p}\|u_{t+\tau}-u_t\|_\infty+\frac{\varepsilon}{p} \end{aligned}$$

for all $t\in\boldsymbol{R}$, $s\in\boldsymbol{R}_-$ and $\beta>0$. Here we have used again the fact that $\|u_{\sigma+\tau}-u_\sigma\|_\infty$ is nondecreasing in σ, and $u\in C_r^*$. It follows from this

$$\|u_{t+\tau}-u_t\|_\infty\leqq\frac{2pr}{p-L}\exp(-p\beta)+\frac{\varepsilon}{p-L}\qquad\text{for all } t\in\boldsymbol{R}.$$

Choose $\beta_0>0$ such that $2rp\exp(-p\beta_0)<\varepsilon$, then

$$\|u(t+\tau)-u(t)\|\leqq\|u_{t+\tau}-u_t\|_\infty\leqq\frac{2\varepsilon}{p-L}\quad\text{for all } t\in\boldsymbol{R}.$$

Thus τ is a $2\varepsilon/(p-L)$−translation number for u, and since $\varepsilon>0$ is arbitrary u is an a. p. solution of $(D. D. E)$.

To show the last assertion of theorem, let v be any solution of (D. D.

E) such that $\|v_{t_0}\|_\infty \leqq M(r)/p$ for some $t_0 \in \boldsymbol{R}$. We note that $\|v_t\|_\infty \leqq M(r)/p$ for all $t \leqq t_0$. Define $\Gamma = \{t \in [t_0, \infty); \|v(s)\| \leqq r$ for all $s \in [t_0, t]\}$ and define $\alpha = \sup \Gamma$. Then $\alpha > t_0$ because $\|v_{t_0}\|_\infty \leqq M(r)/p < r$. Suppose, for contradiction, that $\alpha < \infty$. For sufficiently large integer n there exists a $t_n \in \Gamma$ such that $t_0 < \alpha - \frac{1}{n} < t_n$. In view of (2)–(4) in Lemma 1 and (1.2) in (K_3) we have

$$\begin{aligned} D_+\|v(s)\| &= [v(s), F(s, v(s), v_s)] \\ &\leqq [v(s), F(s, v(s), v_s) - F(s, 0, v_s)] + \|F(s, 0, v_s)\| \\ &\leqq -p\|v(s)\| + M(r) \qquad \text{for all } s \in [t_0, t_n). \end{aligned}$$

From this we get

$$\begin{aligned} \|v(t_n)\| &\leqq \exp(-p(t_n - t_0))\|v(t_0)\| + \frac{M(r)}{p}(1 - \exp(-p(t_n - t_0))) \\ &\leqq \frac{M(r)}{p}. \end{aligned}$$

Letting $n \to \infty$, we conclude that $\|v(\alpha)\| \leqq M(r)/p < r$. By the continuity of v at α, there exists a $\delta > 0$ such that $\|v(\alpha + t) - v(\alpha)\| < r - \|v(\alpha)\|$ for $|t| \leqq \delta$. This implies that $\|v(\alpha + t)\| < r$ for $|t| \leqq \delta$. Choosing a positive integer n such that $\alpha - \delta < t_n$, it can easily be seen that

$$\|v(t)\| \leqq M(r)/p \quad \text{for } t \in [t_0, t_n] \quad \text{and} \quad \|v(t)\| < r \quad \text{for } t \in [t_n, \alpha + \delta].$$

This contradicts to the definition of α. Consequently, $\|v_t\|_\infty \leqq M(r)/p$ holds for all $t \in \boldsymbol{R}$. By the same argument as in the proof of Theorem 2 we see that $v \in S_r$, and hence $v = u$ by the uniqueness of solutions of ($D. D. E$) in C_r^*.

REMARK 2. The conditions (K_1), (K_2) of this paper are the same as those of [4], but (K_3) of this paper is somewhat stronger than that of [4]. The reason to strengthen the condition (K_3) of [4] is that we cannot use the Ascoli-Arzela theorem in a general Banach space. The following condition (K_4) of [4], however, is superfluous in this paper.

(K_4) If for $x^k(t)$, $y^k(t)$, $x(t)$ and $y(t)$ continuous and such that $\|x^k(t)\| \leqq r$, $\|y^k(t)\| \leqq r$ for all $t \in \boldsymbol{R}$ and $k \geqq 1$ and $x^k(t) \to x(t)$, $y^k(t) \to y(t)$ as $k \to \infty$ for $t \in \boldsymbol{R}$, we have

$$F(t, x^k(t), y_t^k) \to F(t, x(t), y_t) \text{ as } k \to \infty \qquad \text{for } t \in \boldsymbol{R}.$$

Acknowledgment. The author would like to express his hearty thanks to the referee for his valuable comments and remarks.

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