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Finite time blowing-up for the Yang-Mills gradient flow in higher dimensions

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1. Introduction.

Let $n\geq 5$, and P be a non-trivial principal G-bundle over S^n with the standard metric g, where G is a compact Lie group satisfying $G\subset SO(N)$. In this paper, we prove the solution of the evolution problem for Yang-Mills connections may blow up in finite time. Yang-Mills connections over P are critical points of the functional

$$E(D)=\frac{1}{2}\int_M|F(D)|^2dV,$$

where $F(D)$ is the curvature form of connection D. If D is a Yang-Mills connection, then it satisfies the Euler-Lagrange equation of E:

$$d_D^*F(D)=0,$$

where d_D^* is the formal adjoint operator of the exterior derivative d_D with respect to the connection D.

In this paper, we consider the Yang-Mills gradient flow:

$$(1.1)\qquad \begin{cases}\dfrac{\partial D}{\partial t}=-d_D^*F(D), & \text{on } M\times[0, T)\\ D(0)=D_0.\end{cases}$$

In the fundamental work of Donaldson [7], he showed the global existence of the heat flow on a holomorphic vector bundle over a compact Kähler manifold. Recently, Kozono, Maeda and the author [8] show the existence of a global weak solution for the heat flow, if dim $M=4$. It is a well-known result that if the initial value D_0 is smooth, then there exists a time-local smooth solution $D(x, t)$ of (1.1) on $M\times[0, T)$ for any compact Riemannian manifold M with arbitrary dimension. In higher dimensional case (dim $M\geq 5$), Bourguignon, Lawson and Simons [2] and Bourguignon and Lawson [1] showed isolation phenomena of Yang-Mills connections over S^n.

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FACT 1.1. (c.f. [1, 2]). *Assume $n \geq 5$. Let P be a principal bundle over S^n. Then there exists no non-flat stable Yang-Mills connection on P.*

FACT 1.2. (c.f. [1, 2]). *Assume $n \geq 5$. Let P be a principal bundle over S^n and D be a Yang-Mills connection over P. Then there exists a constant $\varepsilon_0 > 0$ such that if $|F(D)| < \varepsilon_0$ then D is flat.*

In higher dimensional case, the Yang-Mills functional is not conformally invariant, however, it is conformally invariant in four dimensional case.

In this paper, we prove the following result.

THEOREM 1.3. *Let P be a non-trivial principal G-bundle over S^n, $n \geq 5$. There exists a constant $\varepsilon_1 > 0$, if D_0 satisfies $\|F(D_0)\|_{L^2(S^n)} < \varepsilon_1$, then the smooth solution of* (1.1) *with the initial value D_0 blows up in finite time.*

In the case of harmonic maps heat flow, many authors considered the blow-up and existence of solutions. Chang, Ding and Ye [3] construct solutions over a surface which blow-up in finite time. In higher dimensional case (dim $M \geq 3$), first Coron and Ghidaglia [6] showed the finite time blowing up phenomena for harmonic map heat flow. Later, Chen and Ding [4] gave more general arguments. The method of the proof of the main theorem of this paper is due to the method of Chen and Ding.

2. Preliminaries.

In this section, we prove preliminary lemmas for the proof of main theorem. First, we show the existence of a connection with arbitrary small energy. In the followings, let P be a principal G-bundle over S^n, $n \geq 5$.

PROPOSITION 2.1. *For any positive number $\varepsilon > 0$, there exists a G-connection D over P satisfying $E(D) < \varepsilon$.*

PROOF. Let (r, θ) be poler coordinates on S^n, with $r \in [0, \pi]$ the distance to the pole and $\theta \in S^{n-1}$. On the second copy of S^n, we denote these coordinates by $(\bar{r}, \bar{\theta})$. Consider the map $\varphi_c : S^n \to S^n$ defined by

$$\varphi_c(r, \theta) = \begin{cases} \bar{r}_c(r) = 2 \arctan\left(c \cdot \tan \dfrac{r}{2}\right), \\ \bar{\theta}, \end{cases}$$

for $c > 0$, i.e., $\bar{r}(r, \theta) = \bar{r}_c(r) = 2 \arctan\left(c \cdot \tan \dfrac{r}{2}\right)$, $\bar{\theta}(r, \theta) = \theta$.

Let $\bar{D}$ be any smooth connection on P. On the coordinate $(\bar{r}, \bar{\theta})$, we denote $\bar{D}=d+\bar{A}$. Pulling back $\bar{D}$ by φ_c, we have $D:=\varphi_c^*\bar{D}=\mathrm{d}+\varphi_c^*\bar{A}=d+A$. Notice that D is also a smooth connection on P, since φ_c is a smooth diffeomorphism on M.

Expressing $\bar{A}$ on the coordinates

$$\bar{A}(\bar{r}, \bar{\theta})=\bar{A}_{\bar{r}}(\bar{r}, \bar{\theta})d\bar{r}+\bar{A}_{\bar{\theta}}(\bar{r}, \bar{\theta})d\bar{\theta},$$

then we have

$$A(r, \theta)=\bar{A}_{\bar{r}}(\varphi_c(r, \theta))dr+\bar{A}_{\bar{\theta}}(\varphi_c(r, \theta))d\theta.$$

Elementary calculations in local coordinates show that

$$\begin{aligned} E(D)&=\frac{1}{2}\int_0^\pi\int_{S^{n-1}(\sin r)}|F(D)|^2(r, \theta)drd\theta \\ &=\frac{1}{2}\int_{S^n}\left(\frac{\sin^4\bar{r}}{\sin^4 r}\right)|F(\bar{D})|^2dV \\ &\leq\frac{1}{2}\mathrm{Vol}(S^{n-1})\sup_{S^n}|F(\bar{D})|^2\int_0^\pi\left(\frac{\sin^4\bar{r}}{\sin^4 r}\right)\sin^{n-1}rdr. \end{aligned}$$

For $n\geq 5$, we have

$$E(D)\leq\frac{1}{2}\mathrm{Vol}(S^{n-1})\sup_{S^n}|F(\bar{D})|^2\int_0^\pi\sin^4\bar{r}_c(r)dr.$$

For any $\varepsilon>0$, let

$$\eta:=\frac{2\varepsilon}{\mathrm{Vol}(S^{n-1})\sup_{S^n}|F(\bar{D})|^2},$$

and $\rho:=\pi-\eta/2$, then there exists $K>0$ such that $0\leq\tan r/2\leq K$ on $0\leq r\leq\rho$. Therefore $r_c(r)\leq 2\arctan(c\cdot K)$ on $0\leq r\leq\rho$, and there exists $c_\eta>0$ for which $0<c\leq c_\eta$ implies

$$0\leq\sin^4\bar{r}_c(r)<\frac{\eta}{2\rho},\quad\text{for } r\leq\rho.$$

Hence we have

$$\begin{aligned} \int_0^\pi\sin^4\bar{r}_c(r)dr&=\int_0^\rho\sin^4\bar{r}_c(r)dr+\int_\rho^\pi\sin^4\bar{r}_c(r)dr \\ &\leq\int_0^\rho\frac{\eta}{2\rho}dr+\int_\rho^\pi dr\leq\eta, \end{aligned}$$

and

$$E(D)\leq\frac{1}{2}\mathrm{Vol}(S^{n-1})\sup_{S^n}|F(\bar{D})|^2\int_0^{\pi}\sin^4\bar{r}_c(r)dr$$
$$\leq\frac{1}{2}\mathrm{Vol}(S^{n-1})\sup_{S^n}|F(\bar{D})|^2\frac{2\varepsilon}{\mathrm{Vol}(S^{n-1})\sup_{S^n}|F(\bar{D})|^2}=\varepsilon.$$

Let $D(t)=D(t,x)$ be any smooth solution of (1.1), and let T be the maximal existence time of $D(t)$, where $0<T\leq+\infty$. Set $e(t)=e(t,x)=|F(D)(x,t)|^2$, and $\bar{e}(t)=\sup_{S^n} e(t,x)$.

LEMMA 2.2. *There exists a constant* $\delta>0$ *such that for any* $t_0\in[0, T)$, *we have*

(2.1) $$t_0+\frac{1}{\delta\sqrt{\bar{e}(t_0)}}\leq T,$$

(2.2) $$\bar{e}(t)\leq(\bar{e}(t_0)^{-1/2}-\delta(t-t_0))^{-2},$$

for $0<t-t_0<\dfrac{1}{\delta\sqrt{\bar{e}(t_0)}}$.

PROOF. By Bochner-Weitzenböck formula [11, Lemma 3.1], we see

(2.3) $$\frac{\partial}{\partial t}e(t)\leq\Delta e(t)+Ce(t)^{3/2}.$$

On the point $(x, t)\in M\times(0, T)$ satisfying $e(u)(t,x)=\bar{e}(t)$ we have $\Delta e(t)(t,x)\leq 0$. By (2.3), we have

(2.4) $$\frac{\partial e}{\partial t}(x, t)\leq C\bar{e}(t)^{3/2}.$$

On the other hand, we set

$$D^+\bar{e}(t)=\limsup_{h\to+0}\frac{\bar{e}(t+h)-\bar{e}(t)}{h},$$

then, by (2.4), we have

$$D^+\bar{e}(t)\leq C\bar{e}(t)^{3/2}.$$

By a comparison theorem for an ordinary differential equation, for the solution of

$$y'(t)=Cy(t)^{3/2},\ \text{with } y(t_0)=\bar{e}(t_0),$$

we have $\bar{e}(t)\leq y(t)$. Therefore there exists a constant $\delta>0$ such that $\bar{e}(t)\leq(\bar{e}(t_0)^{-1/2}-\delta(t-t_0))^{-2}$, for $t_0<t<t_0+\dfrac{1}{\delta\sqrt{y(t_0)}}$. ■

Let ρ be a positive constant less than the injectivity radius of S^n, and $\{x^i\}$ a normal coordinate on a geodesic ball $B_\rho(p_0)$. There exists a positive number $C=C(\rho)>0$ such that the metric tensor $\{g_{ij}\}$ of S^n satisfies that

$$g_{ij}(x)=\delta_{ij}+q_{ij}(x),\ g^{ij}(x)=\delta^{ij}+q^{ij}(x),$$
$$\text{with } |q_{ij}(x)|<Cr^2,\ |\partial q_{ij}(x)|<Cr,$$

for $r=|x|<\rho$.

For a smooth solution $D(t)$ of (1.1), we set

$$\Psi(R, D)=\Psi(R)=\frac{1}{2}R^{4-n}\int_{S^n}|F(D)|^2(t_0-R^2, x)G_R(x)\varphi_\rho(x)^2\sqrt{g(x)}\,dx,$$

where φ_ρ is a smooth real valued function satisfying $\varphi_\rho(x)=1$ for $|x|\leq\rho/2$, $\varphi_\rho(x)=0$ for $|x|\geq\rho$, and $0\leq\varphi_\rho\leq 1$ for all x and $G_R(x)=\exp\left(-\frac{|x|^2}{4R^2}\right)$. Hereafter, we denote $E_0=E(D(0))$ for the time-local smooth solution $D(t)$ of (1.1) with the initial value D_0.

LEMMA 2.3. *There exists a constant $C>0$ such that for $0<R_1<R_2\leq R_0=\min\{\rho, \sqrt{t_0}\}$, we have*

$$\Psi(R_1)\leq e^{C(R_2-R_1)}\Psi(R_2)+C(e^{C(R_2-R_1)}-1)E_0.$$

PROOF. For the sake of simplicity, we assume $t_0=0$. By the scaling $\tilde{t}(R)=\tilde{t}=R^2t$, $\tilde{x}(R)=\tilde{x}=Rx$, we have

$$\begin{aligned}&\Psi(R)\\&=\frac{1}{2}R^4\int_{S^n}g^{ij}(\tilde{x})g^{kl}(\tilde{x})F_{ik}(\tilde{t}, \tilde{x})F_{jl}(\tilde{t}, \tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1},\end{aligned}$$

where $G(\tilde{x})=\exp\left(-\frac{|x|^2}{4}\right)$ which is independent from R. Differentiating $\Psi(R)$ by R, we have

$$\begin{aligned}(2.5)\quad &\frac{d}{dR}\Psi(R)=4R^{-1}\Psi(R)\\&+\frac{R^4}{2}\int_{S^n}\frac{d}{dR}(g^{ij}(\tilde{x})g^{kl}(\tilde{x})F_{ik}(\tilde{t}, \tilde{x})F_{jl}(\tilde{t}, \tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})})dx|_{t=-1}.\end{aligned}$$

First, we have

$$\begin{aligned}&(\mathrm{I})\\&:=\frac{R^4}{2}\int_{S^n}\frac{d}{dR}(g^{ik}(\tilde{x})g^{jl}(\tilde{x})F_{ij}(\tilde{t}, \tilde{x})F_{kl}(\tilde{t}, \tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})})dx|_{t=-1}\\&=R^3\int_{S^n}x^m\frac{\partial}{\partial x^m}(g^{ik}(\tilde{x}))g^{jl}(\tilde{x})\ F_{ij}(\tilde{t}, \tilde{x})F_{kl}(\tilde{t}, \tilde{x})G\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\end{aligned}$$

$$
\begin{aligned}
(2.6)\quad & +R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})x^m\frac{\partial}{\partial x^m}(F_{ij}(\tilde{t},\tilde{x}))F_{kl}(\tilde{t},\tilde{x})G\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1} \\
& +2R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})t\frac{\partial}{\partial t}(F_{ij}(\tilde{t},\tilde{x}))F_{kl}(\tilde{t},\tilde{x})G\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1} \\
& +R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})\,F_{ij}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})Gx^m\frac{\partial}{\partial x^m}(\varphi(\tilde{x}))\varphi(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1} \\
& +R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})F_{ij}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})G\varphi^2(\tilde{x})\frac{\partial}{\partial x^m}(g(\tilde{x}))\frac{x^m}{\sqrt{g(\tilde{x})}}dx|_{t=-1} \\
& =:(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})+(\mathrm{V})+(\mathrm{VI}).
\end{aligned}
$$

Now, we remark that

$$R^{4-n}\int_{S^n}|F(\tilde{t},\tilde{x})|^2dV_{\tilde{x}}=\int_{S^n}|F(t,x)|^2dV_x,$$

where $dV_{\tilde{x}}=\sqrt{g(\tilde{x})}\,d\tilde{x}$ and $dV_x=\sqrt{g(x)}\,dx$.

For the integral (II), since

$$\frac{\partial}{\partial x^m}g^{ij}(\tilde{x})\cdot x^m\geq -CR|x|^2g^{ij}(\tilde{x}),$$

we have

$$
\begin{aligned}
& (\mathrm{II}) \\
& \geq -CR^4\int_{S^n} g^{ij}(\tilde{x})g^{kl}(\tilde{x})F_{ik}(\tilde{t},\tilde{x})F_{jl}(\tilde{t},\tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,|x|^2dx|_{t=-1} \\
& \geq -CR^4\int_{S^n} g^{ij}(\tilde{x})g^{kl}(\tilde{x})F_{ik}(\tilde{t},\tilde{x})F_{jl}(\tilde{t},\tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1} \\
(2.7)\quad & \quad -CR^4\int_{S^n} g^{ij}(\tilde{x})g^{kl}(\tilde{x})F_{ik}(\tilde{t},\tilde{x})F_{jl}(\tilde{t},\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,|x|^4\exp\left(-\frac{|x|^2}{4}\right)dx|_{t=-1} \\
& \geq -C\Psi(R)-CR^4\int_{S^n} g^{ij}(\tilde{x})g^{kl}(\tilde{x})\,F_{ik}(\tilde{t},\tilde{x})F_{jl}(\tilde{t},\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1} \\
& \geq -C\Psi(R)-C\int_{S^n}|F(t,x)|^2dV_x|_{t=-1} \\
& \geq -C\Psi(R)-CE_0.
\end{aligned}
$$

For the integral (V), using the estimate

$$\left|\frac{\partial\varphi}{\partial x^m}(\tilde{x})\right|\leq CR\rho^{-1},$$

we have

$$
\begin{aligned}
(2.8)\quad & |(\mathrm{V})|\leq CR^4\int_{S^n}|F(\tilde{t},\tilde{x})|^2|x|\exp\left(-\frac{|x|^2}{4}\right)dV_x|_{t=-1} \\
& \leq C\int_{S^n}|F(t,x)|^2dV_x|_{t=-1}\leq CE_0.
\end{aligned}
$$

For the integral (VI), we have

$$|(\mathrm{VI})| \le CR^4\int_{S^n} g^{ij}(\tilde{x})g^{kl}(\tilde{x})F_{ik}(\tilde{t},\tilde{x})F_{jl}(\tilde{t},\tilde{x})G(\tilde{x})|x|^2\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}.$$

By a similar calculation to the estimate for (II), we have

$$(2.9)\qquad |(\mathrm{VI})| \le C\Psi(R)+CE_0.$$

Using (2.6), (2.7), (2.8) and (2.9), we obtain

$$(2.10)\qquad (\mathrm{I}) \ge -C\Psi(R)-CE_0+(\mathrm{III})+(\mathrm{IV}).$$

For the integral (III), first, we have

$$(2.11)\qquad \begin{aligned} &\frac{\partial F_{mj}}{\partial x^i}=\nabla_i F_{mj}+\Gamma^p_{im}F_{pj}+\Gamma^p_{ij}F_{mp},\\ &\nabla_i x^m=\delta^m_i+\Gamma^m_{ki}x^k,\\ &\langle F,\ [F,\omega]\rangle=0 \text{ for any } \mathfrak{g}\text{-valued function } \omega, \end{aligned}$$

where ∇ denotes the covariant differentiation with respect to a fixed connection. Let $\tilde{\nabla}$ be the covariant differentiation with respect to the connection $A(t)$, $t=-1$. Using the Bianch identity $\tilde{\nabla}_m F_{ij}=\tilde{\nabla}_i F_{mj}-\tilde{\nabla}_j F_{mi}$, $\frac{\partial}{\partial x^i}G=-\frac{x^i}{2}G$ and (2.11), we may calculate that

$$\begin{aligned}
&(\mathrm{III})\\
&=R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})x^m\tilde{\nabla}_m F_{ij}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\
&\quad+R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})x^m\Gamma^p_{im}(x)F_{pj}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\
&\quad+R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})x^m\Gamma^p_{mj}(x)F_{ip}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\
&=-2R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})F_{ij}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\
(2.12)\quad&\quad-2R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})x^m F_{mj}(\tilde{t},\tilde{x})\tilde{\nabla}_i F_{kl}(\tilde{t},\tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\
&\quad-2R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})x^m F_{mj}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})\nabla_i G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\
&\quad-2R^3\int_{S^n} g^{ik}(\tilde{x})g^{jl}(\tilde{x})x^m F_{mj}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})G(\tilde{x})\nabla_i\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\
&=-4R^{-1}\Psi(R)-2R^3\int_{S^n}\langle x\cdot F(\tilde{t},\tilde{x}),\ d^*_D F(\tilde{t},\tilde{x})\rangle G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\
&\quad+R^3\int_{S^n}\langle x\cdot F(\tilde{t},\tilde{x}),\ x\llcorner F(\tilde{t},\tilde{x})\rangle G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}
\end{aligned}$$

$$-4R^3\int_{S^n}\langle x\cdot F(\tilde{t},\tilde{x}),\nabla\varphi\llcorner F(\tilde{t},\tilde{x})\rangle G(\tilde{x})\varphi(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}.$$

Here we set

$$\begin{aligned}&(x\cdot F)_j(\tilde{x})=x^m F_{mj}(\tilde{x}),\\&(x\llcorner F)_l(\tilde{x})=x^i g^{ik}(\tilde{x})F_{kl}(\tilde{x}),\\&(\nabla\varphi\llcorner F)_l(\tilde{x})=\frac{\partial}{\partial x^i}(\varphi(\tilde{x}))g^{ik}(\tilde{x})F_{kl}(\tilde{x}),\end{aligned}$$

which are well-defined only on supp φ.

For the integral (IV), using the equation (1.1), we have

$$\begin{aligned}&(\mathrm{IV})\\&=-2R^3\int_{S^n}tg^{ik}(\tilde{x})g^{jl}(\tilde{x})(d_Dd_D^*F)_{ij}(\tilde{t},\tilde{x})F_{kl}(\tilde{t},\tilde{x})G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\&=2R^3\int_{S^n}|d_D^*F(\tilde{t},\tilde{x})|^2G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\&\quad-R^3\int_{S^n}\langle d_D^*F(\tilde{t},\tilde{x}),x\llcorner F(\tilde{t},\tilde{x})\rangle G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\&\quad+4R^3\int_{S^n}\langle d_D^*F(\tilde{t},\tilde{x}),\nabla\varphi\llcorner F(\tilde{t},\tilde{x})\rangle G(\tilde{x})\varphi(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}.\end{aligned}\tag{2.13}$$

Combining (2.12) and (2.13), we have

$$\begin{aligned}&(\mathrm{III})+(\mathrm{IV})\\&=-4R^{-1}\Psi(R)-4R^3\int_{S^n}\langle\frac{\nabla\varphi}{\varphi}\llcorner F(\tilde{t},\tilde{x}),\ d_D^*F(\tilde{t},\tilde{x})\\&\qquad\qquad-x\cdot F(\tilde{t},\tilde{x})\rangle G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\&\quad+2R^3\int_{S^n}|d_D^*F(\tilde{t},\tilde{x})-x\cdot F(\tilde{t},\tilde{x})|^2G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\&\quad+2R^3\int_{S^n}\langle d_D^*F(\tilde{t},\tilde{x})-x\cdot F(\tilde{t},\tilde{x}),\ x\llcorner F(\tilde{t},\tilde{x})\\&\qquad\qquad-x\cdot F(\tilde{t},\tilde{x})\rangle G(\tilde{x})\varphi(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\&\geq-4R^{-1}\Psi(R)\\&\quad-4R^3\int_{S^n}|\nabla\varphi\llcorner F(\tilde{t},\tilde{x})|^2G(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\&\quad-R^3\int_{S^n}|x\llcorner F(\tilde{t},\tilde{x})-x\cdot F(\tilde{t},\tilde{x})|^2G(\tilde{x})\varphi(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\&=:-4R^{-1}\Psi(R)+(\mathrm{VII})+(\mathrm{VIII}).\end{aligned}\tag{2.14}$$

For the integral (VII), we have

$$(2.15)\quad \begin{aligned}|(\mathrm{VII})|&=R^3\int_{S^n}|\nabla\varphi\llcorner F(\tilde{t},\tilde{x})|^2G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\ &\leq R^3\int_{S^n}|\nabla\varphi|^2|F(\tilde{t},\tilde{x})|^2G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}.\end{aligned}$$

Since $R\leq R_0\leq\rho$, we have $|\nabla\varphi(\tilde{x})|\leq CR\rho^{-1}\leq C$. Therefore, by (2.15), we have

$$(2.16)\quad |(\mathrm{VII})|\leq CR^4\int_{S^n}|F(\tilde{t},\tilde{x})|^2\sqrt{g(\tilde{x})}\,dx|_{t=-1}\leq CE_0.$$

Finally we consider the integral (VIII). Since

$$\begin{aligned}|x\llcorner F(\tilde{t},\tilde{x})-x\cdot F(\tilde{t},\tilde{x})|&\leq|g^{ij}(\tilde{x})-\delta^{ij}||x||F(\tilde{t},\tilde{x})|\\ &\leq|\tilde{x}|^2|x||F(\tilde{t},\tilde{x})|=R^2|x|^3F(\tilde{t},\tilde{x})|,\end{aligned}$$

we may obtain the estimate

$$(2.17)\quad \begin{aligned}|(\mathrm{VIII})|&=R^3\int_{S^n}|x\llcorner F(\tilde{t},\tilde{x})-x\cdot F(\tilde{t},\tilde{x})|^2G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\ &\leq R^3\int_{S^n}|F(\tilde{t},\tilde{x})|^2|x|^6R^4G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\ &=R^7\int_{|x|\leq R^{-1}}|F(\tilde{t},\tilde{x})|^2|x|^6G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\ &\quad+R^7\int_{|x|\geq R^{-1}}|F(\tilde{t},\tilde{x})|^2|x|^6G(\tilde{x})\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\ &\leq R^4\int_{|x|\leq R^{-1}}|F(\tilde{t},\tilde{x})|^2|x|^3\exp\left(-\frac{|x|^2}{4}\right)\varphi^2(\tilde{x})\sqrt{g(\tilde{x})}\,dx|_{t=-1}\\ &\quad+R^{4-n}\int_{|\tilde{x}|\geq1}|F(\tilde{t},\tilde{x})|^2R^{-3}|\tilde{x}|^6\exp\left(-\frac{|\tilde{x}|^2}{4R^2}\right)\varphi^2(\tilde{x})dV_{\tilde{x}}|_{t=-1}\\ &\leq CE_0+R^{4-n}\int_{|\tilde{x}|\geq1}|F(\tilde{t},\tilde{x})|^2R^{-3}|\tilde{x}|^6\exp\left(-\frac{|\tilde{x}|^2}{4R^2}\right)\varphi^2(\tilde{x})dV_{\tilde{x}}|_{t=-1}.\end{aligned}$$

On the other hand, we have

$$(2.18)\quad |\tilde{x}|^6R^{-3}\exp\left(-\frac{|\tilde{x}|^2}{4R^2}\right)\leq C\ \text{for all}\ R>0,\ |\tilde{x}|\geq0.$$

Therefore, by (2.18), we have

$$(2.19)\quad \begin{aligned}&R^{4-n}\int_{|\tilde{x}|\geq1}|F(\tilde{t},\tilde{x})|^2R^{-3}|\tilde{x}|^6\exp\left(-\frac{|\tilde{x}|^2}{4R^2}\right)\varphi^2(\tilde{x})dV_{\tilde{x}}|_{t=-1}\\ &\leq\int_{S^n}|F(t,x)|^2dV_x\leq CE_0.\end{aligned}$$

Combining (2.14), (2.16), (2.17) and (2.19), we have

$$(2.20)\quad (\mathrm{III})+(\mathrm{IV})\geq-4R^{-1}\Psi(R)-CE_0.$$

Finally combining (2.10) and (2.20), we have

$$(\mathrm{I}) \geq -4R^{-1}\Psi(R) - C\Psi(R) - CE_0.$$

Therefore, using (2.5), we have

$$\frac{d}{dR}\Psi(R) \geq 4R^{-1}\Psi(R) - C_1\Psi(R) - C_2E_0 - 4R^{-1}\Psi(R) \tag{2.21}$$
$$= -C_1\Psi(R) - C_2E_0.$$

By (2.21), we have

$$\frac{d}{dR}(e^{C_1R}\Psi(R)) \geq -C_2e^{C_1R}E_0. \tag{2.22}$$

Integrating (2.22), we complete the proof. ■

REMARK. Recently, Chen and Shen [5] also prove a monotonicity formula for Yang-Mills heat flow.

3. Proof of Theorem.

Before the proof of the main theorem, we prepare a lemma for a property of $\bar{e}$.

LEMMA 3.1. *Let $D(t)$ be a smooth solution on* (1.1) *and let T be the maximal existence time of $D(t)$. Then we have*

$$\sup\{\bar{e}(t) : t \in (0, T)\} = +\infty.$$

PROOF. First we assume that

$$\sup\{\bar{e}(t) : t \in (0, T)\} < +\infty. \tag{3.1}$$

Assume $T < +\infty$. If $\bar{e}(t) < C$ for all $t \in (0, T)$, then the solution D smoothly extends beyond the maximal existence time T. Therefore we have $T = +\infty$.

Assuming $T = +\infty$, by the energy equality, we obtain

$$\int_0^\infty \int_{S^n} |d_D^* F(D)|^2 dV\, dt \leq E_0.$$

Thus there exists a sequence $\{t_i\}$, $t_i \to \infty$ satisfying $\|d_D^* F(D)(t_i)\|^2_{L^2(S^n)} \to 0$. By the assumption $\sup_{0<t<\infty} |F(t)| < C$, using a maximum principle for (2.3), we have

$$\int_{S^n} |F(t)|^p dV \leq C \text{ for } p < \infty, \text{ and } t \in [0, \infty).$$

By a Uhlenbeck's result [12, Theorem 3.6], there exist (global) gauge transformations $s_i \in W^{2,p}$ such that

$$s_i^{-1} \circ D(t_i) \circ s_i \rightharpoonup D_\infty \text{ in } W^{1,p} \text{ (weakly).}$$

Because $W^{1,p} \hookrightarrow C^0$ is a compact embedding for $p>n$, we conclude $D_\infty \in C^0$. Since $\|d_D^* F(D)(t_i)\|_{L^2} \to 0$, the connection D_∞ on P is (weakly) Yang-Mills, hence strong.

On the other hand, we have

$$(3.2) \qquad \int_{S^n} |F(D_\infty)|^2 dV \leq \liminf_{i\to\infty} \int_{S^n} |F(D)(t_i)|^2 dV \leq E_0 < \varepsilon_1.$$

Therefore, by Fact 1.2, there exists no Yang-Mills connection satisfying (3.2), since the bundle P is non-trivial. Therefore the claim follows. ∎

PROOF OF THEOREM 1.3. First choosing a sequence $\{t_i\}$ with $t_i \to T$ satisfying

$$\bar{e}(t_i) \to +\infty, \text{ and } \bar{e}(t) \leq \bar{e}(t_i) \text{ for } t \in [0, t_i].$$

For such sequence, we set $\lambda_i^2 = \dfrac{1}{\sqrt{\bar{e}(t_i)}}$.

Let $p_i \in S^n$ be a point satisfying $e(t_i, p_i) = \bar{e}(t_i)$. In the followings, we argue on a local coordinates neighbourhood centerd at p_i.

By Lemma 2.2, choosing δ=(the constant in Lemma 2.2), we have

$$t_i + \lambda_i^2 \delta < T,$$
$$\bar{e}(t) \leq 2\bar{e}(t_i) \text{ for } t_i < t \leq t_i + \lambda_i^2 \delta.$$

Here set $t_{0,i} = t_i + \lambda_i^2 \delta$, and for $t \in [-\lambda_i^{-2} t_i, \delta]$, $x \in B_{\rho\lambda_i^{-1}}$ set $D_i(t, x) = D(t_i + \lambda_i^2 t, \lambda_i x)$, then we have

$$F(D_i)(t, x) = \lambda_i^2 F(D)(t_i + \lambda_i^2 t, \lambda_i x).$$

Moreover D_i satisfies the equation

$$\frac{\partial D_i}{\partial t} = -d_{D_i}^{*} F(D_i)$$

on $[-\lambda_i^{-2} t_i, \delta] \times B_{\rho\lambda_i^{-1}}$, where the formal adjoint operator is defined by the L^2-inner product with respect to the metric tensor $g_{\alpha\beta}^i(x) = g_{\alpha\beta}(\lambda_i x)$. It is easy to see that

$$|F(D_i)|^2(0, 0) = \lambda_i^4 |F(D)|^2(t_i, p_i) = \lambda_i^4 \bar{e}(t_i) = 1.$$

Moreover on $(t, x) \in [-\lambda_i^{-2} t_i, \delta] \times B_{\rho\lambda_i^{-1}} =: Q_i$, it holds that

$$(3.3) \qquad |F(D_i)|^2(t, x) \leq 4|F(D_i)|^2(0, 0) \leq 4.$$

Set $e_i(t, x) := |F(D_i)|^2(t, x)$, then it satisfies

$$\frac{\partial e_i}{\partial t} \leq \Delta_i e_i + \frac{C_i}{2} e_i^{3/2}.$$

In view of (3.3), we see that

$$\frac{\partial e_i}{\partial t} \leq \Delta_i e_i + C_i e_i$$

on any open set $O_i \subset Q_i$. Equivalently, $h_i := \exp(-C_i t) e_i$ satisfies

$$\frac{\partial h_i}{\partial t} \leq \Delta_i h_i.$$

By using a Moser's result [10, Theorem 3], for $O_i := \left(-\min\left\{\frac{\delta}{2}, \frac{\delta}{C_i}\right\}, \frac{\delta}{2}\right) \times B_1$, there exists a constant $C>0$ such that

$$1 < h_i(0, 0) \leq C\left(\frac{2}{\delta \mathrm{Vol}(B_1)} \int_{O_i} h_i^2 dV_i dt\right)^{1/2},$$

for sufficient large i.

Now since $e_i \leq 4$ and $0 \leq h_i \leq e_i \exp(\delta)$, there exists a constant $C_1 = C_1(P)$ such that

$$1 \leq C_1 \int_{O_i} |F(D_i)|^2 dV_i dt.$$

By Lemma 2.3, Ψ satisfies

$$\Psi(R) \leq e^{C(R_0 - R)} \Psi(R_0) + C(e^{C(R_0 - R)} - 1) E_0$$

on $0 < R \leq R_0 = \min\{\rho, \sqrt{t_{0,i}}\}$. Thus we have

$$\begin{aligned} \Psi(R) &\leq e^{CR_0} \Psi(R_0) + Ce^{CR_0} E_0 \\ &\leq e^{CR_0} \Psi(R_0) + C\varepsilon e^{CR_0}, \end{aligned}$$

$$(3.4) \qquad \begin{aligned} \Psi(R_0) &\leq \frac{1}{2} R_0^{4-n} \int_{B_\rho} |F(D)|^2 (t_{0,i} - R^2, x) \varphi^2 dV \\ &\leq R_0^{4-n} E_0 \leq R_0^{4-n} \varepsilon. \end{aligned}$$

By (3.4), we obtain

$$\Psi(R) \leq \varepsilon e^{CR_0} (R_0^{4-n} + C) \text{ for } 0 < R \leq R_0.$$

On the other hand, for $R^2 = \lambda_i^2 S$, where $\sqrt{\delta/2} \leq S \leq \sqrt{\delta + \min\left\{\frac{\delta}{2}, \frac{\delta}{C_i}\right\}}$, we have

$$\lambda_i^{4-n}\int_{B_{\lambda_i}}|F(D)|^2(t_i+\lambda_i^2(\delta-S^2))dV=\left(\frac{R}{S}\right)^{4-n}\int_{B_{R/S}}|F(D)|^2(t_{0,i}-R^2,x)dV \tag{3.5}$$
$$\leq C_2\delta^{\frac{n-4}{2}}e^{\frac{1}{2\delta}}\Psi(R)\leq C_3(\delta)(R_0^{4-n}+C)e^{CR_0}\varepsilon\leq C_4R_0^{4-n}\varepsilon.$$

By scaling back, we have

$$\int_{B_1}|F(D_i)|^2dV_i=\lambda_i^{4-n}\int_{B_{\lambda_i}}|F(D)|^2(t_i+\lambda_i t)dV \tag{3.6}$$

for $-\min\left\{\frac{\delta}{2},\frac{\delta}{C_i}\right\}<t=\delta-S^2<\frac{\delta}{2}$. Hence for $-\min\left\{\frac{\delta}{2},\frac{\delta}{C_i}\right\}<t<\frac{\delta}{2}$, (3.5) and (3.6) imply

$$\int_{B_1}|F(D_i)|^2dV_i\leq C_4R_0^{4-n}\varepsilon.$$

Therefore we may lead

$$1\leq C_4R_0^{4-n}\varepsilon,$$

however, by $R_0=\min\{\rho,\sqrt{t_{0,i}}\}$, if $\rho<\sqrt{t_{0,i}}$, then we have

$$\varepsilon\geq C_4^{-1}\rho^{n-4}.$$

This is a contradiction for sufficient small $\varepsilon>0$. Therefore we have $R_0=\sqrt{t_{0,i}}$. Hence we have $1\leq C_4\left(\sqrt{t_{0,i}}\right)^{4-n}\varepsilon$, we have

$$t_{0,i}^{\frac{n-4}{2}}\leq C_5\varepsilon.$$

Since $t_{0,i}=t_i+\lambda_i^2\delta\to T$ as $i\to\infty$, we conclude

$$T\leq C_6\varepsilon^{\frac{2}{4-n}}.$$

Therefore, the smooth solution $D(t)$ of (1.1) with the initial value D_0 have to blows up in finite-time. ■

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