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An improvement of the C^1 closing lemma for endomorphisms

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Abstract

L. Wen proved the C^1 closing lemma for endomorphisms with finitely many singularities. The arguments and the tools of Wen are available for endomorphisms which have infinitely many singularities but at most finitely many ones in the nonwandering sets. By refining the argument of Wen we prove the C^1 closing lemma for endomorphisms with finitely many singularities in the nonwandering sets. By using this lemma we can slightly improve characterization of C^1 absolutely Ω-stable endomorphisms. That is, for an endomorphism f with finitely many singularities in the nonwandering set, f is C^1 absolutely Ω-stable if and only if f has a neighborhood $\mathscr{U}$ such that every g in $\mathscr{U}$ satisfies weak Axiom A.

1 Introduction

Let M be a compact smooth Riemannian manifold without boundary, and let $\mathrm{End}^1(M)$ be the space of C^1 endomorphisms of M endowed with the C^1 topology. L. Wen proved the C^1 closing lemma for nonsingular endomorphisms [5] and generalized it from nonsingular endomorphisms to endomorphisms with finitely many singularities [6].

THEOREM 1[6]. *Let f be a C^1 endomorphism of M with finitely many singularities, and ω be a nonwandering point of f. Then for any C^1 neighborhood $\mathscr{U}$ of f in* $\mathrm{End}^1(M)$, *there is a $g\in\mathscr{U}$ such that ω is a periodic point of g.*

Recall that a point x in M is a singularity of f if the tangent map T_xf is not injective. Let $S(f)$ be the set of singularities of f. A point x is nonwandering of f if for any neighborhood U of x in M, $f^n(U)\cap U\neq\phi$ for some positive integer n, and periodic of f if $f^m(x)=x$ for some positive integer m. Let $P(f)$ be the set of periodic points of f, and $\Omega(f)$ the set of nonwandering points of f. Remark that in general $f(\Omega(f))\subset\Omega(f)$.

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In this paper we improve Theorem 1(the C^1 closing lemma). By refining the argument in the proof of Theorem 1, we prove the following:

THEOREM A. *Let f be a C^1 endomorphism of M with finitely many singularities in the nonwandering set $\Omega(f)$, and ω be a nonwandering point of f. Then for any C^1 neighborhood $\mathscr{U}$ of f in* $\mathrm{End}^1(M)$, *there is a $g \in \mathscr{U}$ such that ω is a periodic point of g.*

An endomorphism f with finitely many singularities in the nonwandering set $\Omega(f)$ means that f may have infinitely many singularities but has at most finitely many singularities in $\Omega(f)$. The strategy for the improvement is that instead of using the standard argument of Wen we use Lemma 3.1(below) whenever we can utilize it.

By using Theorem A we can slightly improve characterization of C^1 absolutely Ω-stable endomorphisms [1, 2]. We obtain the following:

THEOREM B. *Let f be a C^1 endomorphism of M with finitely many singularities in the nonwandering set $\Omega(f)$. Then f is C^1 absolutely Ω-stable if and only if f has a neighborhood $\mathscr{U}$ in* $\mathrm{End}^1(M)$ *such that every g in $\mathscr{U}$ satisfies weak Axiom A.*

2 Preliminaries

In this section we give some definitions and theorems needed to prove Theorem A. We start with the definition of tree introduced by Wen [5, 6].

By a *tree* $\mathscr{T}=(Q, f)$ we mean an infinite sequence of disjoint non-empty finite sets $Q_0, Q_1, \cdots, Q_n, \cdots$, where Q_0 consists of a single point q_0, together with a map $f: Q-\{q_0\} \to Q$, where $Q=\bigcup_{n=0}^{\infty} Q_n$, such that f maps Q_n into Q_{n-1} for each $n=1, 2, \cdots$. An infinite sequence $q_0, q_1, \cdots, q_n, \cdots, q_n \in Q_n$, is called an *infinite branch* of $\mathscr{T}$ if $f(q_n)=q_{n-1}$ for $n=1, 2, \cdots$. A finite sequence $q_0, q_1, \cdots, q_k$ is called a *finite branch* of $\mathscr{T}$ if $f(q_n)=q_{n-1}$ for each $n=1, 2, \cdots, k$, and if $f^{-1}\{q_k\}$ is empty. A tree $\mathscr{T}=(Q, f)$ is called *complete* if f is onto. It is obvious that $\mathscr{T}$ is complete if and only if $\mathscr{T}$ has only infinite branches. For an infinite branch $\Sigma=\{q_0, q_1, \cdots, q_k, \cdots\}$ of $\mathscr{T}=(Q, f)$, a finite sequence $\{q_0, q_1, \cdots, q_k\}$ is called a *sub-branch* of Σ with length k. By a *tree of isomorphisms* we mean a collection of linear isomorphisms parametrized by a tree $\mathscr{T}$. This means that we associate to each $q \in Q$ an m-dimensional inner product space V_q, and to each $q \neq q_0$ a linear isomorphism $T_q: V_q \to V_{q_0}$.

In the proof of Theorem A we need the following results of Wen [6].

THEOREM 2.1.(ε-kernel avoiding transition theorem). *Given a com-*

plete tree of isomorphisms $(\mathscr{T}, T_q)$ *and* $\varepsilon>0$. *There is a number* $\rho>2$ *and an integer* $\mu\geq 1$ *such that*:
For any finite ordered set $P=\{p_0, p_1, \cdots, p_t\}$ *in* V_{q_0}, *there is a point* $y\in P \cap B(p_t, \rho|p_0-p_t|)$ *such that for any branch* $\Sigma=\{q_0, q_1, \cdots, q_n, \cdots\}$ *of* $\mathscr{T}$, *there is a point* $w\in P\cap B(p_t, \rho|p_0-p_t|)$, *where* w *is before* y *in the order of* P, *together with* $(\mu+1)$*-points* $c_0, c_1, \cdots, c_\mu$ *in* $B(p_t,\rho|p_0-p_t|)$, *not necessarily distinct, satisfying the following two conditions* (a) *and* (b).

(a) $c_0=w$, $c_\mu=y$, *and*

(b) $|T_{q_n}^{-1}(c_n)-T_{q_n}^{-1}(c_{n+1})|\leq \varepsilon d(T_{q_n}^{-1}(c_{n+1}),\ T_{q_n}^{-1}(A))$

for $n=0, 1, \cdots, \mu-1$, *where* T_{q_0} *stands for the identity,* $A=P(w, y)\cup \partial B(p_t, \rho|p_0-p_t|)$, $P(w, y)=\{p\in P|\ p$ *is after* w *and before* $y\}$, *and* d *is the distance on* V_q, $B(s,r)$ *is the ball centered at* s *with radius* r.

For simplicity we assume that M is a Riemannian manifold embedded into some Euclidean space R^d. Then $\mathrm{End}^1(M)$ has a C^1 metric d_1 inherited from $C^1(M, R^d)$ compatible with its C^1 topology. Fix $\zeta>0$ such that $\exp_p$ embeds $\{u\in T_pM||u|\leq\zeta\}$ into M for each $p\in M$. The following Lemma 2.2 is essentially Theorem 6.1 in [4].

LEMMA 2.2.(the ε-kernel lifts) *For any* $\eta>0$, *there is an* $\varepsilon>0$ *such that for any* $f\in\mathrm{End}^1(M)$, *any* $p\in M$, *and any two points* $v_1, v_2\in T_pM$ *with* $B(v_2,|v_1-v_2|/\varepsilon)\subset\{u\in T_pM||u|\leq\zeta\}$, *there is a diffeomorphism* $h=h_{p,\varepsilon,v_1,v_2}$: $M\to M$, *called an* ε*-kernel lift, such that*:

(1) $h(\exp_p(v_2))=\exp_p(v_1)$;

(2) $\mathrm{supp}(h)\subset\exp_p(B(v_2, |v_1-v_2|/\varepsilon))$, *here the support means the closure of the set where* h *differs from the identity*;

(3) $d_1(hf, f)<\eta$.

Before stating Lemma 2.3, we recall some facts and definitions in [5, 6]. The negative orbit of $p\in M$ under f is defined by $\mathscr{O}^-(p)=\mathscr{O}_f^-(p)=\bigcup_{n=0}^{\infty} f^{-n}\{p\}$, where $f^{-n}\{p\}$ denotes the preimage $(f^n)^{-1}\{p\}$. Next we define a *μ-dynamical neighborhood* in somewhat different setting from the case of Wen [5, 6]. We define μ-dynamical neighborhood on complete subsets of $\mathscr{O}_f^-(p)$ without singularities while Wen defined it on $\mathscr{O}_f^-(p)$ itself. Let Q be a subset of $\mathscr{O}_f^-(p)$ such that (Q, f) is a complete tree with $Q_0=\{p\}$ and Q contains no singularities. Then given an integer $\mu\geq 1$, f is a local diffeomorphism near each $q\in\bigcup_{n=1}^{\mu} Q_n$. In this case we may find a neighborhood W of p in M, called a *μ-dynamical neighborhood* of p, such that for each $q\in\bigcup_{n=1}^{\mu}Q_n$ there is a neighborhood $W(q)$ satisfying

(a) $f^m(W(q))=W$ whenever $f^m(q)=p$, $m=1,2,\cdots,\mu$;
(b) $W(q)\cap W(q')=\phi$ if $q\neq q'$.

$W(q)=W_f(q)$ is called the W-component at q.

LEMMA 2.3. *Let $f\in \mathrm{End}^1(M)$, $p\in M$, (Q,f) a complete tree with $Q_0=\{p\}$, and an integer $\mu\geq 1$ be given. Assume that $\bigcup_{n=1}^{\mu+1}Q_n$ contains no singularities of f. Then for any $\eta>0$, there is a $\lambda>0$, and a map $f_1\in \mathrm{End}^1(M)$, called a local linearization of f, with the following properties (1)−(5).*
Write $W'=\{u\in T_pM \mid |u|\leq\lambda\}$, $V'=\{u\in T_pM \mid |u|\leq\lambda/4\}$, $W=\exp_p(W')$, $V=\exp_p(V')$.

(1) *W is $(\mu+1)$-dynamical for both f and f_1, and the W-component for f and f_1 are the same, i.e. $W_f(q)=W_{f_1}(q)$ for each $q\in\bigcup_{n=1}^{\mu+1}Q_n$;*
(2) *$f_1=\exp_{f(q)}(T_qf)\exp_q^{-1}$ on $V_{f_1}(q)$ if $q\in\bigcup_{n=1}^{\mu}Q_n$;*
(3) *$f_1^{\mu+1}=f^{\mu+1}$ on $W(q)$ if $q\in Q_{\mu+1}$. In particular, if $q\in Q_{\mu+1}$ then $f_1=\exp_{f(q)}(T_{f(q)}f^{\mu})^{-1}\exp_p^{-1}f^{\mu+1}$ on $V(q)$. Note that $V_f(q)=V_{f_1}(q)$ here and we have written both of them as $V(q)$;*
(4) *$f_1=f$ on $M-\bigcup\{W(q) \mid q\in\bigcup_{n=1}^{\mu+1}Q_n\}$;*
(5) *$d_1(f_1,f)<\eta$.*

3 Proof of Theorem A

In this section we prove the C^1 closing lemma for endomorphisms with finitely many singularities in the nonwandering sets. We do this by refining the argument of Wen [5, 6].

PROOF of THEOREM A. To prove Theorem A, as pointed out in [3, p 967], it suffices to prove that given any C^1 neighborhood $\mathscr{U}$ of f in $\mathrm{End}^1(M)$ and any neighborhood U of ω in M, there is a $g\in\mathscr{U}$ such that g has a periodic point in U. Because another perturbation allows us to push this periodic point onto ω.

Let $\mathscr{U}$ be any C^1 neighborhood of f in $\mathrm{End}^1(M)$, and U be any neighborhood of ω in M. We assume that ω is not periodic of f. Since ω is nonwandering, $\mathscr{O}^-(\omega)$ is infinite.

In a certain good situation, we can construct a periodic point by a (C^∞) small perturbation (cf. [6]).

LEMMA 3.1. *If there exist $\sigma\in\mathcal{O}^-(\omega)\cap\Omega(f)$, $q\in f^{-1}(\sigma)$, a sequence $\{x_i\}$ and an increasing sequence of positive integers $\{n_i\}$ such that $q\notin\Omega(f)$, $\{x_i\}$ converges to σ and $\{f^{n_i}(x_i)\}$ converges to q, then for any C^1 neighborhood $\mathcal{U}$ of f and any neighborhood U of ω, there is a $g\in\mathcal{U}$ such that g has a periodic point in U.*

PROOF of LEMMA 3.1. Let $n\geq 0$ be an integer with $f^n(\sigma)=\omega$. Since $q\notin\Omega(f)$, there is a ball B around q such that $f^m(B)\cap B=\phi$ for all $m\geq 1$. Take a neighborhood V_j of $f^j(q)$ for each $j=0,1,\cdots,n+1$, such that:

(a) $V_j\cap B=\phi$ for $j=1,\cdots,n+1$,;
(b) $V_0\subset B$ and $V_{n+1}\subset U$;
(c) $V_k\cap V_l=\phi$ for all $0\leq k<l\leq n+1$;
(d) $f(V_j)\subset V_{j+1}$ for $j=0,1,\cdots,n$.

Arbitrarily near σ and q, there are two points x_i and $f^{n_i}(x_i)$ for large i. Hence there is a C^1 small perturbation g of f taking $f^{n_i}(x_i)$ onto x_i. More precisely, we can take diffeomorphisms g_i, h_i of M such that

(i) $g_i=$identity outside V_0;
(ii) $h_i=$identity outside V_1;
(iii) $g_i(f^{n_i}(x_i))=q$;
(iv) $h_i(f(q))=h_i(\sigma)=x_i$.

Since V_0 and V_1 are fixed we can choose $g_i, h_i\rightarrow id$ in the C^∞-sence as $i\rightarrow\infty$. Then $g=h_i\circ f\circ g_i$ C^1-approximates f. Then the g-orbit from x_i to $g^{n_i-1}(x_i)$ is the same as the f-orbit from x_i to $f^{n_i-1}(x_i)$ since B is wandering and g is equal to f outside B. Hence x_i is periodic of g. Since $f=g$ on $\cup\{V_j|1\leq j\leq n\}$, $g^n(x_i)=f^n(x_i)\in V_{n+1}\subset U$. Therefore g has a periodic point $g^n(x_i)$ in U. □

We divide the proof into two cases.

Case 1. There is $\sigma\in\mathcal{O}^-(\omega)\cap\Omega(f)$ such that $f^{-1}\{\sigma\}\cap\Omega(f)=\phi$.
It is easy to see that this case is proved by Lemma 3.1.

Case 2. No such σ exists.
In this case there exists an infinite sequence (ω_j) such that

(i) $\omega_j\in\Omega(f)$ for all $j\geq 0$;
(ii) $\omega_0=\omega$;
(iii) $f(\omega_{j+1})=\omega_j$ for all $j\geq 0$.

Then there is an integer $n_0\geq 0$ such that ω_j is not a singularity of f for all $j\geq n_0$. Suppose not. Since the cardinality of $S(f)\cap\Omega(f)$ is finite, there exist a singularity p of f and two integers $1 \leq j_1 < j_2$ such that $\omega_{j_1} = \omega_{j_2} = p$. Then

$$f^{j_2-j_1}(\omega) = f^{j_2-j_1}(f^{j_1}(\omega_{j_1})) = f^{j_2}(\omega_{j_2}) = \omega,$$

contradicting that ω is not periodic of f.

By the similar argument as above, there is an integer $n_1 \geq n_0$ such that

$$(\mathcal{O}^-(\omega_{n_1}) \cap \Omega(f)) \cap S(f) = \phi.$$

We claim that $f^{-j}\{\omega_{n_1}\} \cap \Omega(f)$ is a finite set for every $j \geq 0$. We prove the claim by induction on j. For $j=0$, it is trivial. Suppose that the claim holds for $j-1$ and not for j. Then there exist a point $q \in f^{-j+1}\{\omega_{n_1}\}$ and a sequence of distinct points $\{x_i\} \subset f^{-j}\{\omega_{n_1}\} \cap \Omega(f)$ such that $f(x_i) = q$ for all i. Let p be a limit point of $\{x_i\}$. Since $\Omega(f)$ is closed, $p \in \Omega(f) \cap f^{-j}\{\omega_{n_1}\}$. Hence $p \notin S(f)$ and f is a local diffeomorphism near p. But for any small neighborhood V of p, $f|V$ is not injective. This is a contradiction.

Since $f(\Omega(f)) \subset \Omega(f)$, there does not exist a sequence (a_j) starting from ω_{n_1} such that:

(i) $a_0 = \omega_{n_1}$, $f(a_{j+1}) = a_j$ for all $j \geq 0$;

(ii) $a_j \in \Omega(f)$ for all $0 \leq j < k$ and $k+1$, $a_k \notin \Omega(f)$, where k is some positive integer.

Suppose that there exists a finite sequence contained in $\Omega(f)$, $a_0 = \omega_{n_1}$, a_1, $\cdots$, a_k such that $f(a_{j+1}) = a_j$ for all $0 \leq j < k$. Since a_k is nonwandering, there exists a preimage of a_k. If $f^{-1}\{a_k\} \cap \Omega(f) = \phi$ then that contradicts the assumption of Case 2. By above observations, we assume that $\mathcal{O}^-(\omega_{n_1}) \cap \Omega(f)$ consists of infinite sequences starting from ω_{n_1}. Therefore we assume that $(\mathcal{O}^-(\omega_{n_1}) \cap \Omega(f), f)$ is a complete tree. Let $Q_j = f^{-j}\{\omega_{n_1}\} \cap \Omega(f)$ for all $j \geq 0$, and $Q = \bigcup_{j=0}^{\infty} Q_j$. Then $\mathcal{T} = (Q, f)$ is a complete tree. Remember that $f^{n_1}(\omega_{n_1}) = \omega$. Let N be a neighborhood of ω_{n_1} such that $f^{n_1}(N) \subset U$ and $N \subset \exp_{\omega_{n_1}}\{u \in T_{\omega_{n_1}} M \| u| \leq \zeta\}$. Take any $\eta > 0$ such that the η-ball of f in $\mathrm{End}^1(M)$ is contained in $\mathscr{U}$. By Lemma 2.2, there is a $\varepsilon > 0$ such that $d_1(hf, f) < \eta/2$ for any $f \in \mathrm{End}^1(M)$, where h is any ε-kernel lift. Denote by $(\mathcal{T}, T_q)$ the tree of isomorphisms, where $q \in Q - \{\omega_{n_1}\}$, and $T_q = T_q f^m$ if $f^m(q) = \omega_{n_1}$.

Let $\rho > 2$, $\mu \geq 1$ be the two numbers obtained by Theorem 2.1 respecting $\{\mathcal{T}, T_q\}$ and $\varepsilon > 0$ above. For the f, ω_{n_1}, μ, there is by Lemma 2.3 a $\lambda > 0$ and a local linearization f_1 with the following properties (1)–(5). Write $W' = \{u \in T_a M \| u| \leq \lambda\}$, $V' = \{u \in T_a M \| u| \leq \lambda / 4\}$, $W = \exp_a(W')$, $V = \exp_a(V')$, where $a = \omega_{n_1}$.

(1) W is $(\mu+1)$-dynamical for both f and f_1, and $W_f(q)=W_{f_1}(q)$ for each $q\in\bigcup_{m=1}^{\mu+1}Q_m$;

(2) $f_1=\exp_{f(q)}(T_qf)\exp_q^{-1}$ on $V_{f_1}(q)$ if $q\in\bigcup_{m=1}^{\mu}Q_m$, where $V_{f_1}(q)=\exp_q\circ(T_qf^m)^{-1}\circ\exp_\alpha^{-1}(V)$, $\alpha=\omega_{n_1}=f^m(q)$;

(3) $f_1^{\mu+1}=f^{\mu+1}$ on $W(q)$ if $q\in Q_{\mu+1}$. In particular, if $q\in Q_{\mu+1}$, then $f_1=\exp_{f(q)}(T_{f(q)}f^\mu)^{-1}\exp_\alpha^{-1}f^{\mu+1}$ on $V(q)$;

(4) $f_1=f$ on $M-\cup\{W(q)|q\in\bigcup_{m=1}^{\mu+1}Q_m\}$;

(5) $d_1(f_1,f)<\eta/2$.

By shrinking W if necessary, we assume that $W\subset N$. Put a metric d' on W by defining

$$d'(p,q)=|u-v|,$$

where $p,q\in N$, $u=\exp_\alpha^{-1}(p)$, $v=\exp_\alpha^{-1}(q)$, $\alpha=\omega_{n_1}$.
Since ω_{n_1} is nonwandering, there exist a sequence $\{x_i\}$ in M and a sequence $\{m_i\}$ of positive integers such that $\{x_i\}$ and $\{f^{m_i}(x_i)\}$ both converge to ω_{n_1}. Then there exists a positive integer K such that

$$B(f^{m_i}(x_i),\rho d'(x_i,f^{m_i}(x_i)))\subset V\qquad\text{for all } i\geq K.$$

Let $P_i=\{x_i,f(x_i),\cdots,f^{m_i}(x_i)\}\cap V$ for all $i\geq K$. Say $P_i=\{p_0^i,p_1^i,\cdots,p_t^i\}$, where t is a positive integer depending on i. Note that $p_0^i=x_i$, $p_t^i=f^{m_i}(x_i)$. Hence $B(p_t^i,\rho d'(p_0^i,p_t^i))\subset V$ for all $i\geq K$. Let $P_i'=\exp_\alpha^{-1}(P_i)$, $\bar p_j^i=\exp_\alpha^{-1}(p_j^i)$, $\alpha=\omega_{n_1}$. Then $P_i'=\{\bar p_0^i,\bar p_1^i,\cdots,\bar p_t^i\}$.
By Theorem 2.1, for each $i\geq K$, there is a point $\bar y_i\in P_i'\cap B(\bar p_t^i,\rho|\bar p_0^i-\bar p_t^i|)$, such that for any branch $\Sigma=\{q_0,q_1,\cdots,q_n,\cdots\}$ of $\mathscr{T}$, there is a point $\bar\omega_i(\Sigma)\in B(\bar p_t^i,\rho|\bar p_0^i-\bar p_t^i|)$, where $\bar\omega_i(\Sigma)$ is before $\bar y_i$ in P_i', together with $(\mu+1)$-points $\bar c_0^i(\Sigma),\bar c_1^i(\Sigma),\cdots,\bar c_\mu^i(\Sigma)$ in $B(\bar p_t^i,\rho|\bar p_0^i-\bar p_t^i|)$, not necessarily distinct, satisfying the following two conditions (a) and (b).

(a) $\bar c_0^i(\Sigma)=\bar\omega_i(\Sigma)$, $\bar c_\mu^i(\Sigma)=\bar y_i$; and

(b) $|(T_{q_n}f^n)^{-1}(\bar c_n^i(\Sigma))-(T_{q_n}f^n)^{-1}(\bar c_{n+1}^i(\Sigma))|$
$\leq\varepsilon d((T_{q_n}f^n)^{-1}(\bar c_{n+1}^i(\Sigma)),(T_{q_n}f^n)^{-1}(A))$

for $n=0,1,\cdots,\mu-1$, where $A=P'(\bar\omega_i(\Sigma),\bar y_i)\cup\partial B(\bar p_t^i,\rho|\bar p_0^i-\bar p_t^i|)$, and $P'(\bar\omega_i(\Sigma),\bar y_i)=\{p\in P_i'|\ p\text{ is before }\bar y_i\text{ and after }\bar\omega_i(\Sigma)\}$.

Let $\omega_i(\Sigma)=\exp_\alpha(\bar\omega_i(\Sigma))$, $y_i=\exp_\alpha(\bar y_i)$. Then $\omega_i(\Sigma)$ and y_i are both in P_i. Hence for each $i\geq K$, there exists an integer $\phi_i(\Sigma)\geq 1$ such that $f^{\phi_i(\Sigma)}(\omega_i(\Sigma))=y_i$. Remark that $\lim_{i\to\infty}p_t^i=\lim_{i\to\infty}f^{m_i}(x_i)=\omega_{n_1}=\alpha$, $\lim_{i\to\infty}d(x_i,f^{m_i}(x_i))=0$, hence $\lim_{i\to\infty}y_i=\alpha$ and $\lim_{i\to\infty}\omega_i(\Sigma)=\alpha$.

For any branch Σ there exists an integer $N>0$ such that for any $i\geq N$

there is a $q\in Q_1$ satisfying $f^{\phi_i(\Sigma)-1}(\omega_i(\Sigma))\in W(q)$. Suppose that the above property does not hold. Then for some branch Σ there is a convergent subsequence $\{f^{\phi_{i'}(\Sigma)-1}(\omega_{i'}(\Sigma))\}$ such that

$$f^{\phi_{i'}(\Sigma)-1}(\omega_{i'}(\Sigma))\notin \cup\{W(q)|q\in Q_1\} \text{ for all } i'.$$

Let $x=\lim_{i'\to\infty}f^{\phi_{i'}(\Sigma)-1}(\omega_{i'}(\Sigma))$. Since $f(x)=\lim_{i'\to\infty}f^{\phi_{i'}(\Sigma)}(\omega_{i'}(\Sigma))=\lim_{i'\to\infty}y_{i'}=\alpha=\omega_{n_1}$, $x\in f^{-1}\{\omega_{n_1}\}$. Moreover $\{\phi_{i'}(\Sigma)\}$ is unbounded. Suppose that $\{\phi_{i'}(\Sigma)\}$ is bounded. Let $\{i''\}$ be a subsequence such that $\phi_{i''}(\Sigma)=k$=constant. Then

$$\omega_{n_1}=f(x)=\lim_{i''\to\infty}f^{\phi_{i''}(\Sigma)}(\omega_{i''}(\Sigma))=\lim_{i''\to\infty}f^k(\omega_{i''}(\Sigma))=f^k(\omega_{n1}).$$

This contradicts that ω_{n_1} is not periodic of f.

Now we have the sequence $\{\omega_{i'}(\Sigma)\}$ converging to ω_{n_1} and the increasing sequence of positive integers $\{\phi_{i'}(\Sigma)-1\}$ such that $f^{\phi_{i'}(\Sigma)-1}(\omega_{i'}(\Sigma))$ converges to x. Hence, by Lemma 3. 1, we can construct a periodic point in any neighborhood U of ω_{n_1} by a sufficiently small C^1 perturbation. By the similar argument as above we can assume that; for any branch Σ, there exists an $N>0$ such that for any $i\geq N$, there is a $q\in Q_{\mu+1}$ satisfying $f^{\phi_i(\Sigma)-\mu-1}(\omega_i(\Sigma))\in W(q)$. Let $z_i(\Sigma)=f^{\phi_i(\Sigma)-\mu-1}(\omega_i(\Sigma))$. Remark that $z_i(\Sigma)$ actually does not depend on Σ, since $\omega_i(\Sigma)$ and y_i are both in P_i, and $f^{\phi_i(\Sigma)}(\omega_i(\Sigma))=y_i$, and since μ and y_i do not depend on Σ. Thus we simply write

$$z_i=f^{\phi_i(\Sigma)-\mu-1}(\omega_i(\Sigma)) \text{ for any branch } \Sigma \text{ of } \mathscr{T}.$$

By taking a subsequence of i if necessary we can assume that

(P) $z_i\in W(\sigma_{\mu+1})$ for all $i\geq 1$, where $\sigma_{\mu+1}$ is some point in $Q_{\mu+1}$.

Now fix a branch Σ of $\mathscr{T}$ that contains $\sigma_{\mu+1}$ above. The property (P) above means that for any i and any branch Γ of $\mathscr{T}$, the orbit $\{f^{\phi_i(\Gamma)-\mu-1}(\omega_i(\Gamma)),\cdots,f^{\phi_i(\Gamma)}(\omega_i(\Gamma))\}$ returns to W along the sub-branch $\{\sigma_{\mu+1},f(\sigma_{\mu+1}),\cdots,f^{\mu+1}(\sigma_{\mu+1})=\omega_{n_1}\}$ of Σ with length $\mu+1$. We can assume that the following claim holds, because if it is not the case, we can construct a periodic point near ω_{n_1} by an arbitrary small perturbation and therefore Theorem A holds.

CLAIM. Let Σ be the above fixed branch of $\mathscr{T}$. Shrinking W if necessary, for any sufficiently large i, if the orbit $\{\omega_i(\Sigma), f(\omega_i(\Sigma)),\cdots, f^{\phi_i(\Sigma)}(\omega_i(\Sigma))\}$ meets a W-component $W(f^k(\tilde{q}))$, $\tilde{q}\in Q_{\mu+1}$, $1\leq k\leq\mu+1$, at $f^j(\omega_i(\Sigma))$, then $j\geq k$ and

$$f^{j-l}(\omega_i(\Sigma))\in W(f^{k-l}(\tilde{q})) \quad \text{for all } 0\leq l\leq k.$$

This Claim means that if the orbit $\{\omega_i(\Sigma), f(\omega_i(\Sigma)), \cdots, f^{\phi_i(\Sigma)}(\omega_i(\Sigma))\}$ returns to W intermediately, the orbit must return along a sub-branch of some branch Γ with length $\mu+1$. If the Claim does not hold, then for any small neighborhood W and any positive integer N, there exist an $i \geq N$ such that $f^{l_i}(\omega_i(\Sigma)) \in W(\tilde{q})$ and $f^{l_i-1}(\omega_i(\Sigma)) \notin \cup\{W_i(q) | q \in \bigcup_{n=1}^{\mu+1} Q_n\}$ for some $1 \leq l_i < \phi_i(\Sigma) - \mu - 1$, and $\tilde{q} \in Q_k$ for some $0 \leq k \leq \mu$. Therefore, taking a subsequence if necessary, we can assume that $f^{l_{i'}}(\omega_{i'}(\Sigma))$ converges to some $q \in Q_k$ as $i' \to \infty$.

Case A. $\{l_{i'}\}$ is bounded.
We take a subsequence $\{l_{i''}\}$ such that $l_{i''} = m =$constant. Then

$$\begin{aligned}\omega_{n_1} &= f^k(q) = f^k(\lim_{i'' \to \infty} f^m(\omega_{i''}(\Sigma))) \\ &= \lim_{i'' \to \infty} f^{k+m}(\omega_{i''}(\Sigma)) = f^{k+m}(\omega_{n_1})\end{aligned}$$

contradicting that ω_{n_1} is not periodic of f.

Case B. $\{l_{i'}\}$ is unbounded.
We take a subsequence $\{f^{l_{i''}-1}(\omega_{i''}(\Sigma))\}$ which converges to some point p. By assumption, $p \notin \Omega(f)$. Since $q \in Q_k$, there exists a sequence $\{z_{i''}\}$ such that $f^k(z_{i''}) = \omega_{i''}(\Sigma)$ for all i''. Since $\omega_{i''}(\Sigma) \to \omega_{n_1}$ as $i'' \to \infty$, we get $z_{i''} \to q$ as $i'' \to \infty$. Now we have that $z_{i''} \to q$, $f^{k+l_{i''}-1}(z_{i''}) \to p$, $k + l_{i''} - 1 \to \infty$ as $i'' \to \infty$. Also we have $p \in f^{-1}(q)$, $p \notin \Omega(f)$. Therefore we can apply Lemma 3.1 and Theorem A holds in this case.

By the above Claim, finally we have only to consider the following case. There exist a $(\mu+1)$-dynamical neighborhood W of ω_{n_1} and a positive integer N such that for any $i \geq N$, the orbit $\{f^j(\omega_i(\Sigma)) | 0 \leq j \leq \phi_i(\Sigma)\}$ has the following properties;

(c) $f^{\phi_i(\Sigma)-\mu-1}(\omega_i(\Sigma)) \in W(\sigma_{\mu+1})$ for $\sigma_{\mu+1} \in Q_{\mu+1}$, where $\sigma_{\mu+1}$ is as in (P);
(d) if $f^j(\omega_i(\Sigma)) \in W(q')$, $q' = f^k(\tilde{q})$, $1 \leq k \leq \mu+1$, where $\tilde{q}$ is some point in $Q_{\mu+1}$, then $j \geq k$ and $f^{j-l}(\omega_i(\Sigma)) \in W(f^{k-l}(\tilde{q}))$ for all $0 \leq l \leq k$;
(e) $f^j(W)$ does not intersect $\cup\{W(p) | p \in \bigcup_{n=1}^{\mu+1} Q_n\}$ for $j = 1, 2, \cdots, n_1$, here n_1 is the integer with $f^{n_1}(\omega_{n_1}) = \omega$.

Now fix i such that properties (c), (d), and (e) above hold and

$$\exp_a B(\bar{p}_t^i, \rho|\bar{p}_0^i - \bar{p}_t^i|) \subset V, \text{ where } a = \omega_{n_1}.$$

For simplicity we omit this i, i.e. $\omega(\Sigma) = \omega_i(\Sigma)$, $y = y_i$, and $\phi(\Sigma) = \phi_i(\Sigma)$. Let $\omega' = \overline{\omega_i}(\Sigma)$, together with $c_0' = \bar{c}_0^i(\Sigma)$, $c_1' = \bar{c}_1^i(\Sigma), \cdots, c_\mu' = \bar{c}_\mu^i(\Sigma)$, be guaranteed by Theorem 2.1 respecting Σ and the finite ordered set $\exp_a^{-1}(\{f^j(\omega(\Sigma)) | 0 \leq j \leq \phi\} \cap V)$, where $a = \omega_{n_1}$, $\phi = \phi(\Sigma)$ is the integer such that $f^{\phi}(\omega(\Sigma))=y$. Let $\{\sigma_0=\omega_{n_1}, \sigma_1, \cdots, \sigma_{\mu+1}\}$ be the sub-branch of Σ with length $\mu+1$. For each σ_n, $n=0, 1, \cdots, \mu-1$, let h_{σ_n} be the ε-kernel lift obtained by treating in Lemma 2. 2 $p=\sigma_n$, $v_1=(T_{\sigma_n}f^n)^{-1}(c'_n)$, and $v_2=(T_{\sigma_n}f^n)^{-1}(c'_{n+1})$. Define a map g by

$$g=\begin{cases} h_{\sigma_n}\circ f_1 \text{ on } W(\sigma_{n+1}),\ n=0, 1, \cdots, \mu-1, \\ \quad f_1 \text{ on the rest of } M. \end{cases}$$

Then $d_1(g, f_1)<\eta/2$. Hence $d_1(g, f)<\eta$. We now verify that $\tilde{\omega}=\omega(\Sigma)=\exp_a(\omega')$ is periodic of g. It suffices to show that $g^{\phi-\mu-1}(\tilde{\omega})=z$ and $g^{\mu+1}(z)=\tilde{\omega}$. Remember that $z=f^{\phi(\Sigma)-\mu-1}(\omega(\Sigma))$. By the condition (b) above, the g-orbit from $\tilde{\omega}$ to z never touches the supports of these lifts. Hence $g^{\phi-\mu-1}(\tilde{\omega})=f_1^{\phi-\mu-1}(\tilde{\omega})$. Moreover $f_1^{\phi-\mu-1}(\tilde{\omega})=f^{\phi-\mu-1}(\tilde{\omega})$ by the conditions (3) and (d) above. Therefore $g^{\phi-\mu-1}(\tilde{\omega})=f^{\phi-\mu-1}(\tilde{w})=z$. It remains to verify that $g^{\mu+1}(z)=\tilde{\omega}$. By the condition (3) above, $g(z)=f_1(z)=\exp_{\sigma_\mu}(T_{\sigma_\mu}f^{\mu})^{-1}\exp_{\sigma_0}^{-1} f^{\mu+1}(z)$ because $z\in V(\sigma_{\mu+1})$. Hence $g(z)=\exp_{\sigma_\mu}(T_{\sigma_\mu}f^{\mu})^{-1}(c'_\mu)$ because $f^{\mu+1}(z)=y$ and $c'_\mu=y'=\exp_{\sigma_0}^{-1}y$. Thus these lifts $h_{\sigma_{\mu-1}}, \cdots, h_{\sigma_0}$ give $g^{\mu}(g(z))=\tilde{\omega}$ by the condition (2) above. Therefore $\tilde{\omega}$ is periodic of g. By the condition (e) $g^{n_1}(\tilde{\omega})=f^{n_1}(\tilde{\omega})$ is a periodic point of g in U. This completes the proof of Theorem A for Case 2. □

4 Application

Recently we have obtained a characterization of absolutely Ω-stable endomorphisms [1, 2]. To be precise we make the following definitions. We say that $f\in\mathrm{End}^1(M)$ is *absolutely Ω-stable* if there exist a neighborhood $\mathscr{U}$ of f in $\mathrm{End}^1(M)$ and a function $\varphi: \mathscr{U}\to C^0(\Omega(f), M)$ and a constant $K>0$ such that

(a) $\varphi(g)$ is a homeomorphism for each $g\in\mathscr{U}$, and $\varphi(f)=i:\Omega(f)\to M$;
(b) $g\circ\varphi(g)=\varphi(g)\circ f$;
(c) $d(\varphi(g), i)\leq Kd_0(f, g)$,

where i is the inclusion map, d is the metric on $C^0(\Omega(f), M)$ defined by $d(j, k)=\sup\{\rho(j(x), k(x))|x\in\Omega(f)\}$, ρ is a metric on M and d_0 is the metric on $\mathrm{End}^1(M)$ defined by

$$d_0(f, g)=\sup\{\rho(f(x), g(x))|x\in M\}.$$

$f\in\mathrm{End}^1(M)$ satisfies *weak Axiom A* if there exist a continuous splitting $TM|\Omega(f)=E^s\oplus E^u$, and a Riemannian norm $|\ |$ on TM, and constants $C>0, 0<\lambda<1$ such that

(a) $(Tf)E^s \subset E^s$, $(Tf)E^u = E^u$;
(b) $|(Tf)^n v| \leq C\lambda^n |v|$ for $x \in \Omega(f)$, $v \in E^s_x$, $n>0$,
$|(Tf)^n v| \geq C\lambda^{-n} |v|$ for $x \in \Omega(f)$, $v \in E^u_x$, $n>0$;
(c) if $x_1, x_2 \in \Omega(f)$, $x_1 \neq x_2$ and $f(x_1)=f(x_2)=y$, then $E^s_y=\{0\}$;
(d) the periodic points of f are dense in $\Omega(f)$.

For a C^1 endomorphism f of M with at most finitely many singularities, absolute Ω-stability of f is characterized by the following property:

(*) f has a neighborhood $\mathscr{U}$ in $\mathrm{End}^1(M)$ such that every $g \in \mathscr{U}$ satisfies weak Axiom A.

By using Theorem A instead of Theorem 1 we can improve this characterization from endomorphisms with finitely many singularities to endomorphisms with finitely many singularities in the nonwandering sets.

REMARK. The requirement that f has finitely many singularities in the nonwandering set $\Omega(f)$ is necessary for "absolute Ω-stability $\Rightarrow$ (*)" [2]. "(*) $\Rightarrow$ absolute Ω-stability" holds for arbitrary C^1 endomorphism f[1].

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