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The Cauchy problem in abstract Gevrey spaces for a nonlinear weakly hyperbolic equation of second order

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§ 1. Introduction

We investigate here the existence of local solutions to the following abstract Cauchy problem:

$$u''+A(t)u=f(t, u(t)) \tag{1}$$
$$u(0)=u_0, \ u'(0)=u_1 \tag{2}$$

in a Hilbert space H, where $A(t)$ is a nonnegative unbounded operator.

In case $A(t)$ satisfies some strict coercivity assumptions, i. e. when (1) is a strictly hyperbolic equation, the local solvability for Pb. (1), (2) is well known, provided $A(t)$ is Lipschitz continuous in time and f is smooth enough. An extensive theory on this problem, embracing most of the concrete results in Sobolev spaces with optimal regularity assumptions, was given by Kato (see [Ka]; see also [LM]).

On the other hand, when $A(t)\geq 0$ is allowed to be degenerate, i.e. when Eq. (1) is of weakly hyperbolic type, then we need much stronger assumptions in order that (1), (2) be locally solvable. This is evident also for linear equations such as

$$u_{tt}=a(t)u_{xx} \tag{3}$$

which may be not locally solvable in C^{∞} for a suitable nonnegative $a(t)\in C^{\infty}$ (see [CS]).

It is possible to overcome this difficulty by requiring that the data and the coefficients are more regular in space variables. Thus in [CJS], [N] it was proved that the equations

$$u_{tt}=\sum_{i,j}a_{ij}(t, x)u_{x_ix_j}+\sum_{j}b_j(t, x)u_{x_j}, \ \sum a_{ij}\xi_i\xi_j\geq 0 \tag{4}$$

are globally sovable in the spaces $\gamma^s(\mathbf{R}^n)$ of Gevrey functions of order s, defined as follows:

$$v(x)\in\gamma^s(\mathbf{R}^n)\Longleftrightarrow\forall K\subset\subset\mathbf{R}^n\ \exists C_K,\ \Lambda_K\geq 0: \ |D^{\alpha}v(x)|\leq C_K\Lambda_K^{|\alpha|}\cdot|\alpha|!^s \quad \text{for } x\in K \tag{5}$$

(see also [OT] for the case of hyperbolic equations of higher order). More precisely, there is an interesting connection among the regularity in space and time: namely, if the coefficients a_{ij} are Hölder continuous in time with exponent λ, and Gevrey of order s in x, then (4) is uniquely solvable in $\gamma^s(\mathbf{R}^n)$ provided

$$s<1+\lambda/2 \tag{6}$$

and locally solvable if there is equality in (6). This holds up to $\lambda=2$, where we mean, for $1<\lambda\leq 2$, that the coefficients are C^1 with first derivative Hölder continuous with exponent $\lambda-1$. A similar relation, involving the multiplicity of characteristics, holds in the case of higher order equations and systems. The above mentioned result for Eq. (4) was extended to the abstract framework in [D].

These remarks inspire the natural conjecture that an equation like

$$u_{tt}=\sum a_{ij}(t, x)u_{x_i x_j}+f(u, u_x) \tag{7}$$

may be locally solvable in Gevrey classes, provided the function f has suitable smoothness properties. Indeed, as far as 1967, Leray and Ohya [LO] (see also [Br], [S]) proved that the Cauchy problem for a general semilinear system is well posed in Gevrey classes, provided the system is (weakly) hyperbolic with smooth characteristic roots. This assumption of smoothness was removed by Kajitani [K1], who further improved the result by showing that it is sufficient to assume Hölder continuity in time of the coefficients, provided (6) holds (see [K2]).

We should also mention that the case $s=1$ is trivial, since it can be regarded as an application of the theorem of Cauchy and Kowalewski in the nonlinear version (see [O], [Y]), and in this case the hyperbolicity assumption is superfluous.

The aim of the present paper is to propose an extension of Kajitani's result to the abstract setting of Gevrey classes, at least for a second order equation like (1) (including (7) as a concrete example). For such an equation, the weak hyperbolicity can be easily expressed in abstract form, and the method of energy is applicable.

In Section 2 we recall the definition of abstract Gevrey classes, and state our main result (Theorem 1), which is proved in Section 3. The last section is devoted to the applications: we prove in particular that the equation

$$u_{tt}=\sum_{ij}a_{ij}(t, x)u_{x_i x_j}+f(t, x, u, \nabla u) \tag{8}$$

is locally solvable in $\gamma^s(\mathbf{R}^n)$ for $s \leq 1+\lambda/2$, $s<2$, provided the a_{ij} are Hölder functions of exponent λ in t and Gevrey of order s in x; as to the nonlinear term $f(t, x, r, p)$, we assume that it is L^1_{loc} in t, γ^s in x, and is a Gevrey function of (r, p) of some order $s'<s$. We remark that in [K2] the coefficients are assumed to be real analytic functions of u.

§ 2. Notations and statement of the Theorem

We recall the main definitions and properties of scales of abstract Gevrey spaces generated by an n-tuple of operators (see [C]; see also [B], [DT]).

Let H be a Hilbert space with norm $|\cdot|$ and product $(\cdot, \cdot)$, and let $\mathbf{B}=(B_1, \cdots, B_n)$ an n-tuple of linear closed commuting operators on H. We define

$$V_j \equiv \{v \in H : \mathbf{B}^\alpha v \in H \ \forall |\alpha|=j\} \tag{9}$$

where we have used the notation $\mathbf{B}^\alpha = B_1^{\alpha_1} \circ \ldots \circ B_n^{\alpha_n}$; we shall also use the notation

$$V_\infty = \bigcap_{j \geq 0} V_j.$$

Moreover, for $v \in V_j$, we can define

$$|v|_j \equiv \Big(\sum_{|\alpha|=j} |\mathbf{B}^\alpha v|^2\Big)^{1/2}. \tag{10}$$

Thus in particular $|\cdot|_0 \equiv |\cdot|$. If we endow $V \equiv V_1$ with the norm $\|v\|_V = |v| + |v|_1$, and identify H with its dual space, we obtain the Hilbert triple $V \subseteq H \subseteq V'$ in the sense of [LM]. Note that the spaces V_j have a natural Hilbert structure, but we shall not use it in the following.

We shall make a further assumption on H, wich will allow us to implement a Faedo-Galerkin approximating scheme for Pb. (1), (2):

$$H \text{ has a countable basis made of common eigenvectors of } B_1 \ldots, B_n; \tag{11}$$

moreover, we shall assume that, for some integer $k_0 \geq 0$,

$$\text{the embedding } V_{k_0} \subseteq H \text{ is compact.} \tag{12}$$

Then, the abstract Gevrey spaces of order $s \geq 0$ generated by $\mathbf{B}$ are the Banach spaces

$$X_r^s = \{v \in V_\infty : \|v\|_{r,s} < +\infty\}$$

with the norms

$$\|v\|_{r,s} \equiv \sup_{j\geq 0} |v|_j \cdot j!^{-s} r^j.$$

We shall call $v \in X_r^s$ a Gevrey vector of order s.

For fixed s, it is easy to see that $\{X_r^s\}_{r>0}$ forms a Banach scale, with norms increasing with r. A special role will be played by the Fréchet space

$$X_{0+}^s = \bigcup_{r>0} X_r^s = \lim_{r\to 0+} \operatorname{ind} X_r^s. \tag{13}$$

We can now precise the assumptions on Eq. (1). In the following, we shall tacitly assume that all the H-valued functions appearing are H-measurable; moreover, we remark that a H-measurable function $u(t)$ whose X_r^s norm is in $L^1_{\mathrm{loc}}(0, T)$ is also X_r^s-measurable (see e.g. [AS]).

The operator $A(t)$ satisfies, for $0<\lambda\leq 2$, $T>0$,

$$A(t) \in C^\lambda([0, T]; \mathscr{L}(V, V')) \tag{14}$$

which for $0<\lambda\leq 1$ means Hölder continuity of exponent λ, and for $1<\lambda\leq 2$ means Hölder continuity of exponent $\lambda-1$ of the first time derivative. Moreover, we assume that (1) is weakly hyperbolic, i.e. for all v, $w\in V$

$$\langle Av, v\rangle \geq 0 \tag{15}$$

$$\langle Av, w\rangle = \overline{\langle Aw, v\rangle}. \tag{16}$$

The following assumption ensures that the operators $A(t)$ have the right order with respect to the scale X_r^s, that is, order 2: there exist constants C_0, $\Lambda\geq 0$ such that, for all $v\in X_{0+}^s$ $t\in[0, T]$,

$$|A(t)v|_j \leq C_0 (j+2)!^s \Lambda^{j+2} \sum_{h=0}^{j+2} \frac{|v|_h}{h!^s \Lambda^h}, \quad j\geq 0. \tag{17}$$

In concrete cases, (17) is satisfied by second order linear operators with coefficients Gevrey of order s in x (see Section 4).

The final assumption on $A(t)$ is an estimate of the commutators $[A(t), \mathbf{B}^\alpha] \equiv A(t)\mathbf{B}^\alpha - \mathbf{B}^\alpha A(t)$ (see [AS], [D]): for all $v\in X_{0+}^s$, $t\in[0, T]$, $j\geq 0$

$$\left(\sum_{|\alpha|=j} |[A(t), \mathbf{B}^\alpha]v|^2\right)^{1/2} \leq C_0(j+2) \left(\sum_{|\alpha|=j} \langle A\mathbf{B}^\alpha v, \mathbf{B}^\alpha v\rangle\right)^{1/2}$$
$$+\Lambda^{j+2}(j+2)!^s \sum_{h=0}^{j} \frac{|v|_h \Lambda^{-h}}{h!^s (h+1)^{s-1} (h+2)^{s-1}}. \tag{18}$$

As we shall see in Section 4, ass. (18) is satisfied by second order operators with Gevrey coefficients and nonnegative characteristic form.

It remains to precise the hypotheses on the nonlinear term. We shall assume that $f:[0, T]\times X_{0+}^{s}\rightarrow V_{\infty}$; moreover, there exist a function $\chi(t)\in L^{1}(0, T)$, an integer constant $k\geq 0$ and, for all bounded subset K of X_{r}^{s} for some $r>0$, a constant $M_{K}\geq 0$ such that, for all $v\in K$, $j\geq 0$, we have

$$|f(t, v)|_{j}\leq\chi(t)j!\sum_{0\leq\nu\leq\mu\leq j}\Lambda^{j-\mu}M_{K}^{\nu}(j-\mu)!^{s-1}\nu!^{s'-1} \sum_{\substack{h_{1}+\ldots+h_{\nu}=\mu\\ h_{\nu}\geq h_{i}\geq 1}}\frac{|v|_{h_{1}+k}\cdots|v|_{h_{\nu-1}+k}\cdot|v|_{k_{\nu+1}}}{h_{1}!\ldots h_{\nu}!} \tag{19}$$

where the parameter s' is strictly less than s, while Λ is the same as in ass. (17), (18). To make formula (19) more clear, we notice that when $\nu=0$ or $\mu=0$ the inner sum is not present, i.e. the set $\{h_{1}, \ldots, h_{\nu}\}$ is empty. Moreover, we shall assume that $f(t, \cdot)$ has a Fréchet derivative as a map from X_{0+}^{s} to H, with values in V_{∞}, and that, for all bounded subset K of X_{r}^{s} for some $r>0$, there exists a constant M_{K} such that, for all $w\in K$, $v\in X_{0+}^{s}$

$$|Df(t, w)v|_{j}\leq\chi(t)(j+1)!^{s}\ M_{K}^{j+1}\sum_{h=0}^{j}\frac{|v|_{h}}{h!^{s}M_{K}^{h}},\ j\geq 0. \tag{20}$$

We can now state our result:

THEOREM 1. *Assume that* (11), (12) *hold, and that* $A(t)$, $f(t, u)$ *satisfy* (14)-(20). *Then, for all* u_{0}, $u_{1}\in X_{0+}^{s}$, *there exists* $T_{0}>0$ *such that Pb.* (1), (2) *has a unique solution in*

$$C^{1}([0, T_{0}]; X_{0+}^{s}), \tag{21}$$

provided

$$1\leq s'<s\leq 1+\frac{\lambda}{2},\ s<2. \tag{22}$$

Moreover $u''\in L^{1}(0, T_{0}; X_{r})$ *for some* $r>0$.

REMARK. By the same method, it is possible to handle a nonlinear term of the form $f(t, x, u, u')$, depending also on the first time derivative of u, obtaining a similar result.

§ 3. Proof of Theorem 1

The proof is based on the method of infinite order energy, introduced in [CDS] in the study of weakly hyperbolic equations in the class of analytic functions. The method was extended in [AS] to the abstract framework, and in [D] and [S] to the setting of Gevrey functions.

In order to make the proof more clear, we shall perform it in the particular case when the n-tuple $\mathbf{B}$ is made of one single operator B; only minor modifications are necessary in the general case (see Remark 2). Moreover, we shall consider in detail only the case $0<\lambda\leq 1$, and list in Remark 1 the changes for the case $1\leq\lambda\leq 2$.

We divide the proof into several steps.

STEP 1: Apriori estmate

Let $\phi(t)$ be a Friedrichs mollifier in $C_0^\infty(\mathbf{R})$, with $\int\phi=1$, and let

$$\phi_j(t)=j^{-1}\phi(t/j). \tag{23}$$

Then, extending $A(t)$ as $A(T)$ for $t\geq T$, $A(0)$ for $t\leq 0$, we can define the convolutions (in the norm of H)

$$A_j(t)=A*\phi_j(t). \tag{24}$$

It is easy to prove, using the λ-Hölder continuity of $A(t)$ (see (14)), that

$$\|A-A_j\|_{L^\infty(0,T;\,\mathscr{L}(V,V'))}\leq Lj^{-\lambda} \tag{25}$$
$$\|A_j'\|_{L^\infty(0,T;\,\mathscr{L}(V,V'))}\leq 2Lj^{1-\lambda} \tag{26}$$

for a suitable constant $L\geq 0$.

Consider now the following Cauchy problem

$$u''+A(t)u+M(t)u=f(t,u(t)) \tag{27}$$
$$u(0)=u_0,\quad u'(0)=u_1,\quad u_0,u_1\in X_{0+}^s \tag{28}$$

which differs from (1), (2) for the presence of a first order term $M(t)u$. The reason for the introduction of $M(t)$ will be clear when dealing with the uniqueness (see Step 3). Besides the assumptions of Thm. 1, we shall assume that

$$M\in L^1(0,T;\,\mathscr{L}(V,H)) \tag{29}$$

and that, for some $\mu(t)\in L^1(0,T)$,

$$|M(t)v|_j\leq C_0(j+1)!^s\Lambda_1^{j+1}\sum_{h=0}^{j+1}\frac{|v|_h}{h!^s\Lambda_1^h},\ j\geq 0 \tag{30}$$

for all $v\in X_{0+}^s$, and some constant $\Lambda_1\geq 0$.

Now let $u(t)$ be a solution to (27), (28). We define (formally) the energy of order j of $u(t)$ as follows:

$$E_j(t)=|u'|_{j-1}^2+\langle A_jB^{j-1}u,\ B^{j-1}u\rangle+j^2|u|_{j-1}^2+j^{-\lambda}|u|_j^2. \tag{31}$$

Since by (27) we can write, applying B^{j-1} to both sides,

$$B^{j-1}u''+A_jB^{j-1}u=(A_j-A)B^{j-1}u+[A, B^{j-1}]u-B^{j-1}Mu+B^{j-1}f, \tag{32}$$

we have, differentiating (31) and using (32),

$$E_j'(t)=\langle A_j'B^{j-1}u, B^{j-1}u\rangle+2\mathrm{Re}\langle j^2B^{j-1}+(Aj-A)B^{j-1}u+[A, B^{j-1}]u, B^{j-1}u'\rangle$$
$$+2\mathrm{Re}j^{-\lambda}\langle B^ju, B^ju'\rangle+2\mathrm{Re}\langle -B^{j-1}Mu+B^{j-1}f, B^{j-1}u'\rangle. \tag{33}$$

We shall now use the inequalities (consequences of def. (31))

$$|u|_{j-1}\leq j^{-1}\sqrt{E_j},\ |u|_j\leq j^{\lambda/2}\sqrt{E_j},$$
$$|u'|_{j-1}\leq\sqrt{E_j},\ |u'|_j\leq\sqrt{E_{j+1}}\ ; \tag{34}$$

by (33), using (34), (30), (25), (26) and dividing by $2\sqrt{E_j}$, we obtain after some passages

$$\sqrt{E_j}'\leq c_1(L)\left[j^{-\lambda/2}\sqrt{E_{j+1}}+j\sqrt{E_j}+\mu(t)j!^sj^{\lambda/2}\sum_{h=1}^{j}\frac{\sqrt{E_h}\,\Lambda_1^{j-h}}{h!^s}\right]$$
$$+|[A, B^{j-1}]u|+|f|_{j-1}.$$

We recall now ass. (18), which implies

$$|[A, B^{j-1}]u|\leq c_0j[\langle(A-A_j+A_j)B^{j-1}u, B^{j-1}u\rangle]^{1/2}$$
$$+c_0\Lambda^{j+1}(j+1)!^s\sum_{h=0}^{j-1}\frac{\sqrt{E_{h+1}}\,\Lambda^{-h}}{(h+1)!^s(h+2)^\sigma}$$

($\sigma=s-1$) and, after some easy passages, using again (34), (25), we find

$$\sqrt{E_j}'\leq c_2\left[j^{-\lambda/2}\sqrt{E_{j+1}}+(\mu+1)(j+1)!^s\sum_{h=1}^{j}\frac{\sqrt{E_h}\,\Lambda_2^{j-h}}{h!^s(h+1)^\sigma}\right]+|f|_{j-1} \tag{35}$$

where $c_2=c_2(L, \Lambda, \Lambda_1, c_0)$ while (see (30))

$$\Lambda_2=\max\{\Lambda, \Lambda_1\}. \tag{36}$$

We can now define (formally) the infinite order energy $\mathscr{E}(t)$ associated to $u(t)$:

$$\mathscr{E}(t)\equiv\sum_{j\geq 1}\frac{\rho^{j-k}}{j!^s}j^{ks}\sqrt{E_j} \tag{37}$$

where $\rho(t)$ is an absolutely continuous functions, which will be chosen in the following; the essential property of $\rho(t)$ will be

$$0<r_0/2\leq\rho(t)\leq r_0<1/\Lambda_2 \text{ on } [0, T^*] \tag{38}$$

for some $T^*\in[0, T]$ to be precised, where $r_0>0$ is such that u_0, $u_1\in X^s_{r_0+\varepsilon}$ for some $\varepsilon>0$; of course r_0 can be taken arbitrarily small. The integer k appearing in the definition of $\mathscr{E}(t)$ is the same as in ass. (19).

Differentiating (37) termwise we have (formally)

$$\mathscr{E}'(t)=\sum_{j\geq 1}\sqrt{E_j}'+\sum_{j\geq 1}\frac{\rho^{j-k-1}}{j!^s}(j-k)j^{ks}\sqrt{E_j}\cdot\rho(t)'. \tag{39}$$

In order to estimate $\mathscr{E}'(t)$, we shall introduce (36) into (39). We obtain several terms, of which only the following one deserves special attention:

$$\sum_{j\geq 1}\rho^{j-k}(j+1)^s j^{ks}\sum_{h=1}^{j}\frac{\sqrt{E_h}\,\Lambda_2^{j-h}}{h!^s(h+1)^\sigma}=\sum_{h\geq 1}\frac{\sqrt{E_h}\,\Lambda_2^{-h}}{h!^s(h+1)^\sigma}\rho^{-k}\sum_{j\geq h}(\rho\Lambda_2)^j j^{ks}(j+1)^s \tag{40}$$

and since

$$\sum_{j\geq h}(\rho\Lambda_2)^j(j+1)^{(k+1)s}\leq c(r_0,\Lambda_2,k,s)(\rho\Lambda_2)^h h^{(k+1)s} \tag{41}$$

where we have used (38), we have

$$\sum_{j\geq 1}\rho^{j-k}(j+1)^s j^{ks}\sum_{h=1}^{j}\frac{\sqrt{E_h}\,\Lambda_2^{j-h}}{h!^s(h+1)^\sigma}\leq c(r_0,\Lambda_2,k,s)\sum_{j\geq 1}\frac{\rho^{j-k}}{j!^s}j^{ks}\cdot j\sqrt{E_j}. \tag{42}$$

Hence by (39), (35) and (42) we obtain

$$\begin{aligned}\mathscr{E}'(t)\leq c_3\sum_{j\geq 1}\frac{\rho^{j-k}}{j!^s}j^{ks}\cdot j^{-\lambda/2}\sqrt{E_{j+1}}+c_3(\mu(t)+1)\sum_{j\geq 1}\frac{\rho^{j-k}}{j!^s}j^{ks}\cdot j\sqrt{E_j}\\ +\sum_{j\geq 1}\frac{\rho^{j-k-1}}{j!^s}(j-k)j^{ks}\rho'\sqrt{E_j}+\mathscr{E}(f)\end{aligned} \tag{43}$$

where we have introduced the notation

$$\mathscr{E}(f)=\sum_{j\geq 1}\frac{\rho^{j-k}}{j!^s}j^{ks}|f|_j. \tag{44}$$

Rearranging the terms in (43) we finally obtain

$$\mathscr{E}'(t)\leq\sum_{j\geq 1}\frac{\rho^{j-k-1}}{(j-1)!^s}j^{ks-\sigma}\sqrt{E_j}\left\{\frac{j-k}{j}\rho'+\mu_1(t)\rho+c_4 j^{\sigma-\lambda/2}\right\}+\mathscr{E}(f) \tag{45}$$

where $c_4=c_4(L,\Lambda_2,s,k,c_0)$ and $\mu_1(t)=c_4(\mu(t)+1)$.

It remains now to estimate the nonlinear term $\mathscr{E}(f)$. Recalling (44) and (19), we have

$$\begin{aligned}\mathscr{E}(f)\leq\chi(t)\sum_{j\geq 1}\frac{\rho^{j-k}}{j!^{s-1}}j^{ks}\sum_{0\leq\nu\leq\mu\leq j}\Lambda^{j-\mu}M_K^\nu(j-\mu)!^{s-1}\nu!^{s'-1}\\ \sum_{\substack{h_1+\ldots+h_\nu=\mu\\ h_\nu\geq h_i\geq 1}}\frac{|u|_{h_1+k}\ldots|u|_{h_{\nu-1}+k}\cdot|u|_{h_\nu+1}}{h_1!\ldots h_\nu!}\end{aligned} \tag{46}$$

where M_K is the constant in (19) associated to the bounded set $K=\{u\}$

(consisting of one single function). Using the notation

$$\eta(j)=\frac{\rho^{j-k}}{j!^{s}}j^{ks}\sqrt{E_{j}}, \tag{47}$$

so that $\mathscr{E}=\sum_{j\geq 1}\eta(j)$, we easily see that

$$\begin{aligned}&\frac{|u|_{h_i+k}}{h_i!}\leq\eta(h_i+k)\frac{h_i!^{s-1}}{h_i+k}\rho^{-h_i},\\ &\frac{|u|_{h_\nu+1}}{h_\nu!}\leq\eta(h_\nu+1)\frac{h_\nu!^{s-1}}{(h_\nu+1)^{(k-1)s}}\rho^{-h_\nu+k-1},\end{aligned} \tag{48}$$

and hence, isolating the terms with $\nu=0$ from the others in (46), we get

$$\mathscr{E}(f)\leq\mathscr{E}_1(f)+\mathscr{E}_2(f) \tag{49}$$

where

$$\mathscr{E}_1(f)\equiv\chi(t)\sum_{j\geq 1}\frac{\rho^{j-k}}{j!^{s-1}}j^{ks}\sum_{0\leq\mu\leq j}\Lambda^{j-\mu}(j-\mu)!^{s-1} \tag{50}$$

and

$$\begin{aligned}\mathscr{E}_2(f)\equiv\chi(t)\sum_{j\geq 1}\sum_{1\leq\nu\leq\mu\leq j}&\frac{\rho^{j-k}}{j!^{s-1}}j^{ks}\Lambda^{j-\mu}M_K^{\nu}(j-\mu)!^{s-1}\nu!^{s'-1}\\ &\sum_{\substack{h_1+\ldots+h_\nu=\mu\\ h_\nu\geq h_i\geq 1}}\eta(h_1+k)\ldots\eta(h_{\nu-1}+k)\eta(h_\nu+1)\\ &\frac{h_1!^{s-1}\ldots h_\nu!^{s-1}}{(h_1+k)\ldots(h_{\nu-1}+k)(h_\nu+1)^{ks}}(h_\nu+1)^{s-1}\rho^{\mu-j+k-1}.\end{aligned} \tag{51}$$

We have easily (since $(j-\mu)!/j!\leq 1/\mu!$)

$$\mathscr{E}_1(f)\leq\chi(t)\sum_{j\geq 1}\rho^{j-k}j^{ks}\sum_{0\leq\mu\leq j}\frac{\Lambda^{j-\mu}}{\mu!^{s-1}}=\chi(t)\rho^{-k}\sum_{\mu\geq 0}\frac{\Lambda^{j-\mu}}{\mu!^{s-1}}\sum_{j\geq\mu}(\rho\Lambda)^{j}j^{ks} \tag{52}$$

and with the same argument as in (41), (42) (see (38))

$$\mathscr{E}_1(f)\leq\chi(t)\sum_{\mu\geq 0}\frac{\rho^{\mu-k}}{\mu!^{s-1}}\mu^{ks}\cdot c(r_0,\Lambda,k,s)\equiv\psi(t). \tag{53}$$

Note that the series in (53) converges no matter the value of $\rho(t)$, hence the function $\psi(t)\in L^1(0,T)$ is well defined, and will depend on our choice of $\rho(t)$.

As to $\mathscr{E}_2(f)$, we shall need the easy inequality

$$\frac{(h_1+\ldots+h_\nu)!}{h_1!\ldots h_\nu!}\geq\nu!\quad\text{if } h_1,\ldots,h_\nu\geq 1 \tag{54}$$

which implies

$$\left(\frac{(j-\mu)!}{j!}h_1!\dots h_\nu!\right)^{s-1}=\binom{j}{j-\mu}^{1-s}\binom{\mu}{h_1,\dots,h_\nu}^{1-s}\le\binom{j}{j-\mu}^{1-s}\nu!^s\le\nu!^s. \tag{55}$$

By (51), (55), we have

$$\mathscr{E}_2(f)\le\chi(t)\sum_{j\ge1}\sum_{1\le\nu\le\mu\le j}j^{ks}\rho^{j-\mu-1}\Lambda^{j-\mu}M_K^\nu\nu!^{s'-s}(h_\nu+1)^{s-1}$$
$$\sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_\nu\ge h_i\ge1}}\left(\frac{j}{h_\nu+1}\right)^{ks}\eta(h_1+k)\dots\eta(h_{\nu-1}+k)\eta(h_\nu+1). \tag{56}$$

The inequalities $h_\nu\ge h_i\ge1$, $\nu\ge1$ imply

$$\mu=\sum h_i\le\nu h_\nu\ \Rightarrow\ \left(\frac{j}{h_\nu+1}\right)^{ks}\le\left(\frac{j}{\mu+1}\right)^{ks}\nu^{ks},$$

and since $\binom{j}{j-\mu}^{1-s}\le1$, we obtain, after a suitable rearrangment of the terms in (56),

$$\mathscr{E}_2(f)\le\chi(t)\sum_{\nu\ge1}\nu^{ks}\nu!^{s'-s}M_K^\nu\sum_{h_\nu\ge1}\eta(h_\nu+1)(h_\nu+1)^{s-1}$$
$$\sum_{\mu\ge\nu,\,h_\nu}\sum_{j\ge\mu}\rho^{j-\mu-1}\Lambda^{j-\mu}\left(\frac{j}{\mu+1}\right)\sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_\nu\ge h_i\ge1}}\eta(h_1+k)\dots\eta(h_{\nu-1}+k). \tag{57}$$

Now we observe that, for fixed ν, $h_\nu\ge1$ $(j\to j+\mu)$

$$\sum_{\mu\ge\nu,\,h_\nu}\sum_{j\ge\mu}\rho^{j-\mu-1}\Lambda^{j-\mu}\left(\frac{j}{\mu+1}\right)\sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_\nu\ge h_i\ge1}}\eta(h_1+k)\dots\eta(h_{\nu-1}+k)\le$$
$$\sum_{j\ge0}\sum_{\mu\ge\nu,\,h_\nu}\rho^{j-1}\Lambda^j\left(\frac{j+\mu}{\mu+1}\right)^{ks}\sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_\nu\ge h_i\ge1}}\eta(h_1+k)\dots\eta(h_{\nu-1}+k) \tag{58}$$

and, dividing the terms of the last sum in two groups, we have, when $\mu\le[j/2]$,

$$\sum_{j\ge0}\sum_{[j/2]\ge\mu\ge\nu,\,h_\nu}\rho^{j-1}\Lambda^j\left(\frac{j+\mu}{\mu+1}\right)^{ks}\sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_\nu\ge h_i\ge1}}\eta(h_1+k)\dots\eta(h_{\nu-1}+k)\le$$
$$\le3^{ks}\sum_{j\ge0}\rho^{j-1}\Lambda^jj^{ks}\mathscr{E}^{\nu-1}\le(1-r_0\Lambda)^{-[ks]-1}\rho^{-1}\mathscr{E}(t)^{\nu-1} \tag{59}$$

(recall (38)), while, for the terms with $\mu>[j/2]$,

$$\sum_{j\ge0}\sum_{\mu\ge\nu,\,h_\nu,\,[j/2]}\rho^{j-1}\Lambda^j\left(\frac{j+\mu}{\mu+1}\right)^{ks}\sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_\nu\ge h_i\ge1}}\eta(h_1+k)\dots\eta(h_{\nu-1}+k)\le$$
$$\le4\sum_{j\ge0}\rho^{j-1}\Lambda^j\mathscr{E}^{\nu-1}\le(1-r_0\Lambda)^{-1}\rho^{-1}\mathscr{E}(t)^{\nu-1}. \tag{60}$$

Hence, by (57)-(60)

$$\mathscr{E}_2(f)\leq c_5(r_0, \Lambda, k, s)\chi(t)\sum_{\nu\geq 1}\nu^{ks}\nu!^{s'-s}\, M_K^\nu \mathscr{E}^{\nu-1}\sum_{h_\nu\geq 1}\eta(h_\nu+1)(h_\nu+1)^{s-1}\rho^{-1}. \tag{61}$$

We now define the function

$$\Psi_K(r)=\sum_{\nu\geq 1}\nu^{ks}\nu!^{s'-s}\, M_K^\nu r^{\nu-1} \tag{62}$$

where the series converges for all values of r, since $s'<s$; then we can write

$$\mathscr{E}_2(f)\leq\chi_1(t)\Psi_K(\mathscr{E}(t))\sum_{j\geq 1}j^{s-1}\eta(j)\rho^{-1} \tag{63}$$

where

$$\chi_1(t)=c_5(r_0, \Lambda, k, s)\chi(t). \tag{64}$$

Finally, from (49), (53), (63), we conclude that

$$\mathscr{E}(f)\leq\psi(t)+\chi_1(t)\Psi_K(\mathscr{E}(t))\sum_{j\geq 1}j^{s-1}\eta(j)\rho^{-1}. \tag{65}$$

We can now come back to estimate (45). We have, using (65),

$$\mathscr{E}'(t)\leq\sum_{j\geq 1}\frac{\rho^{j-k-1}}{(j-1)!^s}j^{ks-\sigma}\sqrt{E_j}\left\{\frac{j-k}{j}\rho'+\mu_1(t)\rho+c_4 j^{\sigma-\lambda/2}\right.$$
$$\left.+\chi_1(t)\Psi_K(\mathscr{E}(t))j^{s-2}\right\}+\psi(t) \tag{66}$$

where we have used the identity

$$j^{s-1}\eta(j)\rho^{-1}\equiv\frac{\rho^{j-k-1}}{(j-1)!^s}j^{ks-\sigma}\sqrt{E_j}\cdot j^{s-2}. \tag{67}$$

We shall now use the formal estimate (66) in order to obtain an effective a priori estimate for solutions of Pb. (27), (28). First of all, we can choose the function $\rho(t)$: it will be defined as the solution to the ODE

$$\frac{1}{2}\rho'(t)+\mu_1(t)\rho+2c_4+\chi_1(t)=0 \tag{68}$$

$$\rho(0)=r_0; \tag{69}$$

recall that $\mu_1(t)=c_4(\mu+1)$, $\chi_1(t)$ is defined in (64) and $r_0<1/\Lambda_2$ is such that u_0, $u_1\in X^s_{r_0+\varepsilon}$ for some $\varepsilon>0$. Thus $\rho(t)$ depends only on the coefficients and the data of the problem. The function $\rho(t)$ is absolutely continuous, nonincreasing, and satisfies an inequality of the form

$$0<\frac{r_0}{2}\leq\rho(t)\leq r_0 \quad \text{for } t\in[0, T^*] \tag{70}$$

for some $T^*>0$, so that (38) is fulfilled on $[0, T^*]$. It is clear that also T^* depends only on the coefficients and the data of the problem.

We are now ready to prove the

LEMMA 1. (a priori estimate). *Let* $u\in C^1([0, T]; X^s_{1/\Lambda_2})$ *be a solution to Pb.(27), (28). Then we can find a time* $\bar{T}>0$ *and a constant* $\bar{C}>0$, *depending only on the coefficients of Eq.* (27) *and on* r_0, *such that*

$$\mathscr{E}(t)\leq\bar{C} \text{ for } t\in[0, \bar{T}]. \tag{71}$$

PROOF. Since $0<\rho<1/\Lambda_2$ on $[0, T^*]$, the energy $\xi(t)$ and all the series used in the computations leading to (66) converge for the solution $u(t)$ under consideration, hence (66) holds.

Assume now that

$$\mathscr{E}(t)\leq C \quad \text{on} \quad [0, T^*] \tag{72}$$

for some constant C. We remark that an inequality like (72) can be used to define a bounded subset of X^s_r as follows. For any vector $v\in X^s_{0+}$, we can consider an energy as in (32) with $u(t)\equiv v$ (and of course $u'\equiv 0$), and define accordingly the infinite order energy $\xi_v(t)$ as in (37). Then, the set K_C of the elements of X^s_{0+} such that (72) holds will be a bounded subset of $X^s_{r_0/2}$, since $\rho\geq r_0/2$, and will be increasing as C increases. Let $M(C)=M_{K_C}$ be the constant given by ass. (19) in correspondence with the set K_C, clearly an increasing function of C, and let $\Psi(C, r)\equiv\Psi_{K_C}(r)$ the corresponding function defined in (62); as it is evident from (62) and the preceding arguments, $\Psi(C, r)$ is increasing in each variable.

Consider now estimate (66). The quantity between braces can be estimated by the following one, using (72) and ass. (22):

$$\frac{j-k}{j}\rho'+\mu_1(t)\rho+c_4+\chi_1(t)\Psi(C, C)j^{s-2}. \tag{73}$$

Since $s<2$, the last term in (73) converges to 0. Recalling (68), we see that the expression (73) is negative as soon as

$$j\geq j_0(C)\geq 2k \tag{74}$$

where $j_0(C)$ is a suitable function, also increasing in C. Hence we can drop the terms for $j\geq j_0$ in (66). As to the remaining terms, we can estimate them as follows:

$$\sum_{j=1}^{j_0(C)} \frac{\rho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \sqrt{E_j} \left\{ \frac{j-k}{j} \rho' + \mu_1(t)\rho + c_4 j^{\sigma-\lambda/2} + \chi_1(t)\Psi(C, C) j^{s-2} \right\}$$
$$\leq c_6(C)\,\mathscr{E}(t) \tag{75}$$

where c_6 depends on r_0, the coefficients of Eq. (28), and of course can be assumed to be increasing in C.

In conclusion we have

$$\mathscr{E}'(t) \leq \psi(t) + c_6(C)\,\mathscr{E}(t) \tag{76}$$

which implies

$$\mathscr{E}(t) \leq e^{c_6(C)t}\left(\mathscr{E}(0) + \int_0^t \psi(s)ds\right). \tag{77}$$

Note that this estimate holds for all sufficiently large constant C, provided (72) holds.

We can now define the constants $\bar{C}$, $\bar{T}$. Let

$$\bar{C} = 2\left(\mathscr{E}(0) + \int_0^{T^*} \psi(s)ds\right) \tag{78}$$

and let $\bar{T}$ be so small that

$$e^{c_6(\bar{C})\bar{T}} \leq \frac{3}{2}. \tag{79}$$

Then we claim that (71) holds. Indeed, $\mathscr{E}(0) \leq \bar{C}/2$ by (78), and by continuity this implies $\mathscr{E}(t) \leq \bar{C}$ on some interval $[0, \varepsilon]$. We define then

$$T_1 = \sup\{t : \mathscr{E}(\tau) \leq \bar{C} \text{ on } [0, t]\}. \tag{80}$$

It is easy to prove that $T_1 \geq \bar{T}$. Assume by contradiction that $T_1 < \bar{T}$; since

$$\mathscr{E}(t) \leq \bar{C} \text{ on } [0, T_1] \tag{81}$$

we can apply estimate (77) with $C = \bar{C}$ on that interval, and by (79) we have

$$\mathscr{E}(T_1) \leq \frac{3}{2}\left(\mathscr{E}(0) + \int_0^{T_1} \psi(s)ds\right) < \bar{C} \tag{82}$$

and this contradicts the maximality of T_1.

The proof of Lemma 1 is complete.

□

REMARK 1. In the case $\lambda \in [1, 2]$, it is not necessary to regularize the coefficient $A(t)$ in time; we define the energy of order j as follows

$$E_j(t)=|u'|_{j-1}^2+\langle AB^{j-1}u,\ B^{j-1}u\rangle+j^2|u|_{j-1}^2+j^{-\lambda}|u|_j^2.$$

In the course of the computation, the only difference is the estimate of the term $\langle A'B^{j-1}u,\ B^{j-1}u\rangle$ which is obtained by applying to the function $\phi(t)=\langle A(t)v,\ v\rangle$ the following property (a proof of which can be found in [CJS], [J]):

Let $\phi\in C^\lambda([0,\ T])$ be a non-negative function, $1\leq\lambda\leq 2$. Then

$$\|\phi^{1/\lambda}\|_{Lip}\leq c(\lambda)\|\phi\|_{C^\lambda}^{1/\lambda}.$$

REMARK 2. In the general case when $\mathbf{B}$ consists of an n-tuple of operators, $n>1$, we define the energy E_j as follows

$$E_j(t)=|u'|_{j-1}^2+\sum_{|\alpha|=j}\langle A_j\mathbf{B}^\alpha u,\ \mathbf{B}^\alpha u\rangle+j^2|u|_{j-1}^2+j^{-\lambda}|u|_j^2$$

and the proof follows exactly the same lines as above.

STEP 2: Local existence

With the a priori estimate (71), it is not difficult to prove that a local solution to Pb. (1), (2) exists. Ass. (11) implies the existence of a sequence of orthogonal projections P_N, with finite dimensional images H_N, commuting with B and strongly converging to the identity. Defining $A_N(t)=P_NA(t)$, $f_N(t,\ u)=P_Nf(t,\ u)$, it is clear that ass. (14)-(20) hold also for A_N, f_N without any modification. Moreover, the image H_N of P_N is contained in X_r^s for all $r>0$; indeed, $B|_{H_N}\equiv P_NBP_N:\ H_N\rightarrow H_N$ is a bounded operator (owing to finite dimension), hence for $v\in H_N$

$$|v|_j=|B^jv|\leq\|B\|_{\mathscr{L}(H_N,\ H_N)}^j|v|_{H_N}. \tag{83}$$

Thus, choosing $M(t)\equiv 0$ (and hence $\Lambda_2=\Lambda$, see (36)), the assumptions of Lemma 1 are uniformly satisfied by the Cauchy problems

$$v''+A_N(t)u=f_N(t,\ v(t)) \tag{84}$$

$$v(0)=P_Nu_0,\quad v'(0)=P_Nu_1, \tag{85}$$

hence the conclusion of Lemma 1 holds true and the constants $\bar{C}$, $\bar{T}$ do not depend on N.

We remark now that Pb. (84), (85) is locally solvable, since it is finite dimensional; the solution $u_N(t)$ belongs to

$$u_N\in C^1([0,\ T_N];\ X_r^s) \tag{86}$$

for some $T_N>0$, and for all $r>0$ (since $u_N(t)\in H_N$). But it is easy to see that, thanks to estimate (71), $u_N(t)$ can in fact be prolonged beyond $\bar{T}$. Indeed, let $[0,\ T]$ be the maximal interval of definition for u_N, then (71) and Eq. (85) itself imply that

$$u_N,\ u_N'\in L^\infty(0,\ T^*;\ X_r^s),\ u_N''\in L^1(0,\ T^*;\ X_r^s) \tag{87}$$

for some $r>0$ small enough (e.g. $r=r_0/2$), hence $T^*\geq \bar{T}$ by a standard continuation argument. Hence, by the above mentioned property of the spaces H_N(see (83)) it follows that

$$u_N(t)\in C^1([0,\ \bar{T}];\ X_r^s) \tag{88}$$

for all $r>0$, and estimate (71) holds.

A consequence of (71) is that, for each $j\geq 0$, the sequences $B^j u_N$, $B^j u_N'$ are bounded in $L^\infty(0,\ \bar{T};H)$ and hence in $L^2(0,\ \bar{T};\ H)$. Thus, by extracting subsequences through a diagonal procedure, we can assume that, for each j, $B^j u_N \rightharpoonup u^{(j)}$, $B^j u_N' \rightharpoonup v^{(j)}$ in the weak topology of $L^2(0,\ \bar{T};\ H)$. It is clear that $v^{(j)}\equiv\dfrac{d}{dt}u^{(j)}$; moreover, by ass. (12) and the continuous embedding $W^{1,2}\subseteq C^0$, by possibly extracting further subsequences we have that $B^j u_N$ converges uniformly in $C^0([0,\ \bar{T}];\ H)$, hence by the closedness of B we conclude that $u^{(j)}=B^j u$ where $u\equiv u^{(0)}\equiv\lim u_N$. Now, recalling ass. (19), (20), by the uniform convergence $B^j u_N\to B^j u$ we deduce

$$f_N(\cdot,\ u_N(\cdot))\to f(\cdot,\ u(\cdot)) \text{ strongly in } L^1(0,\ \bar{T};\ H). \tag{89}$$

By standard arguments, it is easy to conclude that the limit $u(t)$ is a solution to Pb. (1), (2) such that

$$u(t)\in C^1([0,\ \bar{T}];\ X_{0+}^s)$$

and satisfies estimate (71).

STEP 3: Uniqueness

Let $u,\ v$ be two solutions to Pb. (1), (2) such that

$$u,\ v\in C^1([0,\ \bar{T}];\ X_{0+}^s);$$

hence in particular

$$u,\ v\in C^1([0,\ \bar{T}];\ X_{r_1}^s) \tag{90}$$

for some $r_1>0$. Consider now the identity

$$f(t,\ u(t))-f(t,\ v(t))=\int_0^1 Df(t,\ \tau u(t)+(1-\tau)v(t))d\tau\cdot(u(t)-v(t)) \tag{91}$$

where Df is the Fréchet derivative of $f(t,u)$ with respect to u. If we define

$$M(t)=\int_0^1 Df(t,\tau u(t)+(1-\tau)v(t))d\tau, \tag{92}$$

we see that the function $w=u-v$ satisfies the equation

$$w''+A(t)w+M(t)w=0 \tag{93}$$

with vanishing data. Moreover, since the segment $K=[u,v]$ is a bounded subset of $X^s_{r_1}$, we can apply ass. (20) to (92) and we obtain

$$|M(t)w|_j\leq\chi_2(t)(j+1)!^s\ M_K^{j+1}\sum_{h=0}^{j+1}\frac{|w|_h}{h!^s M_K^h}, \tag{94}$$

for some $\chi_2(t)\in L^1(0,T)$. Hence we can regard (93) as an equation of the form (27), satisfying the assumptions of Lemma 1 with $\Lambda_1=M_K$(see (30)). Then, possibly choosing a smaller value of r_1, we can apply estimate (71) to (93), and we obtain that $u\equiv v$ in a neighbourhood of $t=0$. A standard continuation argument shows that $u\equiv v$ on $[0,T]$.

This concludes the proof of Thm. 1.

$\square$

§4. Applications

As a first application of Thm. 1, we prove a local existence result for the Cauchy problem

$$u_{tt}=\sum_{i,j=1}^n a_{ij}(t,x)u_{x_ix_j}+f(t,x,u,\nabla u) \tag{95}$$

$$u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x) \tag{96}$$

in the space $\gamma^{\{s\}}_{L^2}=\gamma^{\{s\}}_{L^2}(\mathbf{R}^n)$ defined as

$$v(x)\in\gamma^{\{s\}}_{L^2}\Longleftrightarrow \exists C_0,\ \Lambda\geq 0:\ \|D^\alpha v\|_{L^2(\mathbf{R}^n)}\leq C_0\Lambda^{|\alpha|}|\alpha|!^s.$$

The following Prop. 1 is essentially a particular case of the result in [K2]; only the assumptions on f are weaker, since we assume $f(t,x,r,p)$ to be a Gevrey function of r,p of order $s'<s$, instead of real analytic as in [K2].

More precisely, we shall assume that, for all $(t,x,\xi)\in[0,T]\times\mathbf{R}^n\times\mathbf{R}^n$, $\alpha\in\mathbf{N}^n$, $\lambda\in]0,2]$,

$$\sum a_{ij}(t,x)\xi_i\xi_j\geq 0,\ a_{ij}=\overline{a_{ji}} \tag{97}$$
$$|D^\alpha a_{ij}(t,x)|\leq c_0\Lambda^{|\alpha|}|\alpha|!^s, \tag{98}$$

$$a_{ij}(t, x) \text{ is } \lambda\text{-Hölder continuous in } t, \text{ uniformly in } x \tag{99}$$

for some constants c_0, $\Lambda \geq 0$, where (99) has the usual meaning for $\lambda \geq 1$. As to the nonlinear term, we shall assume that $f(t, x, r, p): [0, T] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to C$ satisfies the following estimate: for any $R \geq 0$ we can find a constant $M_R \geq$such that, whenever $|r|+|p| \leq R$, $(t, x) \in [0, T] \times \mathbf{R}^n$,

$$|D_x^\alpha D_r^\nu D_p^\beta f(t, x, r, p)| \leq \chi(t) \Lambda^{|\alpha|} M_{\mathbf{R}}^{\nu+|\beta|} |\alpha|!^s (\nu+|\beta|)!^{s'} \tag{100}$$

for a given $\chi(t) \in L^1(0, T)$.

We have then:

PROPOSITION 1. *Assume* (97)-(100) *hold, and let* u_0, $u_1 \in \gamma_{L^2}^{(s)}$ *such that*

$$\|D^\alpha u_j\|_{L^2(\mathbf{R}^n)} \leq C_0 \Lambda_0^{|\alpha|} |\alpha|!^s \tag{101}$$

with $\Lambda_0 > \Lambda$. *Then there exists* $T > 0$ *such that Pb.* (95), (96) *has a unique solution*

$$u(t, x) \in C^1([0, T]; \gamma_{L^2}^{(s)}), \tag{102}$$

provided

$$1 \leq s' < s \leq 1 + \frac{\lambda}{2}, \quad s < 2. \tag{103}$$

PROOF. In the following proof we shall use the property of finite speed of propagation, which is well known for strictly hyperbolic equations, and holds also for the weakly hyperbolic ones, in the Gevrey classes (for a proof, see the Appendix of [D]).

We divide the proof in two steps.

1) *Compactly supported initial data.* From the finite speed of propagation it easily follows that, if the initial data are compactly supported functions, then we can arbitrarily modify the coefficients a_{ij} of Eq. (95) outside the influence domain emanating from the support of the data, without affecting the solution. Analogously, we can multiply the nonlinear term f by a C^∞ function $\phi(t, x)$, vanishing for x outside the same influence domain, without changing the value of the solution. Hence it is clear that, for fixed compactly supported data, we can reduce (95), (96) to an equivalent problem with periodic boundary condition in space variables.

We choose then $H = L^2(\mathbf{T}^n)$, $V = H^1(\mathbf{T}^n)$, $V' = H^{-1}(\mathbf{T}^n)$, $\mathbf{B} = \nabla$. Then it is not difficult to verify that X_r^s is the space

$$X_r^s = \{v \in C^\infty(\mathbf{T}^n) : \exists\, C, \|D^\alpha v\|_2 \leq C r^{-|\alpha|} |\alpha|!^s\}$$

and hence, by (101), $X^s_{0+}=\gamma^{(s)}_{L^2}(\mathbf{T}^n)$ (see [C] for details).

Assumption (11) is evidently satisfied by the functions $e^{in\cdot x}$, and (12) is just the Sobolev embedding ($k_0=[n/2]+1$).

Assumptions (14)-(16) are trivial consequences of (97)-(99). As to (17), (18), we recall the following Lemmas from [D]:

LEMMA 2. *Assume* (97), (98) *hold, and denote by* $A(t)$ *the operator*

$$-\sum_{h,k}^{1,n}\partial_{x_h}(a_{hk}(x,t)\partial_{x_k}).$$

Then, fixed an arbitrary $\Lambda_1>\Lambda$, *there exists a constant* $C=C(n, M, \Lambda_1, \Lambda)$ *such that for every* $v\in H^\infty(\mathbf{R}^n)$

$$\Big(\sum_{|\alpha|=j}\|[A(t),\partial^\alpha]v\|^2\Big)^{1/2}\le C(j+2)\Big(\sum_{|\alpha|=j}(A(t)\partial^\alpha v,\partial^\alpha v)\Big)^{1/2}+$$
$$+C(j+2)!^s\sum_{h=0}^{j}\Big(\sum_{|\beta|=h}\|\partial^\beta v\|^2\Big)^{1/2}\frac{\Lambda_1^{j+2-h}}{h!^s(h+1)^\sigma(h+2)^\sigma}$$

where $\sigma=s-1$, *and* $\|\cdot\|$, $(\ ,\ )$ *denote the norm and the scalar product in* $L^2(\Omega)$.

LEMMA 3. *With the same notations as in Lemma* 2, *let*

$$P=\sum_{|\gamma|\le m}a_\gamma(x,t)\partial^\gamma$$

be a partial differential operator on $\mathbf{R}^n$, *with measurable coefficients, infinitely differentiable in the x-variable, and such that, for some* $\mu(t)\in L^1(0,T)$ *and some* $\Lambda>0$

$$|\partial^\alpha a_\gamma|\le\mu(t)\Lambda^{|\alpha|}(|\alpha|!)^s.$$

Then, for any $\Lambda_1>\Lambda$, *there exists a constant* $C=C(n,\Lambda_1,\Lambda)$ *such that for any* v *in* $H^\infty(\mathbf{R}^n)$

$$\Big(\sum_{|\alpha|=j}\|\partial^\alpha Pv\|^2\Big)^{1/2}\le C\mu(t)(j+m)!^s\sum_{h=0}^{j+m}\frac{\Lambda_1^{j+m-h}}{h!^s}\Big(\sum_{|\beta|=h}\|\partial^\beta v\|^2\Big)^{1/2}.$$

Assumptions (17) and (18) are easy consequences of these lemmas.

Finally, we must verify (19) and (20). To avoid cumbersome computations, we shall consider in detail only the particular case $f=f(x,u_x)$, with space dimension equal to 1; the general case is completely analogous. Moreover, for sake of simplicity we shall assume that $\chi(t)\equiv 1$ in ass.(100).

We have then

$$D^j(f(x,u_x))=\sum_{\mu=0}^{j}\binom{j}{\mu}\mu!\sum_{\nu=1}^{\mu}\frac{D_p^\nu D_x^{j-\mu}f}{\nu!}\sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_i\ge 1}}\frac{D^{h_1+1}u\cdots D^{h_\nu+1}u}{h_1!\cdots h_\nu!}. \quad (104)$$

The last sum is symmetric in $h_1, \dots, h_\nu$, thus

$$\sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_i\geq 1}} \leq \nu. \sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_\nu\geq h_i\geq 1}} \tag{105}$$

and we can further estimate ν with 2^ν. Now, if K is a bounded subset of X_r^s for some $r>0$, then in particular $\|v_x\|_\infty$ is bounded for $v\in K$, say $|v_x|\leq R$, and hence we can apply (100) and we find

$$\|D^j(f(x, u_x))\|_2\leq \sum_{0\leq\nu\leq\mu\leq j} j!(j-\mu)!^{s-1}\nu!^{s'-1}\Lambda^{j-\mu}(2M_R)^\nu \sum_{\substack{h_1+\dots+h_\nu=\mu\\ h_\nu\geq h_i\geq 1}} \frac{\|D^{h_1+1}u\|_\infty\cdots\|D^{h_{\nu-1}+1}u\|_\infty\cdot\|D^{h_\nu+1}u\|_2}{h_1!\cdots h_\nu!}. \tag{106}$$

Now it is sufficient to observe that $|u|_j=\|D^ju\|_{L^2(\mathbf{R}^n)}$, and to use the Sobolev immersion

$$\|D^{h_i+1}u\|_\infty\leq c_n\|D^{h_i+k}u\|_2$$

with $k=[n/2]+2$, which holds true for the functions in $\gamma_{L^2}^{(s)}$; we thus obtain (19), with constants Λ and $2c_nM_R$.

As to (20), the Fréchet derivative of $f(x, u_x)$ with respect to u is given by $D_pf(x, u_x)v_x$; now this can be viewed as a first order operator on $v(x)$ with coefficient $D_p(x, u_x)$. Thus (20) will follow by Lemma 3, as soon as we show that $D_pf(x, u_x)\in\gamma_{L^2}^{(s)}$. Indeed, proceeding as in (104), we have, if $\|u_x\|\leq R$ (which is true for some R if u varies in some bounded subset of X_r^s for some $r>0$),

$$\|D^j(D_pf(x, u_x))\|_2\leq \sum_{0\leq\nu\leq\mu\leq j} j!(j-\mu)!^{s-1}\nu!^{s'-1}(\nu+1)^{s'}\Lambda^{j-\mu}(c_nM_R)^\nu \sum_{\substack{h_1+\cdots+h_\nu=\mu\\ h_i\geq 1}} \frac{\|D^{h_1+k}u\|_2\cdots\|D^{h_\nu+k}u\|_2}{h_1!\dots h_\nu!}. \tag{107}$$

Now, since u is in a bounded set of X_r^s, we can assume that, for some $\Lambda_3>0$,

$$\|D^ju\|_2\leq\Lambda_3^j j!^s j^{-ks-2} \tag{108}$$

hence

$$\frac{\|D^{h_i+k}u\|_2}{h_i!}\leq\Lambda_3^{h_i+k}h_i!^{s-1}(h_i+k)^{-2} \tag{109}$$

and using the inequality (see (54), (55))

$$(j-\mu)!^{s-1}h_1!^{s-1}\dots h_\nu!^{s-1}=\binom{j}{\mu}^{1-s}\binom{\mu}{h_1\dots h_\nu}^{1-s} j!^{s-1}\leq\nu!^{1-s}j!^{s-1}$$

we have

$$\|D^j(D_p f(x, u_x))\|_2 \le \sum_{0\le\nu\le\mu\le j} j!^s \nu!^{s'-s}(\nu+1)^{s'}\Lambda^{j-\mu}(c_n M_R)^\nu \sum_{\substack{h_1+\dots+h_\nu=\mu \\ h_i\ge 1}} \frac{\Lambda_3^{\mu+k\nu}}{(h_1+k)^2\dots(h_\nu+k)^2}. \tag{110}$$

Now we remark that

$$\sum_{\mu\ge 0}\sum_{\substack{h_1+\dots+h_\nu=\mu \\ h_i\ge 1}} \frac{1}{(h_1+k)^2\dots(h_\nu+k)^2} \equiv \left(\sum_{j\ge 1}\frac{1}{(j+k)^2}\right)^\nu \equiv c_k^\nu. \tag{111}$$

Hence (we can assume Λ, $\Lambda_0 \ge 1$)

$$\|D^i(D_p f(x, u_x))\|_2 \le j!^s \sum_{0\le\nu\le j} \nu!^{s'-s}(\nu+1)^{s'}\Lambda^j \Lambda_0^{j+k\nu}(c_n c_k M_R)^\nu, \tag{112}$$

and finally, since $(\nu+1)^{s'}\nu!^{s'-s} \le c(s)$,

$$\|D^j(D_p f(x, u_x))\|_2 \le c(s) j!^s \cdot j \cdot [\Lambda\Lambda_0^{k+1}(c_n c_k M_R+1)]^j \tag{113}$$

which implies (20), as observed above.

Proposition 1 is now a direct consquence of Thm. 1, in the case of compactly supported data.

2) *General data in* $\gamma_{L^2}^{(s)}$. We begin by observing that, if we choose $H = L^2(\mathbf{R}^n)$, $V = H^1(\mathbf{R}^n)$, $\mathbf{B} = \nabla$, then all the assumptions of Lemma 1 are satisfied. Note in fact with this choice of the spaces neither (11) nor (12) hold, but these assumptions are not used in the proof of the a priori estimate. Hence, if we have a sequence u_0^j, u_1^j of initial data belonging to a bounded set of $\gamma_{L^2}^{(s)}$, we can apply the a priori estimate to the corresponding solutions, and the lifespan $\bar{T}$ and the constant $\bar{C}$ given by Lemma 1 will not depend on j.

Now let u_0, $u_1 \in \gamma_{L^2}^{(s)}$; choose a compactly supported Gevrey function $\phi(x)$ such that $\phi(x)=1$ for $|x|\le 1$, $\phi(x)=0$ for $|x|\ge 2$, define $\phi_j(x) = \phi(x/j)$ and $u_0^j = u_0\cdot\phi_j$, $u_1^j = u_1\cdot\phi_j$; finally, let $f_i = f\cdot\phi_j$. Clearly the sequences u_0^j, u_1^j belong to a bounded subset of $\gamma_{L^2}^{(s)}$; moreover, the corresponding solutions $u_j(t, x)$ (which exist by step 1) have a common lifespan, and a common bound, by the above remark. Finally, by the finite speed of propagation, for each fixed t, x the sequence $u_j(t, x)$ is eventually constant. Hence the limit $u(t, x) = \lim_j u_j(t, x)$ is well defined, and satisfies (102) by the common a priori estimate. □

As a second application, we consider the mixed problem for (95), (96) with Dirichlet boundary conditions. We assume that (97)-(100) hold for $(t, x) \in [0, T] \times \bar{\Omega}$, where Ω is a bounded open subset of $\mathbf{R}^n$ with real analytic boundary. Moreover, we assume that

$$D_x^\alpha f(t,x,0,0)\equiv 0 \quad \forall\alpha,\ \forall x\in\partial\Omega. \tag{114}$$

We recall that $\gamma^{(s)}(\bar{\Omega})$ denotes the space of functions such that an inequality like (101) holds, with suitable constants, where the norm is replaced by the $L^2(\Omega)$ norm.

Then we have

PROPOSITION 2. *Under the above assumptions, for all u_0, $u_1\in\gamma^{(s)}(\bar{\Omega})$ with $D^\alpha u_j(x)=0$ on $\partial\Omega$, Pb.* (95), (96) *has a unique local solution $u\in C^1([0,T];\gamma(\bar{\Omega}))$, vanishing with all its derivatives at the boundary of Ω, provided $1\le s'<s\le 1+\lambda/2$, $s<2$.*

PROOF. $H=L^2(\Omega)$, $V=H_0^1(\Omega)$, $V'=H^{-1}(\Omega)$, $\mathbf{B}=\nabla$. The proof is similar to that of Prop. 1; see [AS] and [D] for more details. □

Our final application concerns the non-kowalewskian Cauchy problem

$$u_{tt}+a(t)\Delta^2 u+\sum_{i,j=1}^n m_{ij}(t)u_{x_ix_j}=f(t,x,u,\nabla u,\nabla^2 u) \tag{115}$$

$$u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x) \tag{116}$$

where, for some $\lambda\in]0,2]$,

$$a(t)\in C^\lambda([0,T]), \quad a(t)\ge 0, \tag{117}$$

$$m_{ij}(t)\in L^1(0,T). \tag{118}$$

Then we can prove

PROPOSITION 3. *Assume* (117), (118) *hold, and that the function f satisfies*

$$|D_x^\alpha D_r^\nu D_p^\beta D_q^\gamma f(t,x,r,p,q)|\le\chi(t)\Lambda^{|\alpha|}M_{\mathbf{R}}^{\nu+|\beta|+|\gamma|}|\alpha|!^s(\nu+|\beta|+|\gamma|)!^{s'}. \tag{119}$$

Then, for all u_0, $u_1\in\gamma_{L^2}^{(s)}$ Pb. (115), (116) *has a unique local solution $u(t,x)\in C^1([0,T];\gamma_{L^2}^{(s)})$, provided*

$$\frac{1}{2}\le s'<s<\frac{1}{2}+\frac{\lambda}{4}, \quad s<2. \tag{120}$$

PROOF. $H=L^2(\mathbf{R}^n)$, $V=H^2(\mathbf{R}^n)$, $\mathbf{B}=B=\Delta$. We omit the details, since they are straightforward (see [AS], [D]). □

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