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A decomposition theorem of operators for variegations

Yoshinobu KATO and Yasuo WATATANI
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1. Introduction

We consider a decomposition theorem of (bounded linear) operators on Hilbert spaces by analogy with variety in unversal algebra. It is known that any contraction is decomposed into the direct sum of the unitary part and the completely non-unitary part (B. Sz.-Nazy and C. Foias [9]). A general theory of decompositions of operators are developed by J. Ernest [4], A. Brown, C.-K. Fong and D.W. Hadwin [2], [6]. W. Szymanski [10] also studied the canonical decomposition of operator-valued functions in Hilbert spaces. Following his ideas, M. Fujii, M. Kajiwara, Y. Kato, F. Kubo and S. Maeda considered decompositions of operators ([5], [7], [8]). In this paper we shall give another condition on classes of operators to have the canonical decomposition. Finally we should remark that a recent work [1] by J. Agler is very interesting and has a relation with our paper.

2. Decomposable function

We consider a property $\mathscr{S}$ on operators and identify it with the class of all operators having the property $\mathscr{S}$. Many properties are defined by equations of non-commutative polynomials. A property $\mathscr{S}$ is called *algebraically difinite* if there is a family G of non-commutative polynomials $p(x, y)$ such that $T\in\mathscr{S}\Leftrightarrow p(T, T^*)=0$ for all $p\in G$, cf. [5] and [8]. More generally we consider a class of operators involving more general "functions" called decomposable functions invented by A. Brown, C.K. Fong and D.W. Hadwin [2], [6]. Let $B(K)$ be the set of all bounded linear operators on a Hilbert space K.

DEFINITION ([2]). Let H be a separable, infinite dimensional Hilbert space. A *decomposable function* on H is a function ϕ on $\cup\{B(M); M$ is a subspace of $H\}$ such that

(a) $\phi(B(M))\subset B(M)$ for every subspace M of H,

(b) if $T\in B(H)$ and M is a reducing subspace of T, then M reduces $\phi(T)$ and $\phi(T|_M)=\phi(T)|_M$

(c) if M, N are subspaces of H, $S \in B(M)$ and $U: N \to M$ is unitary, then $\phi(U^*SU) = U^*\phi(S)U$.

In this paper we shall consider a decomposable function acts on an arbitrary operator in the same way that an entire complex function acts on an arbitrary operator following D. W. Hadwin [6]. He showed that if ϕ is a decomposable function on H, then there is a net $\{p_n(x, y)\}$ of non -commutative polynomials such that $p_n(T, T^*) \to \phi(T)$ in the strong operator topology for every T in $B(H)$. Hence the decomposable function ϕ on H can be naturally extended to an arbitrary Hilbert space. A decomposable functionϕ is *norm continuous* if $\phi|B(M)$ is norm continuous for every subspace M of H.

EXAMPLES 1. (1) Let $\phi(T) = T^*T - TT^*$. Then an operator T is normal if and only if $\phi(T) = 0$.
(2) Let $\phi(T) = (T^*T - I)^2 + (TT^* - 1)^2$. Then an operator T is unitary if and only if $\phi(T) = 0$.
(3) Let $\phi(T] = T^*T - TT^* - |T^*T - TT^*|$. Then an operator T is hyponormal if and only if $\phi(T) = 0$.

3. Variegation

The study of classes of operators defined by equations is an interesting topic in operator theory. Similarly a major theme in universal algebra is the study of classes of algebras defined by identities [3]. Recall that a class K of algebras is an *equational class* if there is a set of identities Σ such that $K = M(\Sigma)$, the class of algebras satisfying Σ. A class K of algebras is called a *variety* if it is closed under (1) direct products, (2) holomorphic images and (3) subalgebras.

G. Birkhoff gave a characterization of the classes of algebras defined by the identities (cf. [3]):

THEOREM A (*G. Birkhoff*). *A class K of algebras is a equational class if and only if K is a variety.*

We shall consider an analogous fact of the above Birkhoff theorem in operator theory. We imagine that an equational class in universal algebra corresponds to a class of operators defined by equations involving decomposable functions. We shall introduce a notion of variegation in operator theory, which corresponds to a variety in universal abgebra.

DEFINITION. A class $\mathscr{S}$ of operators is a *varigation* if it is closed under (Var 1) direct sums, (Var 2) images of *-homomorphisms and (Var

3) suboperators, i.e., the restrictions to non-zero reducing subspaces. More precisely $\mathscr{S}$ is a variegation if $\mathscr{S}$ satisfies the following conditions:
(Var 1) If $\{S_\lambda\}_{\lambda\in\Lambda}\subset\mathscr{S}(\Lambda\neq\phi)$ and $\sup_{\lambda\in\Lambda}\|S_\lambda\|<\infty$, then $\bigoplus_{\lambda\in\Lambda}S_\lambda\in\mathscr{S}$.
(Var 2) Let π be a unital $*$-homomorphism of the C^*-algebra $C^*(S)$ generated by S and a unit into $B(K)$. If $S\in\mathscr{S}$, then $\pi(S)\in\mathscr{S}$.
(Var 3) If an opertor S on a Hilbert space H reduces a non-zero subspace K of H and $S\in\mathscr{S}$, then $S|K\in\mathscr{S}$.

REMARK. In the above definition the condition (Var 3) is in fact redundant because (Var 2) clearly implies (Var 3). But we would like to include it in the definition of a variegation to express an analogy with a variety explicitly.

EXAMPLES. (1) Let $\mathscr{S}$ be an algebraically definite class of operators. Then $\mathscr{S}$ is a variegation.
(2) Let $\phi(t)=e^t-1$ and $\mathscr{S}$ be the class of all operators T with $\phi(T)=0$. Then $\mathscr{S}$ is a variegation by D. W. Hadwin [6; Proposition 3.1] and is not an algebraically definite class.

4. Decomposition into parts

Let $\mathscr{S}$ be a property of operators. We say that an operator $T\in B(H)$ is completely non-$\mathscr{S}$ if there exist no non-zero reducing subspaces M of H such that $T|M\in\mathscr{S}$. We denote by $\neg\mathscr{S}$ the property of being completely non-$\mathscr{S}$. In this paper when we study a decomposition into the parts, we do *not* consider operators on zero dimensional Hilbert space $H_0=\{0\}$ to avoid a certain trouble. For example we do not consider whether an operator on $H_0=\{0\}$ is unitary.

DEFINITION. Let T be an operator on a Hiblert space H. Suppose that there exists the largest reducing subspace $M\neq\{0\}$ such that $T|M\in\mathscr{S}$. Then we call $T|_M$ the $\mathscr{S}$-part of T. The $\mathscr{S}$-part may not exist. For example if T is completely non-$\mathscr{S}$, then T has no $\mathscr{S}$-part.

We shall state the main theorem of the paper.

THEOREM 1. *Suppose that a class $\mathscr{S}$ of operators is a variegation. Then any operator $T\in B(H)$ can be decomposed uniquely in the following way:*
(1) *If $T\in\mathscr{S}$, then the $\mathscr{S}$-part of T is T itself and the $\neg\mathscr{S}$-part does not exist.*
(2) *If $T\in\neg\mathscr{S}$, then the $\mathscr{S}$-part of T does not exist and the $\neg\mathscr{S}$-part is T itself.*

(3) *Otherwise there exists a unique reducing subspace* $M \neq \{0\}$ *with* $M^{\perp} \neq \{0\}$ *such that* $T|_M$ *is the* $\mathscr{S}$*-part of* T *and* $T|_{M^{\perp}}$ *is the* $\neg\mathscr{S}$*-part of* T*. Moreover the projection* P_M *of* H *onto* M *is in the center of the von Neumann algebra* $R(T)$ *generated by* T.

PROOF. (1) Suppose that $T \in \mathscr{S}$. Then it is clear that $\mathscr{S}$-part of T is T itself by the definition. We shall shows that the $\neg\mathscr{S}$-part does not exist. It is enough to show that $T|_M \in \mathscr{S}$ for any non-zero reducing subspace M of T. It follows from the condition (Var 3).
(2) It is clear by the definition of $\neg\mathscr{S}$.
(3) Suppose that $T \notin \mathscr{S}$ and $T \notin \neg\mathscr{S}$. Put
$\Lambda = \{K \subset H\,;\, K \neq \{0\}$ is a reducing subspace of T with $T|_K \in \mathscr{S}\}$. Let $M = \vee(K\,;\, K \in \Lambda)$. Then M is a non-zero reducing subspace of T. Put $S = T|_M$. We shall show that S is the $\mathscr{S}$-part of T. It is enough to show that $S \in \mathscr{S}$. By the condition (Var 1), $W = \bigoplus_{K \in \Lambda} T|_K$ is in $\mathscr{S}$. There exists a unital *-homomorphism $\pi : C^*(W) \to B(M)$ such that $\pi(W) = T|_M$. Hence $S = \pi(W)$ is in $\mathscr{S}$ by (Var 2). Thus S is the $\mathscr{S}$-part of T. Take any unitary U in the commutant $R(T)'$ of $R(T)$. Put $N = UM$. Then the projection P_N of H onto N is given by $P_N = UP_MU^*$ and N is also a reducing subspace of $UTU^* = T$. For $K \in \Lambda$ we have $T|_K \in \mathscr{S}$. Hence $T|_{UK} = UTU^*|_{UK} \in \mathscr{S}$ by (Var 2) and (Var 3). Therefore $N = UM$ is the largest reducing subspace such that $T|_N = UTU^*{}_N \in \mathscr{S}$. By the unicity of $\mathscr{S}$-part of T, we get $M = N$. Hence $P_M = UP_MU^*$. It follows that P_M is in $R(T)'' = R(T)$. Since P_M is clearly in $R(T)'$ by the constuction, P_M is in the center of $R(T)$. Since $T \notin \mathscr{S}$, $M \neq H$, i.e., $M^{\perp} \neq \{0\}$. Put $C = T|_{M^{\perp}}$. We shall show that C is the $\neg\mathscr{S}$-part of T. First we shall show that C is completely non-$\mathscr{S}$. Let $N \subset M^{\perp}$ be a reducing subspace of C with $C|_N \in \mathscr{S}$. Since N is also a reducing subspace of T, $N \subset M$ by the difinition of M. Hence $N \subset M \cap M^{\perp} = \{0\}$. Therefore $C \in \neg\mathscr{S}$. Let L be a non-zero reducing subspace of T such that $T|_L \in \neg\mathscr{S}$. We shall show that $L \subset M^{\perp}$. On the contrary suppose that $L \not\subset M^{\perp}$. Then $M \cap L$ is a non-zero reducing subspace of $T|_L$, since P_M and P_L commute. Hence $T|_{M \cap L} \notin \mathscr{S}$. Since $S = T|_M \in \mathscr{S}$, we have $T|_{M \cap L} \in \mathscr{S}$ by (Var 3). This is a contrdiction. Hence $L \subset M^{\perp}$. Thus C is the $\neg\mathscr{S}$-part of T. Now it is clear that the decomposition $T = S \oplus C$ on $M \oplus M^{\perp} = H$ is unique.

Q. E. D.

COROLLARY 2. *If* $\mathscr{S}$ *is a variegation, then* $\neg(\neg\mathscr{S}) = \mathscr{S}$.

A Brown, C.-K. Fong and D. W. Hadwin [2] introduced a notion of part class to study a general decomposition theorem. Let $\mathscr{P}$ be a class of operators and T an operator. Put $\mathscr{P}(T)=\vee\{N\,;N$ reduces T, $T|_N\in\mathscr{P}\}$. Recall that a class $\mathscr{P}$ is a *part class* if it is closed under unitary equivalence and, for each operator T,
(i) $T|_{\mathscr{P}(T)}\in\mathscr{P}$ (i.e., $T|_{\mathscr{P}(T)}$ is the $\mathscr{P}$-part of T), and
(ii) if M reduces T, $T|_M\in\mathscr{P}$ and $\mathscr{P}(T|_{M^\perp})=0$, then $M=\mathscr{P}(T)$.

If $\mathscr{P}$ is a part class, then $\neg\mathscr{P}$ is also a part class and $T|_{\mathscr{P}(T)^\perp}$ is the $\neg\mathscr{P}$-part of T by [2; p. 312 (M)].

LEMMA 3. *Let $\mathscr{P}$ be a class of operators. Then we have the following*:
(1) *$\mathscr{P}$ is a part class if and only if $\mathscr{P}$ is closed under unitary equivalence and satisfies* (*Var* 1) *and* (*Var* 3).
(2) *If $\mathscr{P}$ is a variegation, then $\mathscr{P}$ is a part class. But the converse is not true in general.*

PROOF. (1): It is proved in [2; Theorem 3.3].
(2): If $\mathscr{P}$ is a variegation, then $\mathscr{P}$ is a part class by (1). Let $\mathscr{N}$ be the class of all normal operators and $\mathscr{S}=\neg\mathscr{N}$. Then $\mathscr{N}$ and $\mathscr{S}$ are part classes by [2; p. 312 (M)]. $\mathscr{N}$ is also a variegation but $\mathscr{S}$ is not a variegation. In fact $\mathscr{S}$ contains a simple unilatral shift S on $H=\ell^2(\mathbf{N})$. Let $\pi: B(H)\to B(H)/C(H)$ be the Calkin map. Then S is in $\mathscr{S}$ but $\pi(S)$ is a unitary and is not in $\mathscr{S}$. Q. E. D.

REMARK. Let $\mathscr{S}$ be the class of all non-invertible operators. Then $\mathscr{S}$ satisfies the condition (Var 1), but $\mathscr{S}$ is not a part class. In fact a diagonal operator $T=1\oplus 1/2\oplus 1/3\oplus \dots$ on $\ell^2(\mathbf{N})$ is in $\mathscr{S}$, but $1\notin\mathscr{S}$. So $\mathscr{S}$ does not satisfy (Var 3).

For each $r>0$, let $\mathscr{B}_r$ be the class of all operators T such that $\|T\|\leqq r$. Then it is easy to see that $\mathscr{B}_r$ satisfies (Var 1), (Var 2) and (Var 3), so $\mathscr{B}_r$ is a variegation.

LEMMA 4. *It $\mathscr{S}$ is a part class, then $\mathscr{S}\cap\mathscr{B}_r$ is also a part class.*

PROOF. $\mathscr{B}_r$ is a part class. Using Lemma 3, we easily get that $\mathscr{S}\cap\mathscr{B}_r$ is a part class. Q. E. D.

Finally we shall discuss a characterization of a variegation as an analogy of Theorem A by G. Birkhoff using a nice result by D. W. Hadwin [6].

THEOREM 5. *Let $\mathscr{S}$ be a class of operators. Then the following conditions are equivalent.*

(1) *$\mathscr{S}$ is a variegation.*

(2) *$\mathscr{S}$ is a part class and $\mathscr{S}\cap B(H)$ is norm closed for any Hilbert space H.*

(3) *For any number $r>0$, there exists a norm continuous decomposable function ψ_r such that $\mathscr{S}\cap\mathscr{B}_r$ is the class of all operators S such that $\psi_r(S)=0$.*

PROOF. (1)$\Rightarrow$(2): Suppose that $\mathscr{S}$ is a variegataion. Then $\mathscr{S}$ is a part class by Lemma 3. Let $S_n\in\mathscr{S}\cap B(H)$ and S_n converges uniformly to S. Put $T=\bigoplus_{n=1}^{\infty} S_n\in\mathscr{S}$. Then there exists a unital *-homomorphism π: $C^*(T)\to B(H)$ such that

$$\pi(p(T, T^*))=\pi(\bigoplus_{n=1}^{\infty} p(S_n, S_n^*))=\lim_{n\to\infty} p(S_n, S_n^*))=p(S, S^*)$$

for any non-commutative polynomial $p(x, y)$. In particular $\pi(T)=S$. Since $\mathscr{S}$ is a variegation and $T\in\mathscr{S}$, $S\in\mathscr{S}$. Hence $\mathscr{S}\cap B(H)$ is norm closed.

(2)$\Rightarrow$(3): Take a positive number r and a Hilbert space H. Since an operator on an arbitrary Hilbert space is a direct sum of operators on separable spaces, we may assume that H is separable. Since $\mathscr{S}$ is a part class, $\mathscr{S}\cap\mathscr{B}_r$ is also a part class by Lemma 4. Then by a result of D. W. Hadwin [6; Theorem 5. 1.], there is a continuous decomposable ψ_r such that

$$\mathscr{S}\cap\mathscr{B}_r\cap B(H)=\{S\in B(H)\,;\,\psi_r(S)=0\}.$$

(3)$\Rightarrow$(1): Let $\{S_\lambda\}_{\lambda\in\Lambda}\subset\mathscr{S}$ and $\sup_{\lambda\in\Lambda}\|S_\lambda\|<\infty$ $(\Lambda\neq\phi)$. Then there exists $r>0$ such that $\|S_\lambda\|\leqq r$ for all λ. Since $\psi_r(\bigoplus_{\lambda\in\Lambda} S_\lambda)=\bigoplus_{\lambda\in\Lambda}\psi_r(S_\lambda)=0$, $\bigoplus_{\lambda\in\Lambda} S_\lambda\in\mathscr{S}\cap\mathscr{B}_r\subset\mathscr{S}$. Thus $\mathscr{S}$ satisfies (Var 1). Let $S\in\mathscr{S}$ and $\pi: C^*(S)\to B(H)$ be a unital *-homomorphism. Then there exists a number $r>0$ such that $\|\pi(S)\|\leqq\|S\|\leqq r$. Since ψ_r is a norm continuous decomposable function, $\psi_r(\pi(S))=\pi(\psi_r(S))=\pi(0)=0$ by D. W. Hadwin [6; Proposition 3. 1.]. Thus $\pi(S)\in\mathscr{S}$, so that (Var 2) is verfied. Hence $\mathscr{S}$ is a variegation.

Q. E. D.

REMARK. We do not know whether we can replace the condition (3) in Theorem 5 by the following condition (3′).

(3′) There exists a norm continuous decomposable function ψ such that $\mathscr{S}$ is the class of all operators S such that $\psi(S)=0$.

The question is related with an unsolved problem raised in [6; Remark B].

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Nagaikeso Institute
Kanaoka-cho 1983,
Sakai city, Osaka 591
Japan

Department of Mathematics
Hokkaido University
Sapporo 060
Japan
Present address
Department of Mathematics
College of General Education
Kyushu University
Ropponmatsu, chuo-ku
Fukuoka 810
Japan