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Matrix invariants of binary forms

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Abstract. Let S_n be the vector space of homogeneous polynomials of degree n in two variables. Let $A_d(n)$ be the noncommutative algebra consisting of SL_2-equivariant polynomial maps from S_d to $\mathrm{End}S_n$. We show that generators for $A_d(n)$ are derived from generators for the algebra of covariants of the d-ic forms.

Key words: binary forms, covariants, transvectants, Clebsch-Gordan rule.

Introduction

Let k be a field of characteristic 0. Put $S = k[x_1, x_2]$, the polynomial ring, and let S_n be its homogenous part of degree n. The group SL_2 acts on S canonically. We are concerned about SL_2-invariant polynomial maps from the space S_d to the matrix algebra $\mathrm{End}S_n$. Those maps form an algebra $A_d(n)$ by matrix multiplication. $A_d(0)$ is the algebra of invariants of the d-ic form, and was studied in classical invariant theory. We show that $A_d(n)$ is a deformation of a factor of the algebra of covariants of the d-ic form. In particular, the knowledge of the generators for the algebra of covariants gives that for the algebra $A_d(n)$.

More generally, let R be a commutative algebra with SL_2-action. Then SL_2 acts on the algebra $R \otimes \mathrm{End}S_n$ and let $A(n) = (R \otimes \mathrm{End}S_n)^{SL_2}$ be the invariant algebra. On the other hand, we have the commutative algebra $C = (R \otimes S)^{SL_2}$ with grading given by $C_n = (R \otimes S_n)^{SL_2}$. For $\alpha \in R \otimes S_n$, $\beta \in R \otimes S_m$ and $p \geq 0$, we have the transvectant $(\alpha, \beta)_p \in R \otimes S_{n+m-2p}$, where α, β are regarded as forms with coefficients in R ([1]). Define the map $\phi : \bigoplus_{p=0}^{n} C_{2p} \to A(n)$ by

$$\phi(\alpha)(\gamma) = \frac{n!}{(n-p)!}(\alpha, \gamma)_p$$

for $\alpha \in C_{2p}$, $\gamma \in R \otimes S_n$. Then it is shown that ϕ is an isomorphism and

$$\phi(\alpha)\phi(\beta) - \phi(\alpha\beta) \in \phi(\bigoplus_{r<p+q} C_{2r})$$

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for $\alpha \in C_{2p}$, $\beta \in C_{2q}$ with $p+q \le n$. It follows that if the algebra $\bigoplus_{p=0}^{n} C_{2p}$ is generated by homogenous elements $\alpha_1, \alpha_2, \ldots$, then the algebra $A(n)$ is generated by $\phi(\alpha_1), \phi(\alpha_2), \ldots$.

When R is the coordinate ring of S_d, $A(n)$ becomes $A_d(n)$ and C becomes the algebra of covariants. Using Cayley's determination of C for $d = 3, 4$, we give the generators for $A_3(n)$, $A_4(n)$.

1. SL_2-invariant of matrix algebras

Our result is an immediate consequence of a property of transvectants, which might be a classical fact (Proposition below). Let us review some basic facts about binary forms. The Ω-process is the map

$$\begin{aligned} \Omega : S \otimes S &\to S \otimes S \\ \alpha \otimes \beta &\mapsto \frac{\partial \alpha}{\partial x_1} \otimes \frac{\partial \beta}{\partial x_2} - \frac{\partial \alpha}{\partial x_2} \otimes \frac{\partial \beta}{\partial x_1}. \end{aligned}$$

This is SL_2-linear and takes $S_n \otimes S_m$ into $S_{n-1} \otimes S_{m-1}$. Put $\Omega' = \frac{1}{nm}\Omega : S_n \otimes S_m \to S_{n-1} \otimes S_{m-1}$. For $0 \le p \le n, m$, let τ_p be the composite map

$$\tau_p : S_n \otimes S_m \xrightarrow{\Omega'^p} S_{n-p} \otimes S_{m-p} \xrightarrow{\text{mult}} S_{n+m-2p}.$$

$(\alpha, \beta)_p = \tau_p(\alpha \otimes \beta)$ is called the p^{th} transvectant of α and β ([1, §48]).

The Clebsch-Gordan rule is the decomposition

$$\begin{aligned} S_n \otimes S_m &\cong \bigoplus_{p=0}^{\min(n,m)} S_{n+m-2p} \\ \alpha \otimes \beta &\mapsto ((\alpha, \beta)_p)_p . \end{aligned}$$

Equivalently, we have

$$\begin{aligned} &\mathrm{Hom}_{SL_2}(S_n \otimes S_m, S_l) \\ &\quad = \begin{cases} k\tau_p & \text{if } 0 \le \exists p \le n, m \text{ such that } n+m-2p = l \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Another equivalent form is the isomorphism

$$\begin{aligned} \bigoplus_{p=(m-n)_+}^{m} S_{n-m+2p} &\cong \mathrm{Hom}(S_m, S_n) \\ S_{n-m+2p} \ni \alpha &\mapsto (\alpha, \ )_p. \end{aligned}$$

Here $z_+ = \max(z, 0)$ for $z \in \mathbb{Z}$. Because

$$\begin{aligned} \mathrm{Hom}(S_m, S_n) &\cong S_n \otimes S_m^* \\ &\cong \bigoplus_{l \geq 0} \mathrm{Hom}_{SL_2}(S_l, S_n \otimes S_m^*) \otimes S_l \\ &\cong \bigoplus_{l \geq 0} \mathrm{Hom}_{SL_2}(S_l \otimes S_m, S_n) \otimes S_l \\ &\cong \bigoplus_{p=(m-n)_+}^{m} S_{n-m+2p}. \end{aligned}$$

For each $m, n \geq 0$ let $\varphi : \bigoplus_{p=(m-n)_+}^{m} S_{n-m+2p} \to \mathrm{Hom}(S_m, S_n)$ be the isomorphism taking $\alpha \in S_{n-m+2p}$ to the map $\frac{n!}{(m-p)!}(\alpha, \ )_p$.

Proposition *Let $\alpha \in S_{n-m+2p}$, $\beta \in S_{m-l+2q}$ with $(m-n)_+ \leq p \leq m$, $(l-m)_+ \leq q \leq l$, $p+q \leq l$. Then*

$$\varphi(\alpha)\varphi(\beta) - \varphi(\alpha\beta) \in \varphi(\bigoplus_{r<p+q} S_{n-l+2r}) \quad (\subset \mathrm{Hom}(S_l, S_n)).$$

Proof will be given later.

Let R be a commutative k-algebra with SL_2-action. Put $C = (R \otimes S)^{SL_2}$ and $C_n = (R \otimes S_n)^{SL_2}$. For $m, n \geq 0$ put $A(m, n) = (R \otimes \mathrm{Hom}(S_m, S_n))^{SL_2}$. We have the composition maps $A(m, n) \times A(l, m) \to A(l, n)$. Tensoring R with the SL_2-isomorphism φ and taking the SL_2-invariant, we obtain

Theorem *For each $m, n \geq 0$, we have the isomorphism*

$$\phi : \bigoplus_{p=(m-n)_+}^{m} C_{n-m+2p} \xrightarrow{\sim} A(m, n)$$

$$C_{n-m+2p} \ni \alpha \mapsto \frac{n!}{(m-p)!}(\alpha, \ )_p.$$

For $\alpha \in C_{n-m+2p}$, $\beta \in C_{m-l+2q}$ with $(m-n)_+ \leq p \leq m$, $(l-m)_+ \leq q \leq l$, $p+q \leq l$ we have

$$\phi(\alpha)\phi(\beta) - \phi(\alpha\beta) \in \phi(\bigoplus_{r<p+q} C_{n-l+2r}) \quad (\subset A(l, n)).$$

Corollary *If the factor algebra $\bigoplus_{p=0}^{n} C_{2p}$ of $\bigoplus_{p \geq 0} C_{2p}$ is generated by homogenous elements $\alpha_1, \alpha_2, \ldots$, then the algebra $A(n, n)$ is generated by*

$\phi(\alpha_1), \phi(\alpha_2), \ldots$.

Remark. Let $B = R^{SL_2}$. Then C and $A(m,n)$ are B-modules and the isomorphism ϕ is B-linear. If α_i generate $\bigoplus_p C_{2p}$ over B, then $\phi(\alpha_i)$ generate $A(n,n)$ over B.

Proof of Proposition. By Clebsch-Gordan we have

$$\begin{aligned} S_{n_1} \otimes S_{n_2} \otimes S_{n_3} &\cong \bigoplus_{t=0}^{\min(n_1,n_2)} S_{n_1+n_2-2t} \otimes S_{n_3} \\ &\cong \bigoplus_{t=0}^{\min(n_1,n_2)} \bigoplus_{s=0}^{\min(n_1+n_2-2t,n_3)} S_{n_1+n_2+n_3-2t-2s}. \end{aligned}$$

Hence $\mathrm{Hom}_{SL_2}(S_{n_1} \otimes S_{n_2} \otimes S_{n_3}, S_n)$ has a k-basis consisting of the maps

$$\tau_s(\tau_t \otimes 1) : S_{n_1} \otimes S_{n_2} \otimes S_{n_3} \to S_{n_1+n_2-2t} \otimes S_{n_3} \to S_n$$

for s, t such that

$$\begin{aligned} &0 \le t \le n_1, n_2, \quad 0 \le s \le n_1 + n_2 - 2t, n_3, \\ &n_1 + n_2 + n_3 - 2(s+t) = n. \end{aligned}$$

Likewise, $\mathrm{Hom}_{SL_2}(S_{n_1} \otimes S_{n_2} \otimes S_{n_3}, S_n)$ has a k-basis consisting of the maps

$$\tau_p(1 \otimes \tau_q) : S_{n_1} \otimes S_{n_2} \otimes S_{n_3} \to S_{n_1} \otimes S_{n_2+n_3-2q} \to S_n$$

for p, q such that

$$\begin{aligned} &0 \le q \le n_2, n_3, \quad 0 \le p \le n_1, n_2 + n_3 - 2q, \\ &n_1 + n_2 + n_3 - 2(p+q) = n. \end{aligned}$$

So there must be relations

$$\tau_p(1 \otimes \tau_q) = \sum_{s,t} C_{pq}^{st} \tau_s(\tau_t \otimes 1)$$

with some $C_{pq}^{st} \in k$. Namely,

$$(\alpha, (\beta, \gamma)_q)_p = \sum_{s,t} C_{pq}^{st}((\alpha, \beta)_t, \gamma)_s \tag{$*$}$$

for all $\alpha \in S_{n_1}$, $\beta \in S_{n_2}$, $\gamma \in S_{n_3}$. Thus we have in $\mathrm{Hom}(S_{n_3}, S_n)$ that

$$(\alpha, \ )_p \circ (\beta, \ )_q = \sum_{s+t=p+q} C_{pq}^{st}((\alpha, \beta)_t, \ )_s$$

$$= C_{p,q}^{p+q,0}(\alpha\beta,\ )_{p+q}$$
$$+\ (\text{a linear combination of } (\delta,\ )_s \text{ with } s < p+q).$$

So the proposition follows from □

Lemma *If* $p+q \le n_3$, *then*

$$C_{p,q}^{p+q,0} = \binom{n_3-q}{p}\binom{n_2+n_3-2q}{p}^{-1}.$$

Proof. Following [1], for linear forms $a = a_1x_1 + a_2x_2$ and $b = b_1x_1 + b_2x_2$ we write $(ab) = a_1b_2 - a_2b_1$. Then

$$\Omega'(a^n \otimes b^m) = (ab)a^{n-1} \otimes b^{m-1}.$$

We need the formula [1, §49(v)]

$$(f, gh)_p = \sum_{s+t=p} \frac{\binom{m}{s}\binom{l}{t}}{\binom{m+l}{p}} \nabla\Omega_{12}'^{s}\Omega_{13}'^{t}(f \otimes g \otimes h)$$

for $f \in S_n$, $g \in S_m$, $h \in S_l$. Here $\nabla : S \otimes S \otimes S \to S$ is the multiplication map and $\Omega'_{ij} : S \otimes S \otimes S \to S \otimes S \otimes S$ is obtained by making Ω' act on the (ij)-factor of $S \otimes S \otimes S$.

Now put $w = p+q$. Evaluate the both sides of $(*)$ for $\alpha = a^{n_1}$, $\mp a^{n_2}$, $\gamma = b^{n_3}$ with a, b linear forms. Since $(\alpha, \beta)_t = 0$ for $t > 0$, we have

$$\text{RHS} = C_{pq}^{w0}(a^{n_1+n_2}, b^{n_3})_w = C_{pq}^{w0}(ab)^w a^{n_1+n_2-w} b^{n_3-w}.$$

Using the above formula, we compute

$$\begin{aligned}\text{LHS} &= (a^{n_1}, (ab)^q a^{n_2-q} b^{n_3-q})_p \\ &= (ab)^q \sum_{s+t=p} \frac{\binom{n_2-q}{s}\binom{n_3-q}{t}}{\binom{n_2+n_3-2q}{p}} \\ &\quad \times \nabla\Omega_{12}'^{s}\Omega_{13}'^{t}(a^{n_1} \otimes a^{n_2-q} \otimes b^{n_3-q})\end{aligned}$$

$$= (ab)^q \frac{\binom{n_3-q}{p}}{\binom{n_2+n_3-2q}{p}} \nabla\Omega'^p_{13}(a^{n_1}\otimes a^{n_2-q}\otimes b^{n_3-q})$$

$$= \frac{\binom{n_3-q}{p}}{\binom{n_2+n_3-2q}{p}} (ab)^w a^{n_1+n_2-w} b^{n_3-w}.$$

Hence

$$C^{w0}_{pq} = \frac{\binom{n_3-q}{p}}{\binom{n_2+n_3-2q}{p}}.$$

□

Remark. The theorem of [1, §50(ii)] is of a similar nature to our proposition. Explicit linear relations among $(\alpha,(\beta,\gamma)_q)_p$ and $((\alpha,\beta)_t,\gamma)_s$ are given by Gordan's series (see the next section).

2. Matrix invariants of the cubic and the quartic

Let

$$f = \sum_{i=0}^{d} \binom{d}{i} a_i x_1^{d-i} x_2^i$$

be the general d-ic form. This means that the coefficients a_i are taken as indeterminates. SL_2 acts on the polynomial algebra $R = k[a_0,\dots,a_d]$ in such a way that f is invariant. As before, put $C = (R\otimes S)^{SL_2}$, $C_n = (R\otimes S_n)^{SL_2}$ and $B = R^{SL_2}$. The generators for C for $d\le 6$ were given by Cayley and Gordan. So by Corollary we can know in principle the generators for the algebra $A(n,n) = (R\otimes \mathrm{End} S_n)^{SL_2}$ for such d. Let us see the case $d = 3, 4$.

Case $d = 3$. Put $h = (f,f)_2 \in C_2$, $t = (f,h)_1 \in C_3$, $\Delta = (h,h)_2 \in C_0 = B$. It is known that C is generated by f, h, t, Δ with relation $2t^2 + h^3 + \Delta f^2 = 0$ and that $B = k[\Delta]$ ([1, §88]).

It follows that the subalgebra $\bigoplus_p C_{2p}$ of C is generated by h, f^2, ft

over B. Therefore the B-algebra $A(n,n) = (R \otimes \mathrm{End} S_n)^{SL_2}$ is generated by $(h,\ )_1, (f^2,\ )_3, (ft,\ )_3$. But there is a relation

$$[(h,\ )_1, (f^2,\ )_3] = -\frac{6}{n}(ft,\ )_3,$$

where $[x,y] = xy - yx$. So, $A(n,n)$ is generated by $(h,\ )_1, (f^2,\ )_3$ over B.

To derive the above commutation relation we need to find the constants C^{st}_{pq} in $(*)$ for $(p,q) = (1,3), (3,1)$. A quick way will be the use of Gordan's series [1, §54(IX)]. It is the identity

$$\sum_i (-1)^i \frac{\binom{n_3 - s - k}{i}\binom{r}{i}}{\binom{n_2 + n_3 - 2s - i + 1}{i}} (\alpha, (\beta, \gamma)_{s+i})_{r+k-i}$$
$$= \sum_i (-1)^i \frac{\binom{n_1 - r - k}{i}\binom{s}{i}}{\binom{n_1 + n_2 - 2r - i + 1}{i}} ((\alpha, \beta)_{r+i}, \gamma)_{s+k-i}$$

for $\alpha \in S_{n_1}$, $\beta \in S_{n_2}$, $\gamma \in S_{n_3}$ and nonnegataive integers r, s, k such that $r + k \le n_1$, $s + k \le n_3$, $r + s \le n_2$ and either $k = 0$ or $r + s = n_2$.

Now let $\alpha \in S_2$, $\beta \in S_6$, $\gamma \in S_n$ with $n \ge 4$. Then the identities for $(r,s,k) = (1,3,0), (0,4,0)$ become

$$(\alpha, (\beta, \gamma)_3)_1 - \frac{n-3}{n}(\alpha, (\beta, \gamma)_4)_0 = ((\alpha, \beta)_1, \gamma)_3 - \frac{1}{2}((\alpha, \beta)_2, \gamma)_2,$$
$$(\alpha, (\beta, \gamma)_4)_0 = ((\alpha, \beta)_0, \gamma)_4 - ((\alpha, \beta)_1, \gamma)_3 + \frac{2}{7}((\alpha, \beta)_2, \gamma)_2.$$

Hence

$$(\alpha, (\beta, \gamma)_3)_1 = \frac{n-3}{n}((\alpha, \beta)_0, \gamma)_4 + \frac{3}{n}((\alpha, \beta)_1, \gamma)_3 - \frac{3n+12}{14n}((\alpha, \beta)_2, \gamma)_2.$$

Similarly letting $(r,s,k) = (1,1,2), (0,2,2)$, we have

$$(\beta, (\alpha, \gamma)_1)_3 = \frac{n-3}{n}((\beta, \alpha)_0, \gamma)_4 + \frac{3}{n}((\beta, \alpha)_1, \gamma)_3 - \frac{3n+12}{14n}((\beta, \alpha)_2, \gamma)_2.$$

Since $(\beta,\alpha)_p=(-1)^p(\alpha,\beta)_p$, we obtain

$$[(\alpha,\ )_1,(\beta,\ )_3]=\frac{6}{n}((\alpha,\beta)_1,\ )_3$$

as an operator on S_n. This holds also for $n<4$.

Since $(h,f^2)_1=-ft$, we obtain the desired relation.

Case $d=4$. Let $h=(f,f)_2\in C_4$, $t=(f,h)_1\in C_6$, $i=(f,f)_4\in C_0$, $j=(f,h)_4\in C_0$. Then B is the polynomial algebra $k[i,j]$ and the B-algebra C is generated by f,h,t ([1, §89]). Also $\bigoplus_p C_{2p}=C$. Hence B-algebra $A(n,n)$ is generated by $(f,\ )_2,(h,\ )_2,(t,\ )_3$. But we have

$$[(f,\ )_2,(h,\ )_2]=\frac{8(n-2)}{n(n-1)}(t,\ )_3.$$

If $n=2$, $(t,\ )_3=0$. Consequently, $A(n,n)$ is generated by $(f,\ )_2$ and $(h,\ )_2$ over B.

The commutation relation is proved in the same way. Using the Gordan series for $\alpha,\beta\in S_4$ and $(r,s,k)=(2,2,0),(1,3,0),(0,4,0)$, we obtain the identity

$$\begin{aligned}
&n(n-1)(\alpha,\ )_2\circ(\beta,\ )_2\\
&\quad=(n-2)(n-3)((\alpha,\beta)_0,\ )_4+4(n-2)((\alpha,\beta)_1,\ )_3\\
&\qquad-\frac{2}{7}(n+5)(n-3)((\alpha,\beta)_2,\ )_2-\frac{2}{5}(n+3)((\alpha,\beta)_3,\ )_1\\
&\qquad+\frac{1}{30}(n+3)(n+2)((\alpha,\beta)_4,\ )_0
\end{aligned}$$

as an operator on S_n, and hence

$$\begin{aligned}
n(n-1)[(\alpha,\ )_2,(\beta,\ )_2]\ =\ &8(n-2)((\alpha,\beta)_1,\ )_3\\
&-\frac{4}{5}(n+3)((\alpha,\beta)_3,\ )_1.
\end{aligned}$$

Since $(f,h)_1=t$ and $(f,h)_3=0$, the relation follows.

References

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