Atomic decompositions of weighted Hardy-Morrey spaces

Kwok-Pun Ho

Abstract. We obtain the Fefferman-Stein vector-valued maximal inequalities on Morrey spaces generated by weighted Lebesgue spaces. Using these inequalities, we introduce and define the weighted Hardy-Morrey spaces by using the Littlewood-Paley functions. We also establish the non-smooth atomic decompositions for the weighted Hardy-Morrey spaces and, as an application of the decompositions, we obtain the boundedness of a class of singular integral operators on the weighted Hardy-Morrey spaces.

Key words: Vector-valued maximal inequalities, Morrey-Hardy spaces, Atomic decompositions, Singular integral operator.

1. Introduction and Preliminarily results

This paper consists of two main results. The first one is the Fefferman-Stein vector-valued maximal inequalities on Morrey spaces generated by weighted Lebesgue spaces. The second one is the atomic decompositions of weighted Hardy-Morrey spaces.

The classical Fefferman-Stein vector-valued maximal inequalities are established in [7]. There are several generalizations of these inequalities. The weighted vector-valued maximal inequalities are given in [1]. The vector-valued maximal inequalities associated with Morrey spaces are obtained in [42]. In addition, the vector-valued maximal inequalities on rearrangement-invariant quasi-Banach function spaces and their corresponding Morrey type spaces are provided in [18].

In [8], [9], Frazier and Jawerth offered an application of the vector-valued maximal inequalities on the study of Triebel-Lizorkin spaces. Precisely, they show that the vector-valued maximal inequalities for Lebesgue spaces can be used to assure the boundedness of the ϕ-ψ transform on Triebel-Lizorkin spaces and, hence, to establish the Littlewood-Paley characterization of Triebel-Lizorkin spaces. Moreover, the smooth atomic and molecular decompositions are obtained. In [28], [38], [42], [45], Mazzucato, Sawano,

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Tanaka, Tang, Wang and Xu find that a similar approach can be applied to Morrey spaces. That is, the Triebel-Lizorkin-Morrey spaces (in [42], they are called as Morrey type Besov-Triebel spaces) are well defined, they admit the Littlewood-Paley characterization and posses the smooth atomic and molecular decompositions. In particular, Mazzucato obtained the Littlewood-Paley characterization of Morrey space in [28]. Thus, Triebel-Lizorkin-Morrey spaces cover Morrey spaces as a special case.

An important special case of the Triebel-Lizorkin-Morrey spaces is the family of Hardy-Morrey spaces. A study of the Hardy-Morrey spaces by using the maximal function approach is given in [20] and some applications of the Hardy-Morrey spaces are given in [21]. The non-smooth atomic decompositions for the Hardy-Morrey spaces are established in [20]. The Hardy-Morrey space is also investigated from the viewpoint of Littlewood-Paley characterization by Sawano in [40].

Even though the definition of Muckenhoupt weight functions is well known, for completeness, we state it again in the following.

Let $B(z, r) = \{x \in \mathbb{R}^n : |x - z| < r\}$ denote the open ball with center $z \in \mathbb{R}^n$ and radius $r > 0$. Let $\mathcal{B} = \{B(z, r) : z \in \mathbb{R}^n, r > 0\}$.

Definition 1.1   For $1 < p < \infty$, a locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be an A_p weight if

$$\sup_{B \in \mathcal{B}} \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-p'/p} dx \right)^{p/p'} < \infty$$

where $p' = \frac{p}{p-1}$. A locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C \omega(x), \quad a.e. \ x \in B$$

for some constant $C > 0$. We define $A_\infty = \bigcup_{p \geq 1} A_p$.

For any $\omega \in A_\infty$, let q_ω be the infimum of those q such that $\omega \in A_q$. When $q_\omega \neq 1$, according to the openness property of A_p weight functions for $p > 1$, $A_p = \bigcup_{1 < r < p} A_r$, we have $\omega \notin A_{q_\omega}$.

For any $\omega \in A_\infty$ and any Lebesgue measurable set E, write $\omega(E) = \int_E \omega(x) dx$. We have the following standard characterizations of A_∞ and A_p

weights (see [16, Theorem 9.3.3.(d)] and [16, Proposition 9.1.5.(9)], respectively).

Proposition 1.1 *A locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ belongs to A_∞ if and only if there exist a $\delta_\omega > 0$ and a constant $C_0 > 0$ such that for any $B \in \mathcal{B}$ and all measurable subsets E of B, we have*

$$\frac{\omega(E)}{\omega(B)} \leq C_0 \left(\frac{|E|}{|B|} \right)^{\delta_\omega}. \tag{1.1}$$

Proposition 1.2 *If $\omega \in A_p$, then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$, $r > 0$ and $\lambda > 1$*

$$\omega(B(x, \lambda r)) \leq C \lambda^{np} \omega(B(x, r)).$$

In this paper, we use the Fefferman-Stein vector-valued maximal inequalities on Morrey spaces generated by A_∞-weighted Lebesgue spaces to define and study the weighted Hardy-Morrey spaces. The family of weighted Hardy-Morrey spaces is an extension of the weighted Hardy spaces appeared in [3], [12], [19], [26], [41].

We need a weight function $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ to define the weighted Hardy-Morrey spaces (see Definitions 3.1 and 3.2). For the classical Morrey spaces, it is given by $|B|^{1/p-1/q}$ where $1 \leq p \leq q < \infty$ and B is an open ball in $\mathbb{R}^n$. For the weighted Hardy-Morrey spaces, the underlying measure is an A_∞-weighted Lebesgue measure. We introduce the corresponding family of weight functions associated with A_∞-weighted Lebesgue measure for the weighted Hardy-Morrey spaces in Definition 3.1.

In Section 2, we establish the Fefferman-Stein vector-valued maximal inequalities on Morrey spaces generated by weighted Lebesgue spaces. We define the weighted Hardy-Morrey spaces via the Littlewood-Paley functions in Section 3. The non-smooth atomic decomposition for weighted Hardy-Morrey spaces is given in Section 4. In addition, an application of the non-smooth atomic decomposition on the boundedness of the singular integral operator is presented at the end of Section 4. Some technical results for establishing the non-smooth atomic decomposition are presented in Section 5.

2. Vector-valued inequalities

The main theme of this section is the Fefferman-Stein vector-valued maximal inequalities on Morrey spaces generated by weighted Lebesgue spaces.

Let M denote the Hardy-Littlewood maximal operator. For any sequence of locally integrable functions, $f = \{f_i\}_{i \in \mathbb{Z}}$, let $\mathbb{M}(f) = \{\mathrm{M}(f_i)\}_{i \in \mathbb{Z}}$.

We are now ready to establish the main result of this section. Notice that the generalized Morrey spaces introduced in the next section are not rearrangement invariant, therefore, some existing results, such as the results given in [18, Section 4], cannot be applied to the generalized Morrey spaces. The following theorem is important since it extends the Fefferman-Stein vector-valued maximal inequalities to generalized Morrey spaces even though they are not rearrangement invariant. We modify the techniques developed in [4], [18], [31], [42] to obtain the following theorem.

Theorem 2.1   *Let $1 < p$, $q < \infty$, $\omega \in A_p$ and $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ be a Lebesgue measurable function. If there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u fulfills*

$$\sum_{j=0}^{\infty} \left(\frac{\omega(B(x, r))}{\omega(B(x, 2^{j+1}r))} \right)^{1/p} u(x, 2^{j+1}r) < Cu(x, r), \qquad (2.1)$$

then there exists $C > 0$ such that for any $f = \{f_i\}_{i \in \mathbb{Z}}$, $f_i \in L^1_{loc}(\mathbb{R}^n)$, $i \in \mathbb{Z}$

$$\sup_{\substack{y \in \mathbb{R}^n \\ r > 0}} \frac{1}{u(y, r)} \|\chi_{B(y,r)}\| \mathbb{M}(f)\|_{l^q} \|_{L^p(\omega)}$$

$$\leq C \sup_{\substack{y \in \mathbb{R}^n \\ r > 0}} \frac{1}{u(y, r)} \|\chi_{B(y,r)}\| f\|_{l^q} \|_{L^p(\omega)}. \qquad (2.2)$$

Proof.   Let $f = \{f_i\}_{i \in \mathbb{Z}} \subset L^1_{loc}(\mathbb{R}^n)$. For any $z \in \mathbb{R}^n$ and $r > 0$, write $f_i(x) = f_i^0(x) + \sum_{j=1}^{\infty} f_i^j(x)$, where $f_i^0 = \chi_{B(z,2r)} f_i$ and $f_i^j = \chi_{B(z,2^{j+1}r) \setminus B(z,2^j r)} f_i$, $j \in \mathbb{N}$. Applying the weighted Fefferman-Stein vector valued inequalities shown in [1] to $f^0 = \{f_i^0\}_{i \in \mathbb{Z}}$, we obtain $\|\|\mathbb{M}(f^0)\|_{l^q}\|_{L^p(\omega)} \leq C \|\|f^0\|_{l^q}\|_{L^p(\omega)}$. We have

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}\|\mathbb{M}(f^0)\|_{l^q}\|_{L^p(\omega)} \le C\frac{1}{u(z,2r)}\|\chi_{B(z,2r)}\|f\|_{l^q}\|_{L^p(\omega)}$$

$$\le C\sup_{\substack{y\in\mathbb{R}^n \\ r>0}}\frac{1}{u(y,r)}\|\chi_{B(y,r)}\|f\|_{l^q}\|_{L^p(\omega)}$$

because inequality (2.1) yields $u(z,2r) < Cu(z,r)$ for some constant $C > 0$ independent of $z \in \mathbb{R}^n$ and $r > 0$ and ω is a doubling measure.

As $f_i^j = \chi_{B(z,2^{j+1}r)\setminus B(z,2^j r)}f_i$ and $\text{dist}(B(z,r), B(z,2^{j+1}r)\setminus B(z,2^j r)) = (2^j - 1)r$, there is a constant $C > 0$ such that, for any $j \ge 1$ and $i \in \mathbb{Z}$

$$\chi_{B(z,r)}(x)(\mathrm{M}f_i^j)(x) \le C2^{-jn}r^{-n}\chi_{B(z,r)}(x)\int_{B(z,2^{j+1}r)}|f_i(y)|dy.$$

Since l^q is a Banach lattice, we find that

$$\chi_{B(z,r)}(x)\big\|\{(\mathrm{M}f_i^j)(x)\}_{i\in\mathbb{Z}}\big\|_{l^q}$$

$$\le C2^{-jn}r^{-n}\chi_{B(z,r)}(x)\int_{B(z,2^{j+1}r)}\|\{f_i(y)\}_{i\in\mathbb{Z}}\|_{l^q}dy.$$

Since $\omega \in A_p$, Hölder inequalities assert that

$$\int_{B(z,2^{j+1}r)}\|\{f_i(y)\}_{i\in\mathbb{Z}}\|_{l^q}dy$$

$$\le \left(\int_{B(z,2^{j+1}r)}\|\{f_i(y)\}_{i\in\mathbb{Z}}\|_{l^q}^p\omega(y)dy\right)^{1/p}\left(\int_{B(z,2^{j+1}r)}\omega^{-p'/p}dy\right)^{1/p'}$$

$$\le \frac{2^{(j+1)n}r^n}{(\omega(B(z,2^{j+1}r)))^{1/p}}\left(\int_{B(z,2^{j+1}r)}\|\{f_i(y)\}_{i\in\mathbb{Z}}\|_{l^q}^p\omega(y)dy\right)^{1/p}.$$

Subsequently,

$$\chi_{B(z,r)}(x)\big\|\{(\mathrm{M}f_i^j)(x)\}_{i\in\mathbb{Z}}\big\|_{l^q}$$

$$\le C\chi_{B(z,r)}(x)\frac{1}{(\omega(B(z,2^{j+1}r)))^{1/p}}\|\chi_{B(z,2^{j+1}r)}(y)\|\{f_i(y)\}_{i\in\mathbb{Z}}\|_{l^q}\|_{L^p(\omega)}.$$

Applying the norm $\|\cdot\|_{L^p(\omega)}$ on both sides of the above inequality, we have

$$\|\chi_{B(z,r)}\|\{(\mathrm{M}f_i^j)\}_{i\in\mathbb{Z}}\|_{l^q}\|_{L^p(\omega)}$$

$$\leq C\left(\frac{\omega(B(z,r))}{\omega(B(z,2^{j+1}r))}\right)^{1/p}\|\chi_{B(z,2^{j+1}r)}\|\{f_i\}_{i\in\mathbb{Z}}\|_{l^q}\|_{L^p(\omega)}.$$

Thus,

$$\|\chi_{B(z,r)}\|\mathbb{M}f^j\|_{l^q}\|_{L^p(\omega)}$$

$$\leq C\left(\frac{\omega(B(z,r))}{\omega(B(z,2^{j+1}r))}\right)^{1/p}\frac{u(z,2^{j+1}r)}{u(z,2^{j+1}r)}\|\chi_{B(z,2^{j+1}r)}(y)\|f\|_{l^q}\|_{L^p(\omega)}$$

$$\leq C\left(\frac{\omega(B(z,r))}{\omega(B(z,2^{j+1}r))}\right)^{1/p}u(z,2^{j+1}r)\sup_{\substack{y\in\mathbb{R}^n\\R>0}}\frac{1}{u(y,R)}\|\chi_{B(y,R)}\|f\|_{l^q}\|_{L^p(\omega)}.$$

Hence, using inequality (2.1), we obtain

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}\|\mathbb{M}f\|_{l^q}\|_{L^p(\omega)}\leq\frac{1}{u(z,r)}\sum_{j=0}^{\infty}\|\chi_{B(z,r)}\|\mathbb{M}f^j\|_{l^q}\|_{L^p(\omega)}$$

$$\leq C\sup_{\substack{y\in\mathbb{R}^n\\R>0}}\frac{1}{u(y,R)}\|\chi_{B(y,R)}\|f\|_{l^q}\|_{L^p(\omega)}$$

where the constant $C > 0$ is independent of r and z. Taking the supremum over $z \in \mathbb{R}^n$ and $r > 0$ yields (2.2). $\qquad\square$

The above theorem includes several Fefferman-Stein type vector-valued maximal inequalities [1], [22], [31], [42]. When $u(x,r) = (\omega(B(x,r)))^{\kappa/p}$ for some $0 < \kappa < 1$, inequality (2.2) is the vector-valued version of [22, Theorem 3.2]. If $\omega \equiv 1$, Theorem 2.1 offers a generalization of [31, Theorem 2] to vector-valued inequality. The above result is an extension of [42, Lemma 2.5] which is the Fefferman-Stein vector-valued maximal inequalities for classical Morrey spaces. In addition, the weighted Fefferman-Stein vector-valued inequalities in [1] is also a special case of Theorem 2.1 when $u \equiv 1$.

Moreover, a similar result of Theorem 2.1 is obtained by Sawano in [39, Theorem 2.5]. The above theorem also extends the results given in [22, Theorem 3.2] and [37, Theorem 2.4].

3.   Weighted Hardy-Morrey spaces

We introduce and study the weighted Morrey spaces in this section. Let $1 < q \leq p < \infty$, the classical Morrey space consists of those Lebesgue measurable functions f satisfying

$$\|f\|_{M_q^p} = \sup_{B \in \mathcal{B}} \frac{1}{|B|^{1/q-1/p}} \left(\int_B |f(x)|^q dx \right)^{1/q} < \infty.$$

For the study of the classical Morrey spaces, the reader is referred to [30], [34], [35], [46].

We obtain the weighted Morrey spaces by replacing the Lebesgue measure dx and the component $|B|^{1/q-1/p}$ by an A_p-weighted Lebesgue measure and a Morrey weight function defined in Definition 3.1, respectively.

Definition 3.1   Let $0 < p < \infty$ and $\omega \in A_\infty$. A Lebesgue measurable function $u : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ is said to be a *Morrey weight function for* ω if there exist a $0 \leq \lambda < \frac{1}{p}$ and constants $C_1, C_2 > 0$ so that for any $x \in \mathbb{R}^n$, $u(x, r) > C_1$, $r \geq 1$,

$$\frac{u(x, 2r)}{u(x, r)} \leq \left(\frac{\omega(B(x, 2r))}{\omega(B(x, r))} \right)^\lambda, \quad r > 0, \tag{3.1}$$

$$C_2^{-1} \leq \frac{u(x, t)}{u(y, r)} \leq C_2, \quad 0 < r \leq t \leq 2r \text{ and } |x - y| \leq t. \tag{3.2}$$

We denote the class of Morrey weight functions for ω by $\mathcal{W}_{\omega, p}$.

For any $B = B(x, r)$, $x \in \mathbb{R}^n$, $r > 0$, write $u(B) = u(x, r)$.

The subsequent lemma follows from Proposition 1.1 and (3.1).

Lemma 3.1   *Let* $1 \leq p < \infty$. *If* $\omega \in A_p$ *and* $u \in \mathcal{W}_{\omega, p}$, *then* ω *and* u *satisfy inequality* (2.1).

For any $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$, $Q_{j,k} = \{(x_1, x_2 \ldots, x_n) \in \mathbb{R}^n : k_i \leq 2^j x_i \leq k_i + 1, i = 1, 2, \ldots, n\}$. We write $|Q|$ and $l(Q)$ to be the Lebesgue measure of Q and the side length of Q, respectively. We denote the set of dyadic cubes $\{Q_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ by $\mathcal{Q}$.

For any dyadic cube $Q \in \mathcal{Q}$, write $u(Q) = u(x, r)$ where x is the center of Q and $r = l(Q)/2$.

Definition 3.2   Let $0 < p < \infty$, $\omega \in A_\infty$ and $u \in \mathcal{W}_{\omega,p}$. The *weighted Morrey space* $\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$ is the collection of all Lebesgue measurable functions f satisfying

$$\|f\|_{\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)} = \sup_{z\in\mathbb{R}^n, R>0} \frac{1}{u(z,R)} \|\chi_{B(z,R)} f\|_{L^p(\omega)} < \infty.$$

The family of weighted Morrey spaces in the above definition covers the classical Morrey spaces and the weighted Morrey spaces considered in [22, Definition 2.1].

Condition (3.2) ensures that

$$\|f\|_{\mathcal{M}^p_{\omega,u}(\mathcal{Q})} = \sup_{Q\in\mathcal{Q}} \frac{1}{u(Q)} \|\chi_Q f\|_{L^p(\omega)} \tag{3.3}$$

is an equivalent quasi-norm of $\|\cdot\|_{\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)}$. Moreover, the conditions imposed on u in Definition 3.1 guarantee that $\chi_{B(x,r)} \in \mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$.

Lemma 3.2   *For any $x\in\mathbb{R}^n$ and $r>0$, $\chi_{B(x,r)} \in \mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$.*

Proof.   For any $k\in\mathbb{Z}$, write $B_k = B(z,2^k)$. When $k<0$, $B(z,2^k) \cap B(x,r) = \emptyset$ if $|z-x| > r+2$. Thus, we find that

$$\begin{aligned}
\frac{\|\chi_{B(x,r)\cap B_k}\|_{L^p(\omega)}}{u(z,2^k)} &\leq \left(\frac{\omega(B(z,1))}{\omega(B(z,2^k))}\right)^\lambda \frac{(\omega(B(z,2^k)))^{1/p}}{u(z,1)} \\
&\leq \left(\frac{\omega(B(z,1))}{\omega(B(z,2^k))}\right)^\lambda \left(\frac{\omega(B(z,2^k))}{\omega(B(z,1))}\right)^{1/p} \frac{(\omega(B(z,1)))^{1/p}}{u(z,1)} \\
&\leq C(\omega(B(x,r+2)))^{1/p}
\end{aligned}$$

because $u(z,1) \geq C$ and $\lambda < \frac{1}{p}$. For $k \geq 0$, we have

$$\frac{\|\chi_{B(x,r)\cap B_k}\|_{L^p(\omega)}}{u(z,2^k)} \leq C_1^{-1} \|\chi_{B(x,r)}\|_{L^p(\omega)}.$$

In view of (3.2), the above inequalities guarantee that $\chi_{B(x,r)} \in \mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$. $\square$

Lemma 3.2 plays an important role on the study of Morrey type spaces.

In addition, the reader is referred to [6] for some similar ideas used to study Morrey spaces.

We now ready to define the Hardy-Morrey space via the Littlewood-Paley functions. For the development of the Littlewood-Paley characterization of function spaces, the reader is referred to [8], [9], [10], [13], [18], [24], [38], [44], [45].

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the classes of tempered functions and Schwartz distributions, respectively. Let $\mathcal{P}$ denote the class of polynomials in $\mathbb{R}^n$.

Definition 3.3 Let $0 < p \le 1$, $\omega \in A_\infty$ and $u \in \mathcal{W}_{\omega,p}$. The weighted Hardy-Morrey spaces $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ consists of those $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ such that

$$\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)} = \left\|\left(\sum_{\nu \in \mathbb{Z}} |\varphi_\nu * f|^2\right)^{1/2}\right\|_{\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)} < \infty$$

where $\mathcal{P}$ denotes the set of polynomials on $\mathbb{R}^n$ and $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$, $\nu \in \mathbb{Z}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\operatorname{supp} \hat{\varphi} \subseteq \{x \in \mathbb{R}^n : 1/2 \le |x| \le 2\} \quad \text{and}$$

$$|\hat{\varphi}(\xi)| \ge C, \quad 3/5 \le |x| \le 5/3 \tag{3.4}$$

for some $C > 0$.

In fact, we can also define and study the corresponding local version of weighted Hardy-Morrey spaces. For brevity, we leave the details to the reader. The reader may consult [15] for the definition of local Hardy space.

As demonstrated in [8], any function space having the Littlewood-Paley characterization is associated with a sequence space. This sequence space is introduced in order to study the ϕ-ψ transform.

Definition 3.4 Let $0 < p \le 1$, $\omega \in A_\infty$ and $u \in \mathcal{W}_{\omega,p}$. The sequence space $\mathfrak{h}^p_{\omega,u}$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q \in \mathcal{Q}}$ such that

$$\|s\|_{\mathfrak{h}^p_{\omega,u}} = \left\|\left(\sum_Q (|s_Q| \tilde{\chi}_Q)^2\right)^{1/2}\right\|_{\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)} < \infty,$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$.

We recall the definition of the ϕ-ψ transform introduced by Frazier and Jawerth in [8], [9], [10]. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}\hat\varphi, \ \operatorname{supp}\hat\psi \subseteq \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}, \qquad (3.5)$$

$$|\hat\varphi(\xi)|, \ |\hat\psi(\xi)| \ge C \quad \text{if } 3/5 \le |\xi| \le 5/3 \text{ for some } C > 0, \qquad (3.6)$$

$$\sum_{\nu \in \mathbb{Z}} \overline{\hat\varphi(2^{-\nu}\xi)}\hat\psi(2^{-\nu}\xi) = 1 \quad \text{if } \xi \ne 0 \qquad (3.7)$$

where $\hat\varphi$ denote the Fourier transform of φ and similarly for $\hat\psi$.

Write $\tilde\varphi(x) = \overline{\varphi(-x)}$. We set $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$, $\psi_\nu(x) = 2^{\nu n}\psi(2^\nu x)$ and

$$\varphi_Q(x) = |Q|^{-1/2}\varphi(2^\nu x - k), \ \psi_Q(x) = |Q|^{-1/2}\psi(2^\nu x - k), \quad \nu \in \mathbb{Z}, \ k \in \mathbb{Z}^n$$

for $Q = Q_{\nu,k} \in \mathcal{Q}$. For any $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ and for any complex-valued sequences $s = \{s_Q\}$, we define

$$\mathrm{S}_\varphi(f) = \{(\mathrm{S}_\varphi f)_Q\}_{Q \in \mathcal{Q}} = \{\langle f, \varphi_Q\rangle\}_{Q \in \mathcal{Q}} \quad \text{and} \quad \mathrm{T}_\psi(s) = \sum_Q s_Q \psi_Q.$$

We find that $\mathrm{T}_\psi \circ \mathrm{S}_\varphi = \mathrm{id}$ in $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ because $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ is a subspace of $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ (see [8, Theorem 2.2]). The following theorem is a special case of [18, Theorem 3.1]. Thus, for the sake of brevity, we omit the detail.

Theorem 3.3 *The weighted Hardy-Morrey space $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ is independent of the function φ in Definition 3.3. The operators S_φ and T_ψ are bounded operators on $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ and $\mathfrak{h}^p_{\omega,u}$, respectively. Moreover, we have constants $C_1 > C_2 > 0$ such that, for any $f \in \mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$,*

$$C_2\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)} \le \|\mathrm{S}_\varphi(f)\|_{\mathfrak{h}^p_{\omega,u}} \le C_1\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)}. \qquad (3.8)$$

As $L^r(\omega)$ and l^q satisfy (2.2) when $1 < q, \ r < \infty$ and $\omega \in A_r$, the pair $(l^2, \mathcal{M}^p_{\omega,u}(\mathbb{R}^n))$ is so-called a-admissible with $0 < a < \frac{1}{p}$, in [18]. Thus, in view of Lemma 3.1, Theorem 3.3 is a special case of [18, Theorem 3.1]. Notice that the use of the condition $u \in \mathcal{W}_{\omega,p}$ is given in the general result in [18]. In particular, the reader may consult [18, Theorem 5.5] on the use of

the above condition for the study of the Littlewood-Paley characterization of Morrey type spaces.

We recall the definition of smooth atoms from [10, p. 46]. For each dyadic cube Q, A_Q is a smooth N-atom for $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$, $N \in \mathbb{N}$, if it satisfies

$$\int x^\gamma A_Q(x)dx = 0 \quad \text{for} \quad 0 \le |\gamma| \le N, \ \gamma \in \mathbb{N}^n, \tag{3.9}$$

$$\text{supp } A_Q \subseteq 3Q, \tag{3.10}$$

and for $\gamma \in \mathbb{N}^n$,

$$|\partial^\gamma A_Q(x)| \le C_\gamma |Q|^{-1/2-|\gamma|/n}. \tag{3.11}$$

The validity of the following smooth atomic decomposition follows from the boundedness of the ϕ-ψ transform and the Fefferman-Stein vector-valued maximal inequalities on weighted Morrey spaces. For simplicity, we only provide an outline of the proof for the following result. For the detail of the establishment of the smooth atomic decomposition of function spaces, the reader is referred to [10, p. 46–p. 48].

Theorem 3.4 (Smooth Atomic Decomposition)   *Let $0 < p \le 1$, $\omega \in A_\infty$ and $u \in \mathcal{W}_{\omega,p}$. For any $N \ge [n(q_\omega/p-1)]$ and $N \in \mathbb{N}$, if $f \in \mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$, then there exist a sequence $s = \{s_Q\}_{Q \in \mathcal{Q}} \in \mathfrak{h}^p_{\omega,u}$ and a family of smooth N-atoms $\{A_Q\}_{Q \in \mathcal{Q}}$ such that $f = \sum_{Q \in \mathcal{Q}} s_Q A_Q$ and $\|s\|_{\mathfrak{h}^p_{\omega,u}} \le C\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)}$ for some constant $C > 0$.*

Proof.   According to [10, Lemma 5.12], for any $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, we have $f = \sum_{Q \in \mathcal{Q}} s_Q a_Q$, where each a_Q is a smooth N-atom and s_Q satisfy

$$\sum_{|Q|=2^{-jn}} |s_Q| \tilde{\chi}_Q(x) \le C\big(\text{M}(|\tilde{\varphi}_j * f|^h)(x)\big)^{1/h}$$

for some $\varphi \in \mathcal{S}(\mathbb{R}^n)$ fulfilling (3.3) and some positive h sufficiently close to zero. Thus, the inequality $\|s\|_{\mathfrak{h}^p_{\omega,u}} \le C\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)}$ follows from Theorem 2.1. $\qquad \square$

Theorem 3.4 can be considered as a special case of [36, Theorem 5.8]. Notice that the size of N given by the above decomposition reduces to the usual vanishing moment condition imposed on the smooth atoms for Hardy

spaces when $\omega \equiv 1$ and $u \equiv 1$, see [8, Section 4].

4.   Non-smooth atomic decompositions

One of the remarkable features of the Hardy type spaces is the non-smooth atomic decompositions [5], [8], [9], [17], [25]. The non-smooth atomic decompositions have profound applications on the boundedness of singular integral operators, the reader may refer to [16, Section 6.7] for detail.

We use the approach given in [9] to obtain the non-smooth atomic decompositions for the weighted Hardy-Morrey spaces. We recall some definitions and modify some notations from [9]. For any sequence $s = \{s_Q\}_{Q \in \mathcal{Q}}$, we call

$$g(s) = \left(\sum_{Q \in \mathcal{Q}} (|s_Q| \tilde{\chi}_Q)^2 \right)^{1/2}$$

the Littlewood-Paley function of s. So, $\|s\|_{\mathfrak{h}^p_{\omega,u}} = \|g(s)\|_{\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)}$. We first define the atoms for the sequence spaces $\mathfrak{h}^p_{\omega,u}$.

Definition 4.1   A sequence $r = \{r_Q\}_{Q \in \mathcal{Q}}$ is an ∞-atom for $\mathfrak{h}^p_{\omega,u}$ if there exists a dyadic cube $P \in \mathcal{Q}$ such that $r_Q = 0$ if $Q \not\subset P$ and $\|g(r)\|_{L^\infty} \le \omega(P)^{-1/p}$.

We call P the support of r and write $\text{supp}(r) = P$.

The reader is referred to [9, p. 403] for the definition of ∞-atom for Hardy space.

Definition 4.2   A family of ∞-atoms indexed by $\mathcal{Q}$, $\{r_J\}_{J \in \mathcal{Q}}$, is called an ∞-atomic family for $\mathfrak{h}^p_{\omega,u}$ if $\text{supp}(r_J) = J$.

We now give the sequence spaces associated with the weighted Morrey spaces.

Definition 4.3   Let $0 < p \le 1$, $\omega \in A_\infty$ and $u \in \mathcal{W}_{\omega,p}$. The sequence space $m^p_{\omega,u}$ consists of those complex-valued sequence $t = \{t_Q\}_{Q \in \mathcal{Q}}$ satisfying

$$\|t\|_{m^p_{\omega,u}} = \sup_{Q \in \mathcal{Q}} \frac{1}{u(Q)} \left(\sum_{J \subseteq Q} |t_J|^p \right)^{1/p} < \infty.$$

We follow the idea in [9, Theorem 7.3] to establish the atomic decompositions for the sequence spaces $\mathfrak{h}_{\omega,u}^p$.

Theorem 4.1 *Let* $0 < p \le 1$, $\omega \in A_\infty$ *and* $u \in \mathcal{W}_{\omega,p}$. *For any* $s \in \mathfrak{h}_{\omega,u}^p$, *there exist an* ∞-*atomic family for* $\mathfrak{h}_{\omega,u}^p$, $\{r_J\}_{J \in \mathcal{Q}}$ *and a sequence* $t = \{t_J\}_{J \in \mathcal{Q}} \in m_{\omega,u}^p$ *such that* $s = \sum_{J \in \mathcal{Q}} t_J r_J$ *and* $\|t\|_{m_{\omega,u}^p} \le C \|s\|_{\mathfrak{h}_{\omega,u}^p}$ *for some constants* $C > 0$.

Proof. For any $P \in \mathcal{Q}$, write

$$g_P(s) = \left(\sum_{Q \in \mathcal{Q}, P \subseteq Q} (|Q|^{-1/2}|s_Q|)^2 \right)^{1/2}.$$

We find that whenever $P_1 \subseteq P_2$, $0 \le g_{P_2}(s) \le g_{P_1}(s)$. In addition, for any given $x \in \mathbb{R}^n$, $g_P(s)$ satisfies the following properties

$$\lim_{l(P) \to \infty, x \in P} g_P(s) = 0, \tag{4.1}$$

$$\lim_{l(P) \to 0, x \in P} g_P(s) = g(s)(x). \tag{4.2}$$

For any $k \in \mathbb{Z}$, write

$$\mathcal{A}_k = \{P \in \mathcal{Q} : g_P(s) > 2^k\}.$$

Identity (4.2) assures that

$$\{x \in \mathbb{R}^n : g(s)(x) > 2^k\} = \bigcup_{P \in \mathcal{A}_k} P. \tag{4.3}$$

Moreover, we have

$$\left(\sum_{P \in \mathcal{Q} \setminus \mathcal{A}_k} (|s_P| \tilde{\chi}_P(x))^2 \right)^{1/2} \le 2^k, \quad \forall x \in \mathbb{R}^n. \tag{4.4}$$

We prove (4.4). Whenever $g(s)(x) \le 2^k$, the above inequality is obviously valid. Therefore, we only need to consider $g(s)(x) > 2^k$. We first show that maximal dyadic cubes exist in $\mathcal{A}_k$. If not, there exists a family of dyadic cubes $\{P_j\}_{j \in \mathbb{N}} \subset \mathcal{A}_k$ such that $P_i \subset P_l$, $i < l$ and $\lim_{i \to \infty} l(P_i) = \infty$. Thus, for any $x \in P_0$, we have $\liminf_{i \to \infty, x \in P_i} g_{P_i}(s) > 2^k$ which contradicts

(4.1).

Whenever $g(s)(x) > 2^k$, (4.3) and the above arguments assert that there exists a maximal dyadic cube $P_{\max} \in \mathcal{A}_k$ such that $x \in P_{\max}$. Thus, we have the unique dyadic cube $R \in \mathcal{Q}$ satisfying $P_{\max} \subseteq R$ and $2l(P_{\max}) = l(R)$. The left hand side of (4.4) is precisely $g_R(s)$ and, hence, it is less than 2^k because $R \notin \mathcal{A}_k$.

For any $k \in \mathbb{Z}$, let $\mathcal{B}_k$ denote the set of maximal dyadic cubes in $\mathcal{A}_k \backslash \mathcal{A}_{k+1}$. As maximal dyadic cubes exist in $\mathcal{A}_k$, $\mathcal{B}_k$ is well defined. According to the proof of [9, Theorem 7.3], for any $J \in \mathcal{B}_k$, the family of sequences $\beta_J = \{(\beta_J)_Q\}_{Q \in \mathcal{Q}}$ defined by

$$(\beta_J)_Q = \begin{cases} s_Q, & Q \subseteq J \text{ and } Q \in \mathcal{A}_k \backslash \mathcal{A}_{k+1}, \\ 0, & \text{otherwise}, \end{cases}$$

satisfy $s = \sum_{J \in \mathcal{Q}} \beta_J$ and $|g(\beta_J)| \le 2^{k+1}$.

Let $r_J = \omega(J)^{-1/p} 2^{-k-1} \beta_J$ and $t_J = \omega(J)^{1/p} 2^{k+1}$. As

$$\mathcal{Q} = \left(\bigcup_{k=-\infty}^{\infty} \left(\bigcup_{J \in \mathcal{B}_k} \{Q \in \mathcal{Q} : Q \subset J\} \right) \right) \bigcup \{Q \in \mathcal{Q} : s_Q = 0\}$$

is a disjoint union, we find that $s = \sum_{J \in \mathcal{Q}} t_J r_J$ and $\{r_J\}_{J \in \mathcal{Q}}$ is an ∞-atomic family for $\mathfrak{h}_{\omega,u}^p$. Furthermore, we find that for any $R \in \mathcal{Q}$,

$$\sum_{J \subseteq R} |t_J|^p = \sum_{k \in \mathbb{Z}} 2^{(k+1)p} \sum_{J \in \mathcal{B}_k, J \subseteq R} \omega(J) \le 2^p \sum_{k \in \mathbb{Z}} 2^{kp} \omega \left(\bigcup_{J \in \mathcal{A}_k, J \subseteq R} J \right)$$

$$\le 2^p \sum_{k \in \mathbb{Z}} 2^{kp} \omega(\{x \in R : 2^k < g(s)(x)\}) \le C \|\chi_R g(s)\|_{L^p(\omega)}^p.$$

On both sides, taking the pth root, multiplying by $\frac{1}{u(R)}$ and, then, taking the supremum over $R \in \mathcal{Q}$, we obtain

$$\|t\|_{m_{\omega,u}^p} = \sup_{R \in \mathcal{Q}} \frac{1}{u(R)} \left(\sum_{J \subseteq R} |t_J|^p \right)^{1/p}$$

$$\le C \sup_{R \in \mathcal{Q}} \frac{1}{u(R)} \|\chi_R g(s)\|_{L^p(\omega)} = C \|s\|_{\mathfrak{h}_{\omega,u}^p}. \qquad \square$$

Corollary 4.2 *Let $0 < p \leq 1$, $\omega \in A_\infty$ and $u \in \mathcal{W}_{\omega,p}$. Then,*

$$\|s\|_{\mathfrak{h}^p_{\omega,u}} \approx \inf \left\{ \|t\|_{m^p_{\omega,u}} : s = \sum_{J \in \mathcal{Q}} t_J r_J, \ t = \{t_J\}_{J \in \mathcal{Q}} \ and \right.$$

$$\left. \{r_J\}_{J \in \mathcal{Q}} \ is \ an \ \infty\text{-}atomic \ family \ for \ \mathfrak{h}^p_{\omega,u} \right\}.$$

Proof. It remains to show that for any $t = \{t_J\}_{J \in \mathcal{Q}} \in m^p_{\omega,u}$ and any ∞-atomic family $\{r_J\}_{J \in \mathcal{Q}}$, we have $\|\sum_{J \in \mathcal{Q}} t_J r_J\|_{\mathfrak{h}^p_{\omega,u}} \leq \|t\|_{m^p_{\omega,u}}$.

Since each r_J is an ∞-atom for $\mathfrak{h}^p_{\omega,u}$, we have $\|g(r_J)\|_{L^p(\omega)} \leq 1$. Hence, for any $R \in \mathcal{Q}$, by the p-triangle inequality, we assert that

$$\left\| \chi_R g \left(\sum_{J \in \mathcal{Q}} t_J r_J \right) \right\|^p_{L^p(\omega)} \leq \sum_{J \in \mathcal{Q}, J \subseteq R} |t_J|^p \int |g(r_J)|^p \omega(x) dx$$

$$\leq \sum_{J \in \mathcal{Q}, J \subseteq R} |t_J|^p.$$

Our desired inequality follows by taking the pth root, multiplying $\frac{1}{u(R)}$ and, then, taking the supremum over $R \in \mathcal{Q}$ on both sides. $\square$

In order to present the main result of this section, we recall the definition of non-smooth atoms on weighted function spaces [3], [11], [20], [26], [41].

Definition 4.4 Let $0 < p \leq 1 < r < \infty$ and $\omega \in A_\infty$. For any $N \geq [n(r/p - 1)]$ and $N \in \mathbb{N}$, a family of functions $\{a_Q\}_{Q \in \mathcal{Q}}$ is called a (p, r, N)-atomic family with respect to ω if

$$\operatorname{supp} a_Q \subseteq 3Q, \quad \forall Q \in \mathcal{Q},$$

$$\int x^\gamma a_Q(x) dx = 0, \quad \forall \gamma \in \mathbb{N}^n \text{ with } 0 \leq |\gamma| \leq N,$$

$$\|a_Q\|_{L^r(\omega)} \leq \omega(Q)^{1/r - 1/p}.$$

We now transfer our results of non-smooth atomic decompositions for the sequence spaces $\mathfrak{h}^p_{\omega,u}$ to the corresponding results for function spaces $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$. It consists of two results. The first one, Theorem 4.3, is a decomposition theorem and the second one, Theorem 4.4, is a reconstruction theorem.

Theorem 4.3 *Let $0 < p \leq 1$, $\omega \in A_\infty$, $q_\omega < q < \infty$ and $u \in \mathcal{W}_{\omega,p}$. For any $f \in \mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ and any positive integer satisfying $N \geq [n(q/p - 1)]$, there exist a (p, q, N)-atomic family with respect to ω, $\{a_Q\}_{Q \in \mathcal{Q}}$, and a sequence $t = \{t_Q\}_{Q \in \mathcal{Q}} \in m^p_{\omega,u}$ such that $f = \sum_{Q \in \mathcal{Q}} t_Q a_Q$ and $\|t\|_{m^p_{\omega,u}} \leq C\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)}$ for some $C > 0$.*

Proof. As given by Theorem 3.4, for any $f \in \mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ and $N \geq [n(q/p - 1)]$, there exist a family of smooth N-atoms $\{A_Q\}_{Q \in \mathcal{Q}}$ and a sequence $s = \{s_Q\}_{Q \in \mathcal{Q}} \in \mathfrak{h}^p_{\omega,u}$ so that $f = \sum_{Q \in \mathcal{Q}} s_Q A_Q$ and $\|s\|_{\mathfrak{h}^p_{\omega,u}} \leq C\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)}$.

According to Corollary 4.2, we have $t = \{t_J\}_{J \in \mathcal{Q}} \in m^p_{\omega,u}$ and an ∞-atomic family for $\mathfrak{h}^p_{\omega,u}$, $\{r_J\}_{J \in \mathcal{Q}}$, such that $s = \sum_{J \in \mathcal{Q}} t_J r_J$ and $\|t\|_{m^p_{\omega,u}} \leq 2\|s\|_{\mathfrak{h}^p_{\omega,u}}$. Thus, f can be rewritten as

$$f = \sum_{Q \in \mathcal{Q}} s_Q A_Q = \sum_{Q \in \mathcal{Q}} \left(\sum_{J \in \mathcal{Q}} t_J r_J \right)_Q A_Q = \sum_{J \in \mathcal{Q}} t_J a_J$$

where $a_J = \sum_{Q \subseteq J} (r_J)_Q A_Q$. Since $\operatorname{supp} A_Q \subseteq 3Q$ and $Q \subseteq J$, we have $\operatorname{supp} a_J \subseteq 3J$.

In view of the Littlewood-Paley characterization of weighted Lebesgue spaces $L^q(\omega) = \dot{F}^{02}_q(\omega)$, $\omega \in A_q$, $1 < q$ [24, Theorem 3.1], the boundedness of the φ-transform from $\dot{F}^{02}_q(\omega)$ to $\dot{f}^{02}_q(\omega)$ and the boundedness of the ψ-transform from $\dot{f}^{02}_q(\omega)$ to $\dot{F}^{02}_q(\omega)$ [8, Proposition 10.14], we obtain

$$\|a_J\|_{L^q(\omega)} \leq C\|g(r_J)\|_{L^q(\omega)} \leq C\omega(J)^{1/q - 1/p}$$

for some $C > 0$. The vanishing moment conditions for a_J are inherited from the corresponding conditions from $\{A_Q\}_{Q \in \mathcal{Q}}$. Thus, $\{a_J\}_{J \in \mathcal{Q}}$ is a (p, q, N)-atomic family with respect to ω and $\|t\|_{m^p_{\omega,u}} \leq C\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)}$. $\qquad\square$

We find that for the reconstruction theorem of the atomic decompositions for $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$, we need an extra condition for $u(x, r)$.

Definition 4.5 Let $0 < p \leq 1$, $0 \leq \kappa < \frac{1}{p}$ and $\omega \in A_\infty$. A weight function u belongs to $\mathcal{W}_{\omega,p,\kappa}$ if and only if $u \in \mathcal{W}_{\omega,p}$ and for any $P, Q \in \mathcal{Q}$ with $P \subseteq Q$,

$$\left(\frac{\omega(P)}{\omega(Q)} \right)^\kappa \leq \frac{u(P)}{u(Q)}. \tag{4.5}$$

Roughly speaking, (3.1) controls the "growth of u" in term of ω while (4.5) imposes a restriction on the "decay of u".

Condition (4.5) is related to a technical difficulty generated by the Morrey weight functions. For details, the reader is referred to Lemma 5.4 in Section 5.

Theorem 4.4 *Let $0 < p \le 1$, $\omega \in A_\infty$ and $q_\omega < q$. Suppose that $u \in \mathcal{W}_{\omega,p,\kappa}$ and $t = \{t_Q\}_{Q \in \mathcal{Q}} \in m^p_{\omega,u}$. If $\{a_Q\}_{Q \in \mathcal{Q}}$ is a (p,q,N)-atomic family with respect to ω and q satisfying $\frac{1}{q} < \frac{1}{p} - \kappa$, then $f = \sum_{Q \in \mathcal{Q}} t_Q a_Q \in \mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ and $\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)} \le C\|t\|_{m^p_{\omega,u}}$ for some $C > 0$.*

Proof. Let $f = \sum_{Q \in \mathcal{Q}} t_Q a_Q$ where $t = \{t_Q\}_{Q \in \mathcal{Q}} \in m^p_{\omega,u}$ and $\{a_Q\}_{Q \in \mathcal{Q}}$ be a (p,q,N)-atomic family with respect to ω.

For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying the conditions in Definition 3.3 and for any $h \in \mathcal{S}'(\mathbb{R}^n)$, define the Lebesgue measurable function $\mathcal{G}(h)$ by

$$\mathcal{G}(h) = \left(\sum_{\nu \in \mathbb{Z}} |(h * \varphi_\nu)|^2 \right)^{1/2}.$$

When $x \in \mathbb{R}^n \backslash 4Q$, we use the vanishing moment condition satisfied by a_Q to obtain

$$|(a_Q * \varphi_\nu)(x)| \le \int_{\mathbb{R}^n} \left| a_Q(y) \left(\varphi_\nu(x - y) - \sum_{|\gamma| \le N} \frac{(y - x_Q)^\gamma}{\gamma!} \partial^\gamma \varphi_\nu(x - x_Q) \right) \right| dy.$$

By using the reminder terms of the Taylor expansion of φ_ν, we have

$$|(a_Q * \varphi_\nu)(x)| \le \int_{\mathbb{R}^n} |a_Q(y)| \sum_{|\gamma| = N+1} \left| \frac{(y - x_Q)^\gamma}{\gamma!} \partial^\gamma \varphi_\nu(x - y + \theta(y - x_Q)) \right| dy$$

for some $0 \le \theta \le 1$. Since $y \in Q$, we have $|y - x_Q|^\gamma \le |Q|^{\frac{N+1}{n}}$ for any $|\gamma| = N + 1$. Moreover, for any $y \in Q$,

$$|x - y + \theta(y - x_Q)| \ge |x - x_Q| - (1 - \theta)|y - x_Q| \ge \frac{1}{2}|x - x_Q|.$$

We obtain

$$|(a_Q * \varphi_\nu)(x)| \leq C 2^{(N+n+1)\nu} |Q|^{(N+1)/n} (1 + 2^\nu |x - x_Q|)^{-M} \int_{3Q} |a_Q(y)| dy$$

for some sufficient large $M > 0$. The Hölder inequality and the definition of A_q yield

$$\int_{3Q} |a_Q(y)| dy \leq \left(\int_{3Q} |a_Q(y)|^q \omega(y) dy \right)^{1/q} \left(\int_{3Q} \omega^{-q'/q}(y) dy \right)^{1/q'}$$

$$\leq C \omega(Q)^{1/q - 1/p} |Q| \omega(Q)^{-1/q} = C \omega(Q)^{-1/p} |Q| \qquad (4.6)$$

where q' is the conjugate of q.

Let $K \in \mathbb{Z}$ satisfy $(\log_2 |x - x_Q|^{-1}) - 1 < K \leq \log_2 |x - x_Q|^{-1}$. We find that

$$\sum_{\nu \in \mathbb{Z}} 2^{(N+n+1)\nu} (1 + 2^\nu |x - x_Q|)^{-M}$$

$$= \sum_{\nu = -\infty}^{K} 2^{(N+n+1)\nu} (1 + 2^\nu |x - x_Q|)^{-M}$$

$$+ \sum_{\nu = K+1}^{\infty} 2^{(N+n+1)\nu} (1 + 2^\nu |x - x_Q|)^{-M}$$

$$\leq C \left(\sum_{\nu = -\infty}^{K} 2^{(N+n+1)\nu} + \sum_{\nu = K+1}^{\infty} 2^{(N+n+1-M)\nu} |x - x_Q|^{-M} \right)$$

$$\leq C |x - x_Q|^{-N-n-1}.$$

Since $l^1 \hookrightarrow l^2$, we have

$$\left(\sum_{\nu \in \mathbb{Z}} |a_Q * \varphi_\nu(x)|^2 \right)^{1/2} \leq C |x - x_Q|^{-N-n-1} \omega(Q)^{-1/p} |Q|^{(1+(N+1)/n)}$$

$$\leq C \omega(Q)^{-1/p} \left(1 + \frac{|x - x_Q|}{l(Q)} \right)^{-N-n-1}$$

for some $C > 0$ independent of the family $\{a_Q\}_{Q \in \mathcal{Q}}$.

Therefore,

$$\mathcal{G}(a_Q)(x) \leq \mathcal{G}(a_Q)(x)\chi_{4Q}(x) + C\omega(Q)^{-1/p}\chi_{\mathbb{R}^n \setminus 4Q}\left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-N-n-1}$$

$$= X_Q(x) + Y_Q(x)$$

and

$$\mathcal{G}(f) \leq \sum_{Q \in \mathcal{Q}} |t_Q| X_Q + \sum_{Q \in \mathcal{Q}} |t_Q| Y_Q = X + Y.$$

For any $P \in \mathcal{Q}$, the p-triangle inequality yields

$$\|\chi_P X\|_{L^p(\omega)}^p \leq \sum_{Q \in \mathcal{Q}} |t_Q|^p \int_{P \cap 4Q} |\mathcal{G}(a_Q)|^p d\omega.$$

We use the Hölder inequality and the Littlewood-Paley characterization of $L^q(\omega)$ to obtain

$$\int_{P \cap 4Q} |\mathcal{G}(a_Q)|^p d\omega \leq \left(\int_{\mathbb{R}^n} |\mathcal{G}(a_Q)|^q d\omega\right)^{p/q} \omega(P \cap 4Q)^{1-p/q}$$

$$\leq C\|a_Q\|_{L^q(\omega)}^p \omega(P \cap 4Q)^{1-p/q} \leq C\left(\frac{\omega(P \cap 4Q)}{\omega(Q)}\right)^{1-p/q}$$

for some $C > 0$ independent of Q. Thus, Lemma 5.4, given in the next section, assures that

$$\|X\|_{\mathcal{M}_{\omega,u}^p(\mathbb{R}^n)} \leq C\|t\|_{m_{\omega,u}^p}. \tag{4.7}$$

For the function Y, we have

$$Y \leq \sum_{Q \in \mathcal{Q}} |t_Q| \omega(Q)^{-1/p}\left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-N-n-1}$$

$$\leq C \sum_{\mu \in \mathbb{Z}} \left(M\left(\sum_{l(Q)=2^{-\mu}} |t_Q| \omega(Q)^{-1/p}\chi_Q\right)^{1/h}\right)^h$$

for some $h > 1$ satisfying $hp > q_\omega$. We have such h because $p(N+n+1) > nq$. Hence, for any $P \in \mathcal{Q}$,

$$\frac{1}{u(P)}\|\chi_P Y\|_{L^p(\omega)}$$

$$\leq C\bigg(\frac{1}{(u(P))^{1/h}}\bigg(\int_P \bigg(\sum_{\mu\in\mathbb{Z}}\bigg(\mathrm{M}\bigg(\sum_{l(Q)=2^{-\mu}}|t_Q|$$

$$\times\, \omega(Q)^{-1/p}\chi_Q\bigg)^{1/h}\bigg)^h\bigg)^{(1/h)ph}d\omega\bigg)^{1/ph}\bigg)^h.$$

We are allowed to apply Theorem 2.1 for the pair $(l^h, L^{ph}(\omega))$ because $u \in \mathcal{W}_{\omega,p}$ implies $u^{1/h} \in \mathcal{W}_{\omega,ph} \subset \mathcal{W}_{\omega,q_\omega}$. Consequently, we find that

$$\|Y\|_{\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)} \leq C\sup_{P\in\mathcal{Q}}\frac{1}{u(P)}\bigg(\int_P\sum_{Q\in\mathcal{Q}}|t_Q|^p\omega(Q)^{-1}\chi_Q d\omega\bigg)^{1/p} \leq C\|t\|_{m^p_{\omega,u}}.$$

The above inequality and (4.7) establish our desired result. $\qquad\square$

Theorem 4.4 shows that the atomic series $\sum_{Q\in\mathcal{Q}} t_Q a_Q$ belongs to $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ provided that the family of atoms $\{a_Q\}_{Q\in\mathcal{Q}}$ satisfies a sufficiently high order of integrability.

More precisely, for any $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$, there exists a $q_0 > 1$ such that whenever the family of atoms $\{a_Q\}_{Q\in\mathcal{Q}}$ are elements in $L^q(\omega)$ for $q_0 < q < \infty$, then, for any $\{t_Q\}_{Q\in\mathcal{Q}} \in m^p_{\omega,u}$, $\sum_{Q\in\mathcal{Q}} t_Q a_Q$ belongs to $f \in \mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$.

For the classical Hardy spaces, we have $u \equiv 1$ and, hence, (4.5) is obviously satisfied. Thus, the non-smooth atomic decompositions for the classical Hardy spaces are valid for any (p, q, N)-atomic family with $1 < q < \infty$. In addition, Theorem 4.4 matches with the result in [20] since the atoms used in [20, Definition 1.4] are elements in L^∞. Furthermore, it is also consistent with the results given in Theorem 4.3 because the atoms obtained at there are indeed elements in $L^q(\omega)$ for any $q_\omega < q < \infty$.

We now present an application of the above atomic decomposition to the study of singular integral operator on $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$. We call a linear operator T a Calderón-Zygmund type operator for $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ if its Schwartz kernel $K(x,y)$ satisfying for all $x, z \in \mathbb{R}^n$ with $x \neq z$,

$$\big|(\partial_y^\gamma K)(x,z)\big| \leq C|x-z|^{-n-|\gamma|}, \quad \forall \gamma \in \mathbb{N}^n,\ |\gamma| \leq [n(r/p-1)]+1 \quad (4.8)$$

for some $C > 0$.

The above definition includes those singular integrals on Hardy spaces studied in [16, Section 6.7.3]. In particular, the Schwartz kernel of the Hilbert transform satisfies (4.8). For the details of the definition of non-convolution type Calderón-Zygmund operators and theirs action on function spaces, the reader is referred to [43].

Theorem 4.5   *Let $0 < p \leq 1$, $0 \leq \kappa < \frac{1}{p}$, $\omega \in A_\infty$ and $u \in \mathcal{W}_{\omega,p,\kappa}$. If T is a Calderón-Zygmund type operator for $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$, then T is bounded from $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$ to $\mathcal{M}^p_{\omega,u}(\mathbb{R}^n)$.*

Proof.   For simplicity, we just sketch the proof. Define $L \in \mathbb{N}$ by $L = [n(r/p - 1)]$. Pick $q > q_\omega$. For any $f \in \mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$, Theorem 4.3 yields $f = \sum_{Q \in \mathcal{Q}} t_Q a_Q$ and $\|t\|_{m^p_{\omega,u}} \leq C\|f\|_{\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)}$ for some $C > 0$ where $\{a_Q\}_{Q \in \mathcal{Q}}$ is a (p, q, N)-atomic family with respect to ω and $t = \{t_Q\}_{Q \in \mathcal{Q}} \in m^p_{\omega,u}$.

For any $x \in \mathbb{R}^n \backslash 4Q$, the definition of Schwartz kernel and the vanishing moment condition satisfied by a_Q conclude that

$$T(a_Q)(x) = \int_{3Q} a_Q(y)K(x,y)dy$$

$$= \int_{3Q} a_Q(y)\left[K(x,y) - \sum_{|\gamma| \leq L} (\partial_y^\gamma K)(x, x_Q)\frac{(y - x_Q)^\gamma}{\gamma!}\right]dy.$$

The reminder form of the Taylor expansion and (4.8) yield

$$|T(a_Q)(x)| \leq \frac{C}{|x - x_Q|^{L+1+n}} \int_{3Q} |a_Q(y)||y - x_Q|^{L+1}dy$$

$$\leq \frac{C}{|x - x_Q|^{L+1+n}}|Q|^{(L+1)/n} \int_{3Q} |a_Q(y)|dy.$$

By using (4.6), for any $x \in \mathbb{R}^n \backslash 4Q$, we obtain

$$|T(a_Q)(x)| \leq C\omega(Q)^{-1/p}\left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-L-n-1}.$$

Therefore,

$$|T(a_Q)(x)| \leq |T(a_Q)(x)|\chi_{4Q}(x)$$

$$+ C\omega(Q)^{-1/p}\chi_{\mathbb{R}^n \setminus 4Q}(x)\left(1 + \frac{|x - x_Q|}{l(Q)}\right)^{-L-n-1}.$$

As T is bounded on $L^p(\omega)$, $1 < p < \infty$ (see [16, Corollary 9.4.7]), we find that the rest of the proof is similar to the proof of Theorem 4.4. Thus, for the sake of brevity, we leave it to the reader. $\qquad\square$

The boundedness result may not only consider at the level of atoms, see [2]. Even though Theorem 4.3 only imposes a weaker condition on u, we need a stronger requirement on u for the validity of the preceding theorem because Lemma 5.4 is involved in the proof of Theorem 4.5.

5. Technical Results

In this section, we state and prove an important technical result for our non-smooth atomic decompositions for $\mathfrak{H}^p_{\omega,u}(\mathbb{R}^n)$. We need this technical lemma as the supports of the non-smooth atoms are the dilated dyadic cubes $3Q$ where $Q \in \mathcal{Q}$. The family of dilated dyadic cubes has a serious drawback. It does not possess the nested property. That is, for any arbitrary $Q, P \in \mathcal{Q}$, we have neither $3P \subset 3Q$ nor $3P \cap 3Q = \emptyset$. Furthermore, for any $Q \in \mathcal{Q}$, there exists a $P \in \mathcal{Q}$ such that

$$P \not\subset 3Q \quad \text{and} \quad P \cap 3Q \neq \emptyset.$$

To overcome this obstacle, we consider the λ-neighborhood of the dyadic cubes.

For any $\lambda > 1$ and any $Q \in \mathcal{Q}$, we call a family of dyadic cubes $\{Q^k\}_{k=1}^{2^n}$ the λ-neighborhood of Q if

$$\lambda Q \cap Q^k \neq \emptyset, \quad 1 \leq k \leq 2^n \qquad (5.1)$$

$$l(Q^j) = l(Q^i), \quad 1 \leq i,\ j \leq 2^n \qquad (5.2)$$

$$\lambda Q \subseteq \bigcup_{k=1}^{2^n} Q^k \qquad (5.3)$$

and for any family of dyadic cubes $\{P^k\}_{k=1}^{2^n}$ satisfying (5.1)–(5.3),

$$l(Q^k) \leq l(P^k), \quad 1 \leq k \leq 2^n. \tag{5.4}$$

We use $N_\lambda(Q)$ to denote the λ-neighborhood of Q.

The following results are straightforward consequences of the definition of the λ-neighborhood of $\mathcal{Q}$. For brevity, we leave the proof to the reader.

Lemma 5.1   *Let $\lambda > 1$ and $Q \in \mathcal{Q}$. For any $Q^k \in N_\lambda(Q)$, we have*

$$1 \leq \frac{l(Q^k)}{l(Q)} \leq \lambda, \quad 1 \leq k \leq 2^n \tag{5.5}$$

$$|c_{Q,i} - c_{Q^k,i}| \leq \frac{\lambda + 1}{2} l(Q), \quad 1 \leq k \leq 2^n, \ 1 \leq i \leq n \tag{5.6}$$

where $c_Q = (c_{Q,1}, \ldots c_{Q,n})$ and $c_{Q^k} = (c_{Q^k,1}, \ldots c_{Q^k,n})$ are the centers of Q and Q^k, respectively.

The following lemma is obtained by using Propositions 1.1 and 1.2.

Lemma 5.2   *Let $\lambda > 1$, $Q \in \mathcal{Q}$ and $\omega \in A_\infty$. For any $Q^k \in N_\lambda(Q)$, we have a constant $C > 0$ such that*

$$C^{-1}\omega(Q) \leq \omega(Q^k) \leq C\omega(Q), \quad 1 \leq k \leq 2^n. \tag{5.7}$$

Lemma 5.3   *Let $\lambda > 1$ and $Q \in \mathcal{Q}$. We have*

$$card(\{P \in \mathcal{Q} : Q \in N_\lambda(P)\}) \leq 2^n(1 + [\log_2 \lambda]).$$

Lemma 5.3 follows from (5.4) and (5.5). Lemma 5.4 is crucial to formulate and establish the non-smooth atomic decompositions of weighted Hardy-Morrey spaces, it is inspired by [20, Proposition 3.1].

Lemma 5.4   *Let $0 < p \leq 1$ and $q_\omega < q$. Suppose that ω and u satisfy the conditions given in Theorem 4.4. Then, for any $\lambda > 1$ and $t = \{t_Q\}_{Q \in \mathcal{Q}} \in m_{\omega,u}^p$, we have*

$$\sup_{P \in \mathcal{Q}} \frac{1}{(u(P))^p} \sum_{Q \in \mathcal{Q}} |t_Q|^p \left(\frac{\omega(P \cap \lambda Q)}{\omega(Q)} \right)^{1-p/q} \leq C\|t\|_{m_{\omega,u}^p}^p$$

for some $C > 0$ independent of t.

Proof.   For any $P \in \mathcal{Q}$, the $(1 - \frac{p}{q})$-inequality and (5.7) guarantee that

$$\sum_{Q \in \mathcal{Q}} |t_Q|^p \left(\frac{\omega(P \cap \lambda Q)}{\omega(Q)} \right)^{1-p/q} \leq \sum_{Q \in \mathcal{Q}} \sum_{Q^k \in N_\lambda(Q)} |t_Q|^p \left(\frac{\omega(P \cap Q^k)}{\omega(Q^k)} \right)^{1-p/q}.$$

Thus, Lemma 5.3 ensures that

$$\sum_{Q \in \mathcal{Q}} |t_Q|^p \left(\frac{\omega(P \cap \lambda Q)}{\omega(Q)} \right)^{1-p/q}$$

$$\leq 2^n (1 + [\log_2 \lambda]) \sum_{Q \in \mathcal{Q}} |t_Q|^p \left(\frac{\omega(P \cap Q)}{\omega(Q)} \right)^{1-p/q}$$

$$\leq 2^n (1 + [\log_2 \lambda]) \left(\sum_{Q \subseteq P} |t_Q|^p + \sum_{P \subset Q} |t_Q|^p \left(\frac{\omega(P)}{\omega(Q)} \right)^{1-p/q} \right).$$

Since $|t_Q| \leq \|t\|_{m^p_{\omega,u}} u(Q)$ for any $Q \in \mathcal{Q}$, by using (4.5), we have

$$\frac{1}{u(P)^p} \sum_{P \subset Q} |t_Q|^p \left(\frac{\omega(P)}{\omega(Q)} \right)^{1-p/q} \leq \|t\|^p_{m^p_{\omega,u}} \sum_{P \subset Q} \left(\frac{\omega(Q)}{\omega(P)} \right)^{p\kappa} \left(\frac{\omega(P)}{\omega(Q)} \right)^{1-p/q}.$$

As for any $l \in \mathbb{N}$ there exists an unique $Q \in \mathcal{Q}$ with $P \subset Q$ and $2^{nl}|P| = |Q|$ and $\kappa < \frac{1}{p} - \frac{1}{q}$, by using Proposition 1.1, we find that

$$\sup_{P \in \mathcal{Q}} \frac{1}{(u(P))^p} \sum_{Q \in \mathcal{Q}} |t_Q|^p \left(\frac{\omega(P \cap \lambda Q)}{\omega(Q)} \right)^{1-p/q} \leq C\|t\|_{m^p_{\omega,u}}. \qquad \square$$

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Kwok-Pun Ho
Department of Mathematics and Information Technology
The Hong Kong Institute of Education
10, Lo Ping Road, Tai Po, Hong Kong, China
E-mail: vkpho@ied.edu.hk